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AN APPLICATION OF THE METHOD OF MANUFACTURED SOLUTIONS INTO A SPALART ALLMARAS EQUATION CODE

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ABSTRACT – Based on the Method of Manufactured Solutions, a finite volume code for the solution of an equivalent Spalart-Allmaras equation (S-A) received code verification. All operators in the S-A equation received coordinates exchange for the curvilinear coordinate system. Then, the diffusive terms were approximated by the CDS-2 method while the advective ones by the UDS method. Succeeding the creation of the Manufactured Solution, the resulting linear system was solved iteratively by the MSI method. After a grid refinement study, the effective order of the computational solution for the finest grid converged to the minor asymptotic order of truncation error value between CDS-2 and UDS methods. This process proves that the code is verified and suitable for CFD applications.

1. INTRODUCTION

The Method of Manufactured Solutions (MMS) is a mathematical procedure widely applied in code verification, targeting error evaluation. This method, described by Roache (2002), generates an analytical solution to any set of equations, where the main objective is a construction of a Manufactured Solution (MS). All terms in the MS should be evaluated by the process and cannot assume trivial values. Usually, the MS is built through symbolical manipulation and uses smooth analytical functions. Moreover, the MS emerges as a benchmark solution for the discrete operators or discretized equations without any compromise regarding physical accuracy or logical implementation mistakes. The current procedure evaluates only the code itself.

The equation of interest is the Spalart-Allmaras (S-A) turbulence model equation. Described by Allmaras et al. (2012) as an equivalent conservation form of the classical S-A model, originated by combining it with the mass conservation equation (in their paper, Equation 9). Américo (2021) described the coordinate exchange processes to cylindrical coordinates and curvilinear coordinates. Additionally described the discretization processes that originated the matrix of coefficients and source terms used in this work.

This Finite Volume Code, written in Fortran 2008, was generated by mixed spatial discretization schemes. Diffusive terms received a second-order Central Differencing Scheme (CDS-2). The advective terms received an Upwind-Differencing scheme (UDS). A Dirichlet boundary condition was applied based on the manufactured function for the variable of interest.

2. MANUFACTURED GRID AND SOLUTION

The S-A equation, written in cylindrical coordinates, is represented by Equation 1. The Trip term is not present in this analysis due to the absence of wall detachments consideration. All variables with subscripts represent partial derivatives.

$$
\begin{aligned}
0 = &+ \frac{\partial \rho \hat{\nu}}{\partial t} + \frac{J}{y^p}\left[\frac{\partial\left(y^p \rho \hat{\nu} U\right)}{\partial \xi} + \frac{\partial\left(y^p \rho \hat{\nu} V\right)}{\partial \eta}\right] - \rho\left(P - D\right) \\
&- \frac{J}{\sigma y^p}\left\{\frac{\partial}{\partial \xi}\left[y^p \rho\left(\hat{\nu}+\nu\right) J\left(\hat{\nu}_\xi \alpha - \hat{\nu}_\eta \beta\right)\right] + \frac{\partial}{\partial \eta}\left[y^p \rho\left(\hat{\nu}+\nu\right) J\left(\hat{\nu}_\eta \gamma - \hat{\nu}_\xi \beta\right)\right]\right\} \\
&- \frac{c_{b2}}{\sigma}\rho J^2\left[\hat{\nu}_\xi^2 \alpha + \hat{\nu}_\eta^2 \gamma - 2\hat{\nu}_\eta \hat{\nu}_\xi \beta\right] + \frac{1}{\sigma}\left(\nu + \hat{\nu}\right) J^2\left[\rho_\xi \hat{\nu}_\xi \alpha + \rho_\eta \hat{\nu}_\eta \gamma - \rho_\eta \hat{\nu}_\xi \beta - \rho_\xi \hat{\nu}_\eta \beta\right]
\end{aligned} \tag{1}
$$

where ρ is density; $\hat{\nu}$ is the auxiliary variable to obtain the turbulent viscosity, treated as the variable of interest; t is time; y^p is the distance between the volume centroid and the symmetry line; $J(\xi,\eta)$, $\alpha(\xi,\eta)$, $\beta(\xi,\eta)$, and $\gamma(\xi,\eta)$ are grid parameters, calculated as described by Maliska (2004) ; U and V are the contravariant velocities and ν is the kinematic viscosity. All other terms like P, D, σ and c_{b2} are described by Allmaras et al. (2012).

