Journal of Scientific Computing, Vol. 1, No. 1, 1986

Renormalization Group Analysis of Turbulence. I. Basic Theory

Victor Yakhot[1] **and Steven A. Orszag**[1]

Received May 14, 1986

We develop the dynamic renormalization group (RNG) method for hydrodynamic turbulence. This procedure, which uses dynamic scaling and invariance together with iterated perturbation methods, allows us to evaluate transport coefficients and transport equations for the large-scale (slow) modes. The RNG theory, which does not include any experimentally adjustable parameters, gives the following numerical values for important constants of turbulent flows: Kolmogorov constant for the inertial-range spectrum $C_K = 1.617$; turbulent Prandtl number for high-Reynolds-number heat transfer $P_t = 0.7179$; Batchelor constant $Ba = 1.161$; and skewness factor $\bar{S}_3 = 0.4878$. A differential K–$\bar{\varepsilon}$ model is derived, which, in the high-Reynolds-number regions of the flow, gives the algebraic relation $\nu = 0.0837K^2/\bar{\varepsilon}$, decay of isotropic turbulence as $K = O(t^{-1.3307})$, and the von Karman constant $\kappa = 0.372$. A differential transport model, based on differential relations between K, $\bar{\varepsilon}$, and ν, is derived that is not divergent when $K \to 0$ and $\bar{\varepsilon}$ is finite. This latter model is particularly useful near walls.

KEY WORDS: Renormalization group; turbulence theory; inertial range; turbulence transport; Reynolds number; large-eddy simulation; computational fluid dynamics.

1. INTRODUCTION

Turbulent flows occur in many circumstances, differing by geometry, driving mechanisms, and the physicochemical processes that take place within them. Perhaps the most distinguishing characteristic of high-Reynolds-number turbulent flows is their large range of excited space and time scales. It is well known (e.g., Landau and Lifshitz, 1982) that, in homogeneous

[1] Applied & Computational Mathematics and Mechanical & Aerospace Engineering, Princeton University, Princeton, New Jersey 08544.

0885-7474/86/0600-0003$05.00/0

turbulence, dissipation-scale eddies are of order $R^{3/4}$ times smaller than the energy-containing eddies, where R is a (microscale) Reynolds number. In order to solve the Navier–Stokes equations accurately for such a turbulent flow, it is necessary to retain order $(R^{3/4})^3$ spatial degrees of freedom. Also, since the time scale of significant evolution of turbulent flow is of the order of the turnover time of the energy-containing eddies, it is necessary to perform $R^{3/4}$ time steps to calculate a significant time evolution of the flow. Even if these calculations require only $O(1)$ arithmetic operations per time step, the requirement for computer storage would be $O(R^{9/4})$ and for computational work $O(R^3)$. In this case, even a mere doubling of the Reynolds number would require an order-of-magnitude improvement of computer capability. With this kind of operation and storage count, it is not likely that foreseeable advances in computers will allow the full simulation of turbulent flows at Reynolds numbers much larger than $R = O(100\text{–}1000)$ already achieved.

The second distinguishing characteristic of turbulence is the approximate universality of the properties of scales much smaller than the integral scale L in the flow. High-Reynolds-number turbulent flow is characterized by three different ranges of spatial scales:

1. For wavenumbers $k = O(\pi/L)$ the energy spectrum is strongly anisotropic and is not universal. The integral scale L reflects both the geometry of the flow and the physicochemical processes taking place on these scales.
2. At much smaller scales, with wavenumbers satisfying $\pi/L \ll k \ll k_d = R^{3/4}L^{-1}$, the velocity fluctuation spectrum $E(k)$ is nearly universal and is approximately given by the Kolmogorov energy spectrum:

$$E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3} \tag{1.1}$$

   with the Kolmogorov constant $C_K = 1.3 - 2.3$. Here $\bar{\varepsilon}$ is the rate of energy dissipation per unit volume in the flow.
3. In the dissipation range, $k > O(k_d)$, the energy spectrum decreases exponentially with k.

The existence of the universal inertial range characterized by the Kolmogorov law (1.1) has been checked experimentally for a large variety of turbulent flows. The Kolmogorov law has been confirmed experimentally in fluid and gas shear flows, in atmospheric boundary layers and in the ocean, in hydromagnetic and bouyancy-influenced flows, in jets, and in turbulence behind a grid (e.g., Monin and Yaglom, 1975).

The universality of the small scales can be formulated dynamically as follows. Fluid motions are governed by the Navier–Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu_0 \nabla^2 \mathbf{v} \tag{1.2}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{1.3}$$

and are subject to initial and boundary conditions. Here $\mathbf{v}(\mathbf{x}, t)$ is the fluid velocity, ρ is the density, p is the pressure, and ν_0 is the molecular kinematic viscosity. The inertial-range spectrum (1.1) does not depend directly on geometry or boundary conditions; geometry and boundary conditions do affect the structure of large scales as well as the value of $\bar{\varepsilon}$ that appears in (1.1). Boundary conditions can be considered from the viewpoint of small scales as a source of energy injected into the large scales, which subsequently cascade to the small scales. Using the analogy with equilibrium statistical mechanics, in which small-scale fluctuations are independent of the details of the interaction of the system with a heat bath, we propose to replace (1.2) by the more general, but equivalent, equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu_0 \nabla^2 \mathbf{v} \tag{1.4}$$

where $\mathbf{f}$ is a random force (noise) chosen to generate the velocity field described by the spectrum (1.1) in the limit of large wavenumber k.

