ON THE GRID REFINEMENT RATIO FOR ONE-DIMENSIONAL ADVECTIVE PROBLEMS WITH NONUNIFORM GRIDS

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Abstract. *The purpose of this work is to analyse the usual definition of grid refinement ratio used with irregular grids (unstructured, nonuniform and nonorthogonal), as weel as to define its aplicability. The main reasons for this subject are as follows: the grid refinement ratio is a parameter that directly affect the estimated discretization error made by several estimators; and irregular grids are largely used to obtain numerical solutions. The mathematical model used in this work is the one-dimensional advection of a scalar with source term. This model has known analytical solution, which is compared to numerical solution obtained through the finite difference method. It was verified that the usual definition of grid refinement ratio used with irregular grids is incorrect, even for the simpliest irregular grid, i.e. one-dimensional and nonuniform grid. But it is correct if a nonuniform grid is refined of a uniform way.*

Keywords: *numerical error, CFD, finite difference scheme, numerical simulation*

1. Introduction

The true numerical error (E) of a variable of interest is the difference between its exact analytical solution (Φ) and its numerical solution (ϕ). Then, an ideal numerical solution is the exact analytical solution of the problem, i.e, that in which the numerical error is void. The true numerical error value does not depend of experimental results, but is only obtained when the analytical solution of the mathematical model is known, i.e., (Ferziger e Peric, 2001)

$$E(\phi) = \Phi - \phi \tag{1}$$

In the unknown analytical solution cases, instead of the true numerical error, an estimated numerical error (U) is made, which is evaluated through the difference between the estimated analytical solution (ϕ_∞) and the numerical solution (ϕ), i.e.,

$$U(\phi) = \phi_\infty - \phi \tag{2}$$

It is considered that the numerical error is caused by four errors sources (Marchi and Silva, 2002): Truncation error, Iteration error, Round-off error and Programming error. The truncation errors arises from the numeric approximations used in the discretization of a mathematical model (Tannehill et al., 1997; Ferziger e Peric, 2001; Roache, 1998); the iteration error is the difference between the exact solution of the discretized equations and numerical solutions in a given iteration (Ferziger e Peric, 2001); the round-off errors are due mainly to a finite number of digits in arithmetic computations; and the programming errors includes the errors caused by persons while implementing and using a computational program. This work considers only truncation errors over the numerical solution. In this circunstance, the numerical error calculated through Eq. (1) is called discretization error (Ferziger e Peric, 2001). Other errors are negligible.

A way to evaluate the discretization error of a numerical solution is through the Richardson estimator, which can found in many references (Ferziger and Peric, 2001; Roache, 1998)

$$U_{Ri}(\phi_f) = \frac{(\phi_f - \phi_g)}{(q^p - 1)} \tag{3}$$

in which ϕ_f and ϕ_g are numerical solutions obtained by two grids with different number of elements, and each one of these grids can be represented through its elements size (h), i.e, h_f = fine grid e h_g = coarse grid; p is the asymptotic order (p_L) of the discretization error (Roache, 1994) or the apparent order (p_U); and q is the refinement ratio between two grids.

The usual definition of refinement ratio used with irregular grids (nonuniform, nonorthogonal and unstructured) is given by (Roache, 1994; Celik, 2005)

$$q = \left(\frac{N_f}{N_g} \right)^{\frac{1}{D}} \tag{4}$$

in which N_f and N_g represent, respectively, the number of elements of fine and coarse grids, and D is the spatial dimension of the problem, being equal to 1, 2 or 3, respectively for one, two and three-dimensional cases. In one-dimensional uniform grids the above equation is reduced to

$$q = \frac{N_f}{N_g} \tag{5}$$

The q value can suffer variation as locally between coarse and fine grids, as well as between three grids, as follows (Roache, 1998): the coarse grid related to the fine grid gives a refine value q, and the supercoarse grid related to the coarse grid a refine value q'. Being q constant between three grids, i.e, $q=q'$, the apparent order (p_U) gives (De Vahl Davis, 1983)

$$p_U = \frac{\log\left(\dfrac{\phi_g - \phi_{sg}}{\phi_f - \phi_g} \right)}{\log(q)} \tag{6}$$

in which ϕ_{sg} is the numerical solution obtained in the supercoarse grid (h_{sg}). This work considers q value as constant between grids. Another way to obtain the estimated discretization error of the numerical solution is through the GCI (Grid Convergence Index) estimator (Roache, 1994).

