SOLUTIONS OF THE 2D LAPLACE EQUATION WITH TRIANGULAR GRIDS AND MULTIPLE RICHARDSON EXTRAPOLATIONS

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Abstract. *Numerical solutions for the two-dimensional Laplace equation using isosceles right triangular cell-centred and square control volumes are compared in this work. The methodology employed for the square grids is the one related to unstructured grids, while for the square volumes the discretization process is based on the structured grids methodology. For both geometries, the number of control volumes varies from 4 up to 16,777,216, with a refinement ratio of 2. In order to speed up the obtainment of the numerical results, the algebraic multigrid method was employed for the triangular grids and the geometric multigrid method for the square ones. An unexpected observation was that, for a grid with the same number of control volumes, the triangular grid has a worse performance than the square one, exhibiting a numerical error 2.3 - 2.4 greater. With the intention of reducing the discretization error, providing more accurate results, a strategy based on Richardson Extrapolations was employed, called Multiple Richardson Extrapolations (MRE). According to previous works, this methodology was successfully used in square grids, but its use is not a common practice for triangular ones. It was verified that MRE efficiently reduces the discretization error in triangular grids, although error magnitudes are considerably higher than the ones achieved for a square grid with the same number of control volumes and MRE. The main results of this work can be summarized as follows: (1) although triangular grids are more adaptable than square ones, the last ones should be used as frequent as possible due to the lower discretization errors involved; (2) MRE is efficient and can be used for the reduction of discretization errors in triangular grids; (3) for the same approximation scheme, numerical errors with MRE can not be lessening under than a limit value: there is always a dependence on the performance of the original results.*

Keywords*: Finite Volume Method, Multiple Richardson Extrapolations, discretization error reduction*

1. INTRODUCTION

The continuous improvement of the computer resources leads to the opportunity of describing natural phenomena at previously unimaginable scales. This access and this opportunity have served as strong drivers for computational sciences and engineering, especially in the last 20 years (Ghanem, 2009). In order to achieve accurate results, however, verification procedures are required. The numerical error $E(\phi)$ related to the numerical solution ϕ can be evaluated by the following expression:

$$E(\phi) = \Phi - \phi \tag{1}$$

in which Φ is the exact analytical solution. Numerical errors are composed by four elements: truncation, iteration, round-off and programming errors (Marchi and Silva, 2002). When the numerical error consists on the contribution of none but the truncation one, it is also called discretization error (Tannehill *et al*., 1997).

Since the beginnings of the 20th Century, procedures for discretization error estimates were proposed by Richardson (Richardson, 1910; Richardson and Gaunt, 1927). Besides the common use of Richardson extrapolations only as uncertainty estimator, the technique provided by Richardson can also be used to reduce the discretization error, as made by him for the two-dimensional heat diffusion problem (Richardson and Gaunt, 1927). In this case, Richardson extrapolations were employed recursively for two grid levels, providing more accurate results. Other authors (Benjamin and Denny, 1979; Schreiber and Keller, 1983, Erturk *et al.*, 2005) also employed Richardson extrapolations recursively for a higher number of grid levels (but four at maximum), intending the reduction of the discretization errors in CFD problems. More usual, nevertheless, is the use of only one Richardson extrapolation for the reduction of the discretization error, as made by Wang and Zhang (2009a, 2009b) and Ma and Ge (2010).

Marchi *et al*. (2008) and Marchi and Germer (2009), however, employed Richardson extrapolations recursively for several grid levels, in a process named Multiple Richardson Extrapolations (MRE), for the two-dimensional Laplace equation and one-dimensional advection-diffusion equation, respectively, using structured grids. For both cases, discretization errors were substantially reduced. According to both works, MRE should be used to: (1) for a given discretization error magnitude, reduce the computational requirements by the use of coarser grids; or (2) for a given grid, considerably reduce the magnitude of the discretization errors in order to obtain *benchmark* results.