It is important to emphasize that no initial and boundary conditions are needed in (1.4), since the fluid described by (1.4) is stirred by the force $\mathbf{f}$ so that a statistically steady state with $\mathbf{v} \neq 0$ can be achieved. The relation between the stirring force $\mathbf{f}$ and initial and boundary conditions will be discussed further below. Equation (1.4) is a model that is statistically equivalent to the original Eq. (1.2) in the inertial range. This *correspondence principle* is the basis for the RNG method discussed in this paper.

It is also known that Eq. (1.4) with the Gaussian random force defined by the wavevector–frequency correlation

$$\langle f_i(\hat{k})\, f_j(\hat{k}') \rangle = 2D_0 (2\pi)^{d+1} k^2 P_{ij}(\mathbf{k})\, \delta(\hat{k} + \hat{k}')$$

with

$$D_0 = \nu_0 k_B T/\rho$$

and

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2; \qquad \hat{k} = (\mathbf{k}, \omega) \tag{1.5}$$

describes both the static and dynamic properties of a fluid in thermal equilibrium independently of details of the fluid history and conditions on

its boundary (Landau and Lifshitz, 1982; Forster *et al.*, 1977). Here d is the dimension of space, k_B is Boltzmann's constant, and T is the temperature of the fluid.

On this basis, we may postulate that Eqs. (1.3) and (1.4) with a properly chosen random force provides a correct description of the small-scale-motion of a wide class of turbulent flows. In the inertial range, solutions to Eqs. (1.3) and (1.4) are statistically equivalent to solutions to the original Eqs. (1.2) and (1.3) with initial and/or boundary conditions. Equation (1.4) can be viewed as a general model describing small-scale properties of turbulent flow in the inertial range. This equation will be used for the development of turbulence models using the renormalization group method.

The present paper is organized as follows: The basic ideas of the renormalization group method are described in Section 2 following Forster *et al.* (1977). Modifications of this theory that enable us to evaluate the Kolmogorov constant as $C_K \approx 1.617$ are also described in this section. In Section 3, a subgrid-scale turbulence model is derived using the RNG method. It is shown that this subgrid model is very close to the model used by Deardorff (1970) in the high-Reynolds-number regions of turbulent channel flow far from walls.

In Section 4, the RNG procedure is applied to the evolution of a passive scalar in a turbulent flow. The results of this analysis include the prediction of the turbulent Prandtl number $P_t = 0.7179$ and the Batchelor constant $Ba = 1.161$ [see (4.3)]. In Section 5, the RNG method is used to derive turbulent transport approximations. The values of some basic constants of turbulent flows are found, including the skewness factor $\bar{S}_3 = 0.4878$ and the von Karman constant $\kappa = 0.372$. The high-Reynolds-number version of the RNG form of the K–$\bar{\varepsilon}$ transport model is also derived in Section 5. It is shown that $\nu = 0.0837K^2/\bar{\varepsilon}$, where K and $\bar{\varepsilon}$ are the turbulent kinetic energy and mean dissipation rate, respectively. This K–$\bar{\varepsilon}$ model is very close to the algebraic two-equation K–$\bar{\varepsilon}$ models often used in turbulence modeling. This model also leads to the energy decay law $K = O(t^{-1.3307})$ for homogeneous, isotropic turbulence. In Section 6, differential transport models are derived that are based on *differential* relations between K, $\bar{\varepsilon}$, and ν. This model includes the important effects of destructive interference between molecular and eddy viscosities.

The results of this work, which are summarized in Section 7, are in good agreement with available experimental data. They give some hope that the RNG method may provide a rational, yet workable, basis for turbulence theory in a variety of circumstances. In following papers, we shall present applications of these RNG-based turbulence closures.

2. RENORMALIZATION GROUP ANALYSIS OF FLUID DYNAMICS IN AN UNBOUNDED MEDIUM

2.1. Introductory Remarks and Basic Models

Renormalization group (RNG) methods were first developed in the context of quantum field theory. Wilson (1971) applied RNG ideas to the theory of critical phenomena and was able to solve the problem of the universality of critical exponents and the Kondo problem (Wilson and Kogut, 1974). In the mid-1970s, the theory of dynamic critical phenomena was developed. This theory deals with universal features of dynamics in the vicinity of the critical point (Hohenberg and Halperin, 1977). Dynamic RNG methods developed by Ma and Mazenko (1975) have been used by Forster *et al.* (1977) to investigate velocity fluctuations governed by the Navier–Stokes equation driven by a Gaussian random force. Their ideas have been developed by others (de Dominicis and Martin, 1979; Fournier and Frisch, 1978, 1983; Yakhot, 1981) to deal with the problem of hydrodynamic turbulence. In this section, we outline the basic ideas of the RNG method. The RNG method will be used in later sections to derive both turbulence transport equations for resolvable scales and subgrid models for large-eddy numerical simulations of turbulence.