This aim of this work is to show that: (i) the usual definition of the refinement ratio used with irregular grids, Eq. (4), is correct only in the limit case of uniform grids; (ii) it is incorrect even for the simpliest irregular grid, i.e one-dimensional nonuniform grid; (iii) and if there is a mixed kind of grid between the uniform grid and the irregular grid in which Eq. (4) is valid.

This work validity is as follows: (i) the grid refinement ratio is a parameter that affects directly the estimated discretization error made by the Richardson estimator, Eq. (3); (ii) this estimator is recommended (Celik, 2005) by the Fluids Engineering Division of the ASME (American Society of Mechanical Engineers); and (iii) irregular grids are largely used to obtain numerical solutions.

2. Method

2.1. Numerical approximations

The Finite Difference Method (MDF) is the numerical method used in this work (Tannehill et al., 1997). Its principle is to approximate at each grid node, through the Taylor series (Kreyszig, 1999) each term of a mathematical model describing a problem, in each grid node. For example, a numerical approximation (T^i_{UDS})j in the grid node j, Fig. 1, to a first-order derivative by one-point upstream is given by

$$(T^i_{UDS})_j = \frac{(T_j - T_{j-1})}{h_j} \tag{7}$$

The truncation error (ε) of this approximation is

$$\varepsilon(T^i_{UDS})_j = T^{ii}_j \frac{h_j}{2} - T^{iii}_j \frac{h_j^2}{6} + T^{iv}_j \frac{h_j^3}{24} - \dots \tag{8}$$

where the superscripts are indicating, respectively, derivatives of 2nd, 3rd e 4th orders of T in the node j; hj is the distance between two consecutive grid nodes; and the three points indicate an infinite series.

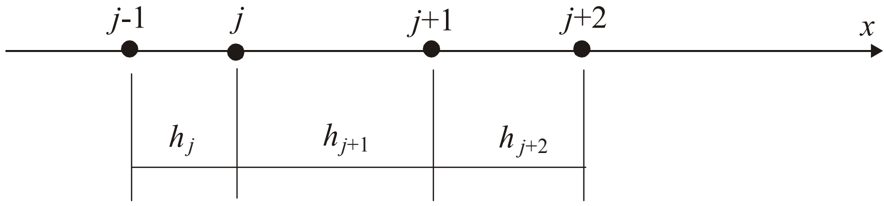

Figure 1. A nonuniform and one-dimensional grid.

It is necessary to see that, having an analytical solution for the mathematical model, it is possible to known previously which is the number of terms appearing in the Eq. (8), that leads to a direct analysis of the discretization error order.

2.2. True, asymptotic and effective order

By comparision of the truncations error expression, Eq. (8), it is accepted that the discretization error E(ϕ) is given by (Ferziger e Peric, 2001; Roache, 1998)

$$E(\phi) = C_1 h^{p_L} + C_2 h^{p_2} + C_3 h^{p_3} + \ldots \quad (9)$$

in which ϕ is the variable of interest; h is the grid spacing sizes; C_1, C_2, C_3, ... are coefficients that do not change with h; p_L, p_2, p_3, ... are the integer and positive numbers representing the true orders of the discretization error; and p_L is the asymptotic order of the discretization error. p_L is always ≥ 1 and shows the error curve inclination in a log(|E|) versus log(h) graph to $h \to 0$.