As suggested by Roache (2002), the MS was generated in the physical domain (x, y) and then translated to the computational domain (ξ, η). Values for ξ and η represents the grid displacement for each direction (horizontal and vertical, respectively). The grid conversion process is given by:

$$X\left(\xi,\eta\right) = 1.2\, C \cos\left(arc\right);\; Y\left(\xi,\eta\right) = -0.5\, C \sin\left(arc\right) \tag{2}$$

$$arc = \frac{0.5\,\pi\ \left(\eta - 1.0\right)}{ny - 2.0} - 0.75\,\pi;\; C = \frac{1.0\ \left(\xi - 1.0\right)}{nx - 2.0} + 0.5 \tag{3}$$

where nx is the number of volumes in the ξ direction; ny is the number of volumes in the η direction; $X\left(\xi,\eta\right)$ and $Y\left(\xi,\eta\right)$ are the curvilinear grid displacement. As a matter of simplification, from now on they are referred as x and y, respectively.

All auxiliary equations required by the S-A model (e.g., P, D and modified vorticity) can be quickly obtained from the manufactured equations. Since the MMS do not require physical accuracy, for the MS were considered: dynamic viscosity $\mu = \rho^2$; $\nu = \rho$; the distance to the closest wall equal to unity ($d = 1$) for all volumes. Furthermore, Equation 4 brings the manufactured $\hat{\nu}\left(x,y\right)$ and $\rho\left(x,y\right)$. Equation 5 depicts two auxiliary terms, $g\left(x,y\right)$ and $\zeta\left(x,y\right)$. The manufactured contravariant velocities $U\left(x,y\right)$ and $V\left(x,y\right)$ are written in Equation 6, on it $mcc = 0.1$. This value represents a small perturbation, useful to check the robustness of the numerical scheme. Once the robustness is verified, mmc can be set to zero.

$$\hat{\nu}\left(x,y\right) = 0.3\ \sin\left(x\right)\ \cos\left(y\right) + 1.0;\; \rho\left(x,y\right) = 0.2\ \sin\left(x\, y\right) + 1.0 \tag{4}$$

$$g(x,y) = -0.5\, y\ \left[0.25\ \sin\left(x\right)\ \cos\left(y\right) + 1.0\right];\; \zeta(x,y) = \rho(x,y)\ \left[\hat{\nu}(x,y) + \rho(x,y)\right] \tag{5}$$

$$U\left(x,y\right)=-\frac{\left[y\,g\left(x,y\right)\right]_{y}\left[1.0+mcc\,cos(x)\right]}{y\,\left[\rho(x,y)\,\hat{\nu}(x,y)\right]};\;V\left(x,y\right)=\frac{\left[g(x,y)\right]_{x}\left[1.0+mcc\,sin(x)\right]}{\left[\rho(x,y)\,\hat{\nu}(x,y)\right]}\tag{6}$$

All four manufactured operators, present in the S-A equation, can be seen from Equation 7 to Equation 10. Including the vorticity S in Equation 10.

$$Op\,Div(x,y)=\frac{1}{y}\left\{\left[y\,\rho(x,y)\,\hat{\nu}(x,y)\,U(x,y)\right]_{x}+\left[y\,\rho(x,y)\,\hat{\nu}(x,y)\,V(x,y)\right]_{y}\right\}\tag{7}$$

$$Op\,Lap\,\zeta(x,y)=\frac{1}{y}\left\{\left[y\,\zeta(x,y)\left[\hat{\nu}(x,y)\right]_{x}\right]_{x}+\left[y\,\zeta(x,y)\,\left[\hat{\nu}(x,y)\right]_{y}\right]_{y}\right\}\tag{8}$$

$$\nabla\rho\cdot\nabla\nu=\left[\rho\left(x,y\right)\right]_{x}\,\left[\hat{\nu}\left(x,y\right)\right]_{x}+\left[\rho\left(x,y\right)\right]_{y}\,\left[\hat{\nu}\left(x,y\right)\right]_{y}\tag{9}$$

$$\left(\nabla\nu\right)^{2}=\left\{\left[\hat{\nu}\left(x,y\right)\right]_{x}\right\}^{2}+\left\{\left[\hat{\nu}\left(x,y\right)\right]_{y}\right\}^{2};\;S=\left|\left[V(x,y)\right]_{x}-\left[U(x,y)\right]_{y}\right|\tag{10}$$