The aim of this work is to investigate the use of MRE in the reduction of discretization errors of the two-dimensional Laplace equation, in a unitary square domain (Fig. 1), discretized with isosceles right triangular cell-centred grids; the results might be compared to the ones obtained in square volumes grids. Triangular volumes are related to unstructured grids, which are the most general form of grid arrangement for more complex geometries (Versteeg and Malalasekera, 2007). The Laplace equation was chosen in this work by its simplicity and the fact that, for such problem, there is an analytical solution, which allows numerical verification. The use of Richardson extrapolations to reduce the discretization error in triangular grids, nevertheless, is not a common practice. Works like the one of Jyotsna and Vanka (1995), in which the Richardson extrapolation was employed to obtain more accurate results for the velocity pattern (and consequently, for the flow field) in triangular grids are yet exceptions.

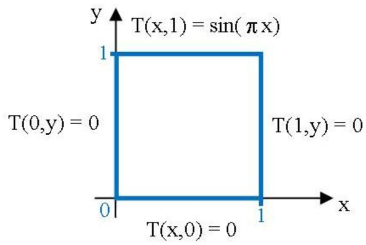

Figure 1. Physical domain and boundary conditions.

2. MATHEMATICAL MODEL

The mathematical model considered in this work is related to the two-dimensional Laplace equation with Dirichlet boundary conditions, schematically given in Fig. 1:

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0; \quad 0 < x, y < 1 \\ T(x,1) = \sin(\pi x); \quad T(0,y) = T(1,y) = T(x,0) = 0 \end{cases} \tag{2}$$

where: x and y are the spatial coordinates and T is the temperature. This equation can be physically related to the heat diffusion on a two-dimensional plate in steady state with constant thermal properties and absence of heat generation (Incropera *et al*., 2008), whose analytical solution is given by $T(x,y) = \sin(\pi x)\sinh(\pi y)/\sinh(\pi)$.

The variables of interest in this problem includes: (1) the temperature at the domain centre (Tc), in other words, the temperature at position x = 1/2 and y = 1/2; (2) the average temperature (Tm) of the whole domain; and the heat transfer rates at the four boundaries, namely: (3) y = 1 (Qn), (4) y = 0 (Qs), (5) x = 1 (Qe), and (6) x = 0 (Qw). The variables Tm, Qs and Qe (Qn and Qw are defined in an analogous way) are defined by the following expressions:

$$Tm = \int_0^1 \int_0^1 T(x,y)\,dx\,dy, \quad Qs = -k\int_0^1 \left(\frac{\partial T}{\partial y}\right)_{y=0} dx, \quad Qe = -k\int_0^1 \left(\frac{\partial T}{\partial x}\right)_{x=1} dy \tag{3}$$

where k is the thermal conductivity, which is assumed to have unitary value in this work.

3. NUMERICAL MODEL

3.1. Numerical solutions without MRE

The unitary side square domain is discretized using the finite volume method (Maliska, 2004; Versteeg and Malalasekera, 2007), using both isosceles right triangular cell-centred and square grids, as shown at Fig. 2. While the methodology applied to square grids is the one related to structured grids, for triangular grids the methodology was the same employed to unstructured ones. For both cases, second order approximation schemes (central differencing scheme, CDS) were used. In order to speed up the convergence of the numerical codes, two different multigrid methods were employed: for triangular grids, an algebraic multigrid (AMG) algorithm, adapted from Ruge and Stüben (1986); and for square grids, a geometric multigrid (GMG) algorithm (Briggs *et al.*, 2000; Trottemberg *et al.*, 2001).

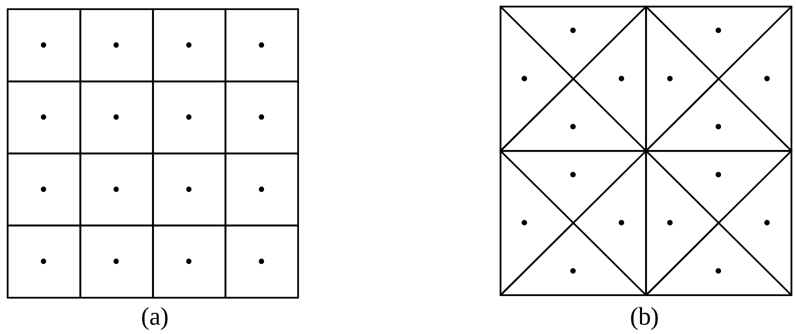

Figure 2. (a) Square and (b) isosceles right triangular grids, both with 16 real control volumes.