The effective order of the true error is given by (Marchi, 2001)

$$p_E = \frac{\log\left[\dfrac{E(\phi_g)}{E(\phi_f)}\right]}{\log(q)} \quad (10)$$

According to Eq. (10), the effective order (p_E) is a true error function of the variable of interest as well as of the refinement ratio (q) between the grids. For problems with known analytical solution, it can be used to verify if $p_E \to p_L$ while $h \to 0$, given validity to the used numerical approximations.

2.3 Definition of problems

The mathematical models used in this work are one-dimensional advection problems of a scalar with different source terms, shown by the following equations

$$\frac{dT^A}{dx} = 2x \quad (11)$$

$$\frac{dT^B}{dx} = \frac{e^x}{e-1} \quad (12)$$

In Eqs. (11) and (12), the superscripts represent the problems A and B; T is the dependent variable; and x is the independent variable. The boundary condition in the problems is $T(0) = 0$ and the problem domain length is $L = 1$.

The exact analytical solutions of problems A and B are

$$T^A = x^2 \quad (13)$$

$$T^B = \frac{e^x - 1}{e - 1} \quad (14)$$

2.4 Discretization of mathematical models

Applying Eq. (7) to problems *A* and *B*, Eqs. (11) and (12), one obtains

$$T_j^A = 2x_j h_j + T_{j-1}^A \quad (15)$$

$$T_j^B = \frac{e^{x_j} h_j}{e - 1} + T_{j-1}^B \quad (16)$$

These problems were chosen because, by the literature, there are no doubt that the asymptotic order (p_L) for them is $p_L = 1$, being them grids uniform or nonuniform, beyond representing examples of local variable.

To the A problem, with its analytical solution, Eq. (13), and the truncation error (ε) of the numerical approximation (T_{UDS}^i)j, Eq. (8), the following equation can be obtained for the discretization error, Eq. (9):

$$E(T^A)_j = (C_1)_j h_j \quad (17)$$

That is, the discretization error equation has only one term for the problem *A*. In the problem *B*, the discretization error equation has infinite terms of errors. Consequently: (i) for the problem *A*, $p_E = 1$ in any *h*; (ii) for the problem *B*, $p_E \to 1$ when $h \to 0$.

3. Results

The results showed in this section were obtained for the mathematical model given by Eqs. (13) and (14), using: (i) uniform coarse grid and uniform and nonuniform fine grid; (ii) nonuniform coarse grid and nonuniform fine grid. With these results it is possible to analyze which kind of refine (uniform or nonuniform) reproduces the theoretical asymptotic order of the numerical approximation, given by Eqs. (17).

3.1 Uniform coarse grid

Firstly, the results were obtained with a uniform grid, as seen in Fig. 2. To understand the effect of grid refinement in the discretization error order, the coordinate of point 2 was keep fixed in the coarse grid (h_g) and in the fine grid (h_f), having been made a variation only in position of floating point (*F*) of the fine grid, between $a = 0$ and $a = L/2$. Then, when a = L/4, the refine of the grid is uniform; on the contrary the refine is called nonuniform. As the exact analytical solution is also known to each case, Eq.(10) was used to calculate the effective order (p_E) at point 2. Equation (5) was used to calculate the refinement ratio (q).

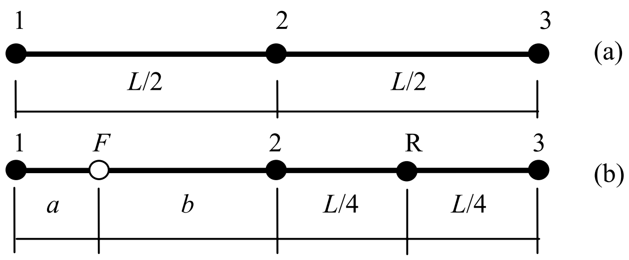

Figure 2. a) uniform coarse grid (h_g); b) nonuniform fine grid (h_f).

Figures 3 and 4 show the results of *T* and p_E for problem *A* at point 2. Here, the effective order (p_E) is equal to asymptotic order (p_L) only when $a = L/4$, i.e., when the refinement is uniform. It can be seen that as the numerical solution approaches the analytical solution, greater is the value of effective order (p_E). In Fig. 4, for $2a/L$=0.5 the effective order value at points 2 and 3 is p_E= 1.