Consider the Navier–Stokes Eqs. (1.4) for incompressible flow subject to the random force $\mathbf{f}(\mathbf{x}, t)$. Here we consider a random force specified by the two-point correlation:

$$\langle f_j(\mathbf{k}, \omega) f_i(\mathbf{k}', \omega') \rangle = 2D_0 k^{-y} (2\pi)^{d+1} P_{ij}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \quad (2.1)$$

where the parameter y is an arbitrary number. As mentioned in Section 1, the case $y = -2$ describes fluid in thermal equilibrium driven by thermal noise. Since we are interested in studying strongly nonequilibrium flows, we concentrate on the case $y > -2$.

We introduce the Fourier decomposition of the velocity fields with an ultraviolet cutoff $\Lambda = O(k_d)$

$$v_i(x, t) = \int_{k \leqslant \Lambda} \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\omega}{2\pi} v_i(\mathbf{k}, \omega) \exp(i\mathbf{k}x - i\omega t) \quad (2.2)$$

The space-time Fourier-transformed equation of motion (1.4) is

$$v_l(\hat{k}) = G^0(\hat{k}) f_l(\hat{k}) - \frac{i\lambda_0}{2} G^0(\hat{k}) P_{lmn}(\mathbf{k}) \int v_m(\hat{q}) v_n(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \quad (2.3)$$

where

$$P_{lmn}(\mathbf{k}) = k_m P_{ln}(\mathbf{k}) + k_n P_{lm}(\mathbf{k})$$
$$G^0(\mathbf{k}, \omega) \equiv G^0(\hat{k}) = [-i\omega + \nu_0 k^2]^{-1} \tag{2.4}$$

Here we have introduced the formal parameter λ_0 $(=1)$ to facilitate the perturbation solution of (2.3) given below. Introducing the ultraviolet cutoff Λ in (2.2), we assume that the Fourier modes $\mathbf{v}(\hat{k})$ vanish when $k > \Lambda$. This assumption is quite plausible, at least for the forcing (2.1) with $y > 0$, since the modes $\mathbf{v}(\hat{k})$ corresponding to wavenumbers $k > O(k_d)$ are overdamped by the viscosity term in the equation of motion (2.3).

In principle, we can use the zeroth-order $(\lambda_0 = 0)$ solution of (2.3),

$$v_i^0(\hat{k}) = G^0(\hat{k}) f_i(\hat{k}) \tag{2.5}$$

as the basis to construct the pertubation expansion of $\mathbf{v}$ in powers of λ_0. This problem has been solved formally by Wyld (1961) and Kraichnan (1961), although the resulting series is too complex to give very useful answers (see also Monin and Yaglom, 1975). However, it is less problematic to answer the following question: How are the long-wavelength modes $\mathbf{v}^<(\hat{k})$ belonging to the interval $0 < k < \Lambda e^{-r}$ affected by the short-wavelength modes $\mathbf{v}^>(\hat{k})$ from a narrow wavevector band near the ultraviolet cutoff $\Lambda e^{-r} < k < \Lambda$? This leads directly to the renormalization group method, which enables us, in some cases, to find the infrared $(k \to 0)$ asymptotics of correlations generated by the model (2.1)–(2.3).

2.2. Elimination of Small Scales

Following Ma and Mazenko (1975) and Forster *et al.* (1977), the RNG procedure consists of two steps. First, we write Eq. (2.3) in terms of the two components $\mathbf{v}^>$ and $\mathbf{v}^<$ of the velocity $\mathbf{v}$:

$$v_l(\hat{k}) = G^0(\hat{k}) f_l(\hat{k}) - \frac{i\lambda_0}{2} G^0(\hat{k}) P_{lmn}(\mathbf{k}) \int [v_m^<(\hat{q}) v_n^<(\hat{k}-\hat{q})$$
$$+ 2v_m^>(\hat{q}) v_n^<(\hat{k}-\hat{q}) + v_m^>(\hat{q}) v_n^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{2.6}$$

In order to eliminate modes from the interval $\Lambda e^{-r} < q < \Lambda$, all terms $\mathbf{v}^>(\hat{q})$ in (2.6) should be removed by repeated substitution of (2.3) for $\mathbf{v}^>$ back into (2.6). This generates an infinite expansion for $\mathbf{v}^<$ in powers of λ_0 in which $\mathbf{v}^>$ does not formally appear.

Second, averages are taken over the part of the random force $\mathbf{f}^>$ belonging to the strip $\Lambda e^{-r} < q < \Lambda$. This procedure formally eliminates the modes $\Lambda e^{-r} < q < \Lambda