The AMG features employed to achieve the numerical results are: correction scheme (CS) (Brandt, 1986; Ruge and Stüben, 1986; Stüben, 2001); V-cycle; parameter of the strength of connection (θ) equals to 0.25; parameter of the strong dependence on the coarser grid (ε) equals to 0.35. Otherwise, for GMG, the main features used are: full approximation scheme (FAS) (Briggs *et al*., 2000; Trottemberg *et al*., 2001); V-cycle; and grid-size ratio of 2. For both multigrid methods: lexicographic Gauss-Seidel (Burden and Faires, 2008) was employed as smoother (with one internal iteration); the number of cycles was high enough to achieve the machine round-off error; double precision operations were used for all the calculations; and null temperature was employed for the whole domain as initial guess.

In order to evaluate numerically the integrals related to the average temperature of the whole domain and the heat transfer rates on the four boundaries, rectangle rule (Kreyszig, 1999) was employed. Moreover, in the evaluation of the derivatives related to the heat transfer rates, upstream differencing scheme (UDS) or downstream differencing scheme (DDS) (Tannehill *et al*., 1997) was adopted, depending on the case. Otherwise, the temperature at the domain centre is evaluated by the arithmetical mean of the temperatures of the volumes with one of the vertices at the coordinates $x = 1/2$ and $y = 1/2$. This procedure is needed once neither in triangular cell-centred nor in square grids, there is a nodal point which is placed exactly on the domain geometric centre.

3.2. Numerical solutions with MRE

Once the numerical solutions are obtained, Richardson extrapolations can be used for reducing the numerical errors associated to the discretization process according to the following expression:

$$\phi_{g,m} = \phi_{g,m-1} + \frac{\phi_{g,m-1} - \phi_{g-1,m-1}}{r^{p_{m-1}} - 1} \tag{4}$$

where: ϕ is the numerical solution of a given variable of interest; the index g refers to the grid in which the numerical solution is evaluated; the index m is the number of Richardson extrapolations; r is the refinement ratio ($r=h_{g-1}/h_g$); and p_m are the true orders of the discretization error (Marchi and Silva, 2002). Equation (4) is valid for $g=[2,G]$ and $m=[1,g-1]$, in which g=1 refers to the coarsest grid, g=G is the most refined grid, m = 0 refers to the numerical solution without any extrapolation and m = 1 is related to the standard Richardson extrapolation. For each value of $\phi_{g,m}$ in Eq. (4), numerical solutions of ϕ in two different grids (g and g – 1) in the m – 1 extrapolation are needed.

For a given value of g, Eq. (4) can be used recursively g – 1 times, providing m Richardson extrapolations. In this work, Multiple Richardson Extrapolations results are obtained when $m>1$. The values of the true orders (p_m) are related to the exponents of the truncated terms of the Taylor series employed in the approximation schemes for the derivatives. More details about Eq. (4) and/or MRE theory can be found in Marchi *et al.* (2008).

4. NUMERICAL RESULTS

Twelve different grids are employed in this work for both triangular cell-centred and square grids: from grids with only 4 real volumes (2^2) up to 16,777,216 real volumes (2^{24}), respecting a (two-dimensional) refinement ratio of 2.

Double precision are used for all the operations and the number of multigrid cycles for both grid geometries were high enough to minimize the iteration error. Numerical results for the six variables of interest are compared to the values of the analytical solutions, with 30 significant figures, obtained in Maple. These comparisons allow the evaluation of the real numerical error in order to study the efficiency of MRE in reducing numerical errors.

The consistency of both triangular and square volumes can be observed in Fig. 3: as expected, for both cases, the mean l_1-norm of the numerical error of the temperature decreases with grid refinement (h represents the two-dimensional grid spacing). Curiously, considering the same number of volumes for both triangular and square grids, triangular norm is always higher than the corresponding square one, by a factor of about 2.3 - 2.4 (except by the two coarsest grids). This result indicates that triangular cell-centred grids present higher discretization errors than their square counterparts. Based on this, despite the adaptability of triangular grids for complex geometries, square structured grids should be employed as frequent as possible, once the discretization errors are smaller for such grids.