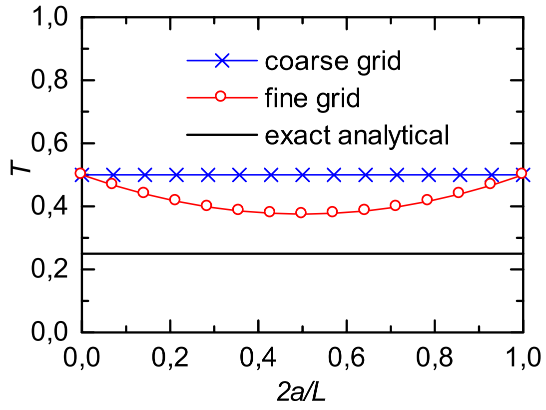

Figure 3. Solutions (T) for problem A at point 2 with grids of Fig. 2.

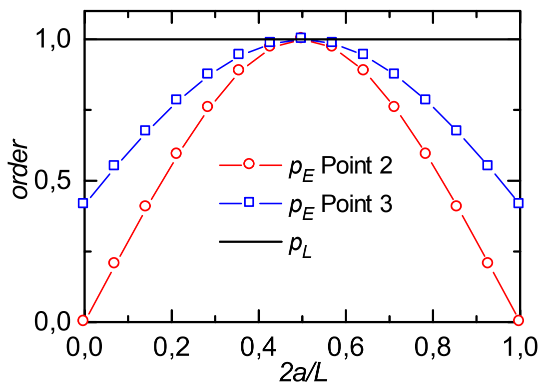

Figure 4. Effective order (p_E) for problem A with grids of Fig. 2.

3.2 Nonuniform coarse grid

As coarse grid of Fig. 2 is uniform, the next step of the tests was concentrated in the refinement shown in Fig. 5. The coarse grid is nonuniform and suffers a uniform and nonuniform refinement, according to the values of c, a and b.

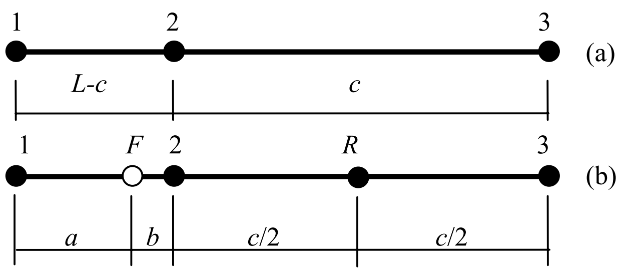

Figura 5. a) nonuniform coarse grid (h_g); b) nonuniform fine grid (h_f).

Figure 6 shows the results obtained for the effective order (p_E) for problem A, with $c = 3L/4$. Here, for $a/(L-c) = 0.5$ the effective order at points 2 and 3 is $p_E = 1$.

From Eq. (10), it was obtained for each problem what should be the correct value of the refinement ratio (q) that would result in the value of asymptotic order (p_L). Figure 7 shows these results. It was seen that there is a different curve to the value of q specific to this problem, and then, there is not a unique rule for its calculation. But in case of refine to be uniform, i.e., $a = (L-c)/2$, the curve pass through the point where $q = 2$.

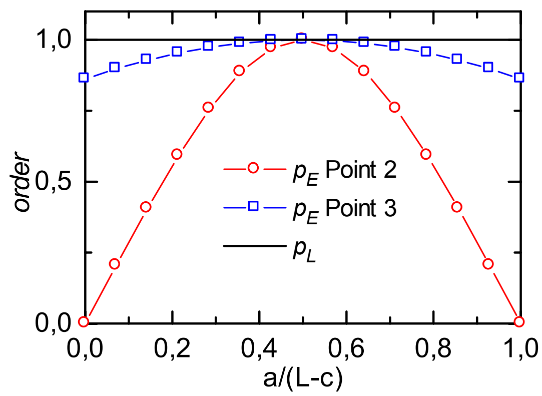

Figure 6. Effective order (p_E) for problem *A* for grids of Fig. 5 with $c = 3L/4$.