The time-dependent term becomes part of the source term of the MS, Equation 11, as $\rho\hat{\nu}$, calculated in the previous iteration. After the discretization of the divergent operator, additional components incorporate the MS. Then, to effectively calculate the value of $\hat{\nu}$, the linear system composed of the matrix of coefficients and source are interactively calculated by the Modified Strong Implicit Method (MSI).

$$\begin{aligned}Source=&-Op\,Div(x,y)+\frac{1}{\sigma}\,Op\,Lap\,\zeta(x,y)-\frac{(\rho+\hat{\nu})}{\sigma}\nabla\rho\cdot\nabla\nu\\&+\frac{(c_{b2}\rho)}{\sigma}\left(\nabla\nu\right)^{2}+\rho(P-D)\end{aligned}\tag{11}$$

3. RESULTS

After grid convergence tests, the local inclination of the discretization error curve, the effective order (pE), was calculated as described by Marchi (2001). Table 1 shows the number o volumes in each direction; grid characteristic size (h); the L2 norm of the error after code convergence (when the difference between two consecutive L2 norms, calculated by the linear system solver that applies the Modified Strongly Implicit procedure, is equal to 1.0E-14), plus values for pE. Figure 1 shows the effective order (pE) versus the grid characteristic size (h).

4. CONCLUSION

Due to the combination of two spatial discretization methods with different asymptotic orders of the truncation error (p_o) (CDS-2, $p_o = 2$ and UDS, $p_o = 1$), a degeneration from the highest p_o to the lowest p_o occurred, as expected. Also, the value for the pE of the MS became as close to unity as $h \to 0$. This behaviour suggests that the code is verified, and the evaluated modules and subroutines can be applied to compute the turbulent viscosity in finite volume codes. Please, be free to contact the first author by the contact email above abstract to receive the verified source code.

Table 1 – Grids, h, L2 norms and pE values.

Grid (ξ x η)	h	L2 norm	pE
20x40	1.25000E-03	2.28562E-04	-
40x80	3.12500E-04	6.22659E-05	1.87607E+00
80x160	7.81250E-05	2.12499E-05	1.55098E+00
160x320	1.95313E-05	8.53860E-06	1.31539E+00
320x640	4.88281E-06	3.80128E-06	1.16751E+00
640x1280	1.22070E-06	1.79107E-06	1.08567E+00
1280x2560	3.05176E-07	8.69131E-07	1.04318E+00

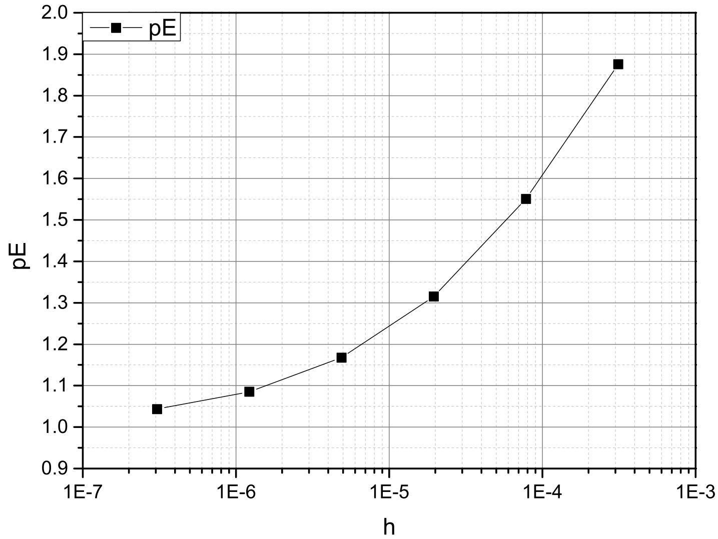

Figure 1 – Effective order.

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