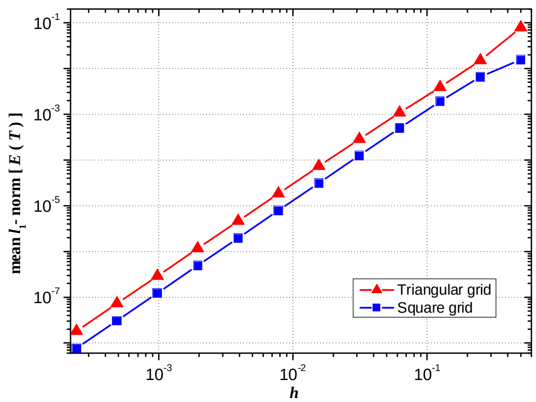

Figure 3. Mean l_1-norm of the numerical error $E(T)$.

The use of several grid sizes allows the evaluation of apparent orders (p_U) for all the variables of interest. Apparent orders (De Vahl Davis, 1983) should be used for ***a posteriori*** verification of the values obtained ***a priori*** for the true orders of numerical error. Details about the evaluation of the true orders and the use of them in MRE are explained in Marchi *et al*. (2008). Results of apparent orders (p_U) for the heat transfer rate at x = 1 (Qe) for both grid volume geometries are presented in Fig. 4. When m = 0, results are related to the asymptotic error order, while for m =1 and m = 2, results are related to the second and third true error orders, respectively. Apparent orders tend to values 2, 4 and 6, being in concordance with the results of Giacomini and Marchi (2009) for the first order approximations (UDS) of derivatives. This result, however, comes from a type of order degeneration, once the UDS presents as asymptotic error order the unitary value and not the value of 2. Nevertheless, as the apparent orders tend to 2, 4, 6 and so on, these values were employed as true orders for MRE. Similar behaviour was seen for the other heat transfer rates (Qn, Qs, Qw) and also for the other variables of interest (Tc, Tm), although for these two last ones, the expected true order values were the ones found.

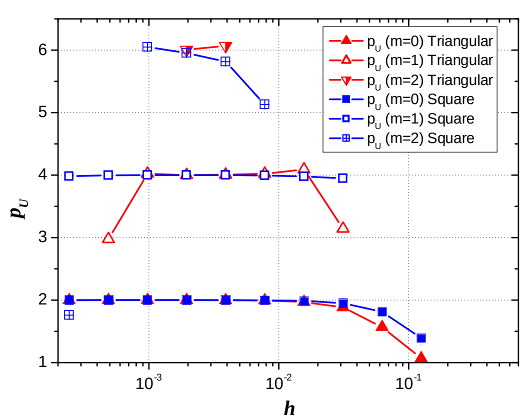

Figure 4. Apparent orders for Qe.

The discretization error results for the variables of interest are presented in Fig. 5; numerical results for the heat transfer rates at $y = 1$ (Qn) and $x = 0$ (Qw) are similar to the ones of the heat transfer rate at $x = 1$ (Qe) and are therefore omitted. In all the cases, both results for triangular cell-centred and square grids are presented, where Eh is the numerical error without MRE and Emre is the numerical error with MRE. As anticipated by the l_1-norm (Fig. 3), in all the cases the numerical error observed, without MRE, in square grids are smaller than the counterparts for triangular grids. Even for Qe, Fig. 5(c), square grid results are a little better to the ones of triangular grids, although the curves for both cases are almost coincident.