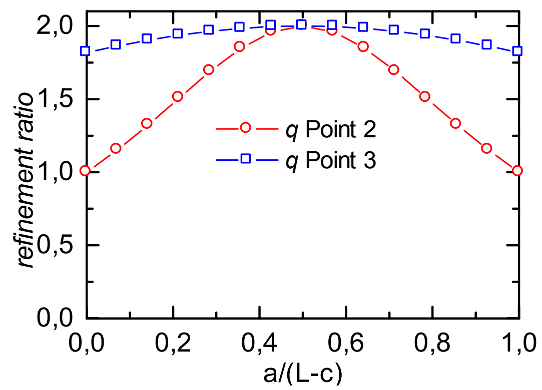

Figura 7. Refinement ratio (q) that reproduce asymptotic order (p_L) for grids of Fig. 5 with $c = 3L/4$.

3.3 Grid with h → 0

To finish the tests, problem *B* was solved, whose discretization error equation, Eq.(9), has infinite terms. In this problem, the effective order (p_E) tends to asymptotical order ($p_L = 1$) when $h \to 0$. The grid used here is shown in Fig. 8, and differs from grid of Fig. 5 only in the number of nodes, these being greater here.

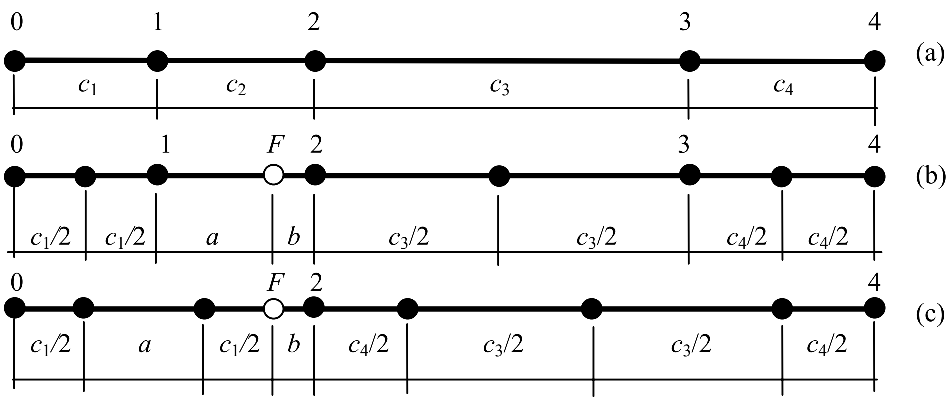

Figure 8. Grids (a) nonuniform coarse (h_g); (b) nonuniform fine (h_f); (c) shuffle nonuniform fine (h_{fe}).

Figures 9 and 10 show the results obtained. Three pairs of grids of kind (a) and (b) according to Fig. 8 were used: (1) grids with 4 and 8 elements; (2) grids with 8 and 16 elements; and (3) grids with 16 and 32 elements. The floating point *F* is always the point at left of point 2 and the distance between them is (b). Point at left of *F* is at a distance (a).

Note that when $h \to 0$, $p_E \to p_L = 1$ for any value of $a/(a+b)$. In Fig. 10, to $a/(a+b) = 0.5$, the effective order at point 2 is: $p_E = 1.01523$ for grids with 4 and 8 elements; $p_E = 1.00768$ for grids with 8 and 16 elements; and $p_E = 1.00385$ for grids with 16 and 32 elements.

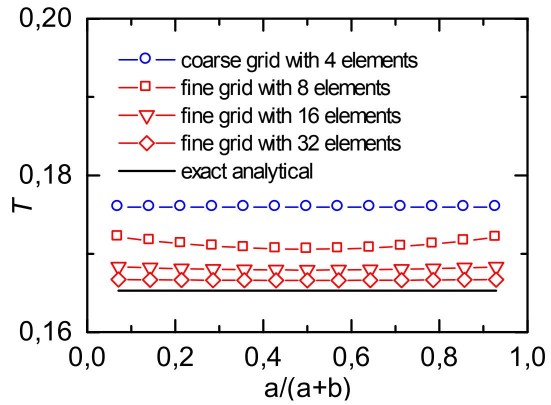

Figure 9. Solutions (T) for problem B at point 2 for three grid pairs of type shown in Fig. 8a and 8b.

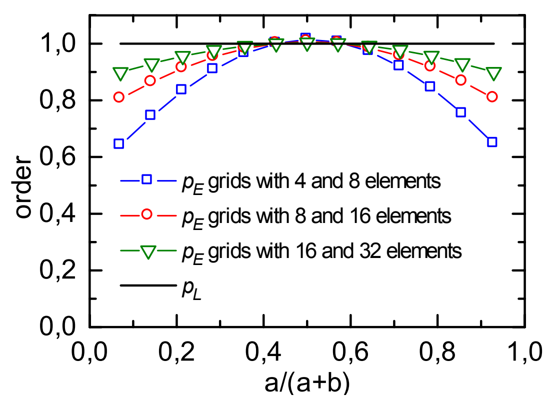

Figure 10. Effective order (p_E) for problem B at point 2 for three grid pairs of type shown in Fig. 8a and 8b.

It was also seen that when $h \to 0$ for fine grid shown in Fig. 8b, it is possible to change freely the position of grid elements, to the left as well to the right from point 2, as shown in Fig. 8c, without any alteration in curve $p_E \to p_L = 1$. The results are shown in Fig. 11, at points 2 and 3, obtained for grids with 4, 8, 16, 32, ..., 8192 and 16384 elements, using the refine of type shown in Fig. 8c.

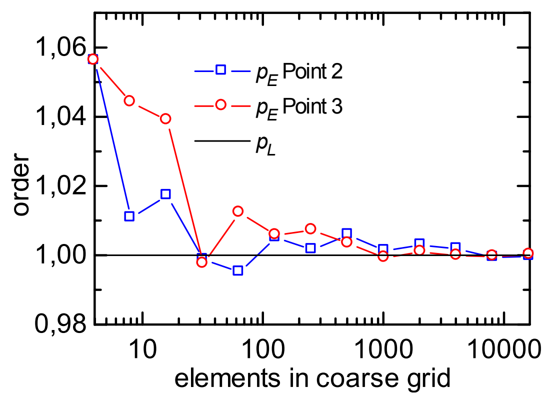

Figura 11. Effective order (p_E) for problem B with thirteen grid pairs of type shown in Fig. 8a and 8c.

From Fig. 10 and 11 it could be seen that when $h \to 0$, there is no matter the order in which fine grid elements are arranged. It is important only if them were obtained through uniform refinement. In Fig. 11, for the grid with 16384 elements, the value of the effective order at point 2 is $p_E = 0.99964$, and at point 3 is $p_E = 1.00009$. And also it is easy to verify that, in any case, for the greater value of h at coarse and fine grids, or for minor value of h at coarse and fine grids, the ratio h_g / h_f is equal to two.

4. Conclusion

To the usual definition of refinement ratio used with irregular grids, Eq. (4), it was showed that:

1) It is incorrect even for the simplest irregular grid, i.e., one-dimensional and nonuniform grid;
2) It is correct when a nonuniform grid is refined in a uniform way, i.e., when each element of the coarse grid is divided in an integer number of segments with equal size to produce the fine grid, being this number constant along the grid and equal to the refinement ratio (q) itself.

According to results of this work, it is believed that the two points above can also be applied to the refinement of multidimensional irregular grids (nonorthogonal, unstructured and nonuniform).

When the goal is to calculate effective and apparent orders, and to estimate the value of discretization error, it is recommended to proceed with uniform refine of irregular grids. In this case, for graphs of error versus h, or order versus h, the suitable metric can be the minor or the greatest size (h) of elements of each grid.

5. References

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