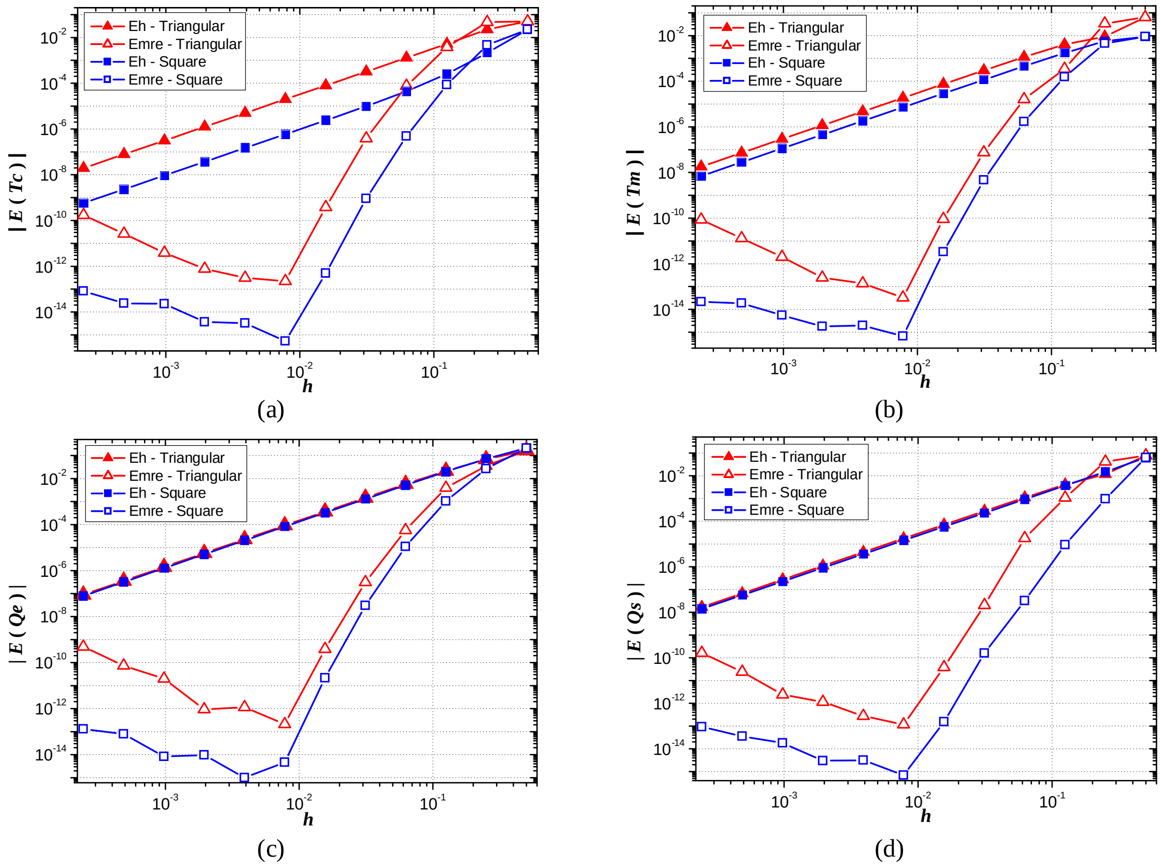

Figure 5. Modulus of the numerical error with (Emre) and without (Eh) MRE for: (a) *Tc*, (b) *Tm*, (c) *Qe* and (d) *Qs*.

As can be seen for all the variables presented in Fig. 5, MRE substantially reduces the numerical error for both the triangular cell-centred and the square grids. While the numerical error is smaller than 10^{-12} for all the variables in a grid with $2^{14} = 16{,}384$ volumes ($h \approx 8\text{x}10^{-3}$) for both geometries using MRE, even in the finest grid, with $2^{24} = 16{,}777{,}216$ volumes ($h \approx 2\text{x}10^{-4}$), the numerical error without MRE is greater than 10^{-10} (it is, mostly, about 10^{-7} - 10^{-9}). The use of MRE for the problem presented, however, does not reduce the numerical error for grids with more than 2^{14} volumes: as seen for all the variables, numerical error achieves a minimum for such grid. In fact, this is an effect of the round-off errors: as the discretization errors associated to MRE reduces much faster than the ones obtained by simple grid refinement, the use of a small amount of grids is enough to achieve the round-off error related to double precision calculations. Because of this, in spite of the numerical error keep on lessening with the grid refinement, it grows as a consequence of the increasing of the round-off error. Such situation can be avoided by the use of the quadruple precision, whose round-off error is about 10^{-30} - 10^{-32}, as presented by Marchi *et al*. (2008).

Considering again the results for the grid with $2^{14} = 16{,}384$ volumes, for triangular cell-centred grids, it can be seen in Fig. 5 that numerical errors are under 10^{-12}, for all the variables, using MRE. In comparison, taking the same grid, but not employing MRE, numerical errors are about 10^{-4} or 10^{-5}. In this case, the use of MRE could reduce numerical errors of about 7 or 8 orders of magnitude, proving the efficiency of MRE in lessening the numerical error for triangular grids. This effect is similar to the one observed to square grids: taking the same grid (with 16,384 volumes), the numerical error without MRE is about 10^{-4} to 10^{-6}, while the use of MRE provides numerical errors of about 10^{-14} or 10^{-15}. Comparing both results, the use of MRE could reduce numerical errors of about 8 to 10 orders of magnitude, as previously observed by Marchi *et al*. (2008) and Marchi and Germer (2009).

One of the possible uses of MRE is illustrated in Tab. 1: the reduction of the grid refinement necessary to achieve a given error magnitude. In this case, numerical error was taken constant and equal to 10^{-6} for three different variables of interest (*Tc*, *Tm* and *Qe*). As can be seen, taking the same geometric volume shape, the ratio of the number of volumes needed to achieve the error magnitude without and with MRE is at least equal to 64 (for *Tc*, using square grids). This ratio, nevertheless, can be as high as 1,024, as seen for *Qe*. Hence, as a consequence of the use of coarser grids to achieve a given error magnitude, the need for CPU time and RAM associated to refined grids are significantly reduced by the employment of MRE.

Observing both results presented in Tab. 1 and Fig. 5, the effect of the use of MRE on the numerical errors for both triangular cell-centred and square grids is clear. Similarly to previous results for square grids (Marchi et al., 2008; Marchi and Germer, 2009), MRE application for triangular grids is very effective for the reduction of numerical errors. It must be noted, nevertheless, that numerical errors associated to triangular volumes are greater to the counterparts with square grids, with or without the use of MRE. As the numerical approximations adopted for triangular and square grids are the same (second order, Dirichlet boundary conditions applied with ghost-cells), the behaviour of numerical error is similar in both cases. The use of MRE is not effective to reduce numerical errors associated to triangular grids to values smaller than the ones of square grids with MRE. So, the dependency of MRE on the obtained numerical solutions is obvious: MRE procedure efficiently reduces numerical errors but this reduction has a limit – if a discretization technique provides better numerical results without MRE than a second one, the results for this first technique with MRE keeps on being better than the second one with MRE. It is clearly represented by the comparison between triangular cell-centred and square grids.

Table 1. Quantity of volumes needed to achieve a given numerical error magnitude.

Volume geometry	**Variable and Error magnitude**		
	***Tc*, $E \approx 10^{-6}$**	***Tm*, $E \approx 10^{-6}$**	***Qe*, $E \approx 10^{-6}$**
Triangular, without MRE	2^{18} = 262,144	2^{18} = 262,144	2^{20} = 1,048,576
Triangular, with MRE	2^{10} = 1,024	2^{10} = 1,024	2^{10} = 1,024
Square, without MRE	2^{14} = 16,384	2^{16} = 65,536	2^{20} = 1,048,576
Square, with MRE	2^{8} = 256	2^{8} = 256	2^{10} = 1,024

5. CONCLUSION

Isosceles right triangular cell-centred and square grids were employed in the discretization of the two-dimensional Laplace equation with Dirichlet boundary conditions by the finite volume method in order to study the efficiency of MRE. The numerical model implemented includes: second order approximation scheme (CDS); boundary conditions applied with ghost-cells; the discretization with triangular grids made according to the methodology for unstructured grids; the discretization with square grids made according to the procedures for structured grids; algebraic multigrid for triangular grids and geometric multigrid for square ones, in order to speed up the numerical convergence; Gauss-Seidel smoother; number of multigrid cycles high enough to achieve the machine round-off error; double precision calculations.

The main results of this work are:

1. Despite the versatility of the triangular grids, the use of square grids is recommended (if the geometry of the domain allows its employment) by the smaller discretization errors associated to this geometry.
2. MRE is efficient to reduce numerical errors in triangular grids.
3. Considering different grid types but the same approximation scheme, MRE results depend on the original numerical errors: even if numerical errors can be reduced with the use of MRE, they can not be lessening more than a limit value. In this work, as triangular volume grids present higher levels of discretization error than the square ones without MRE, so the results with MRE present the same tendency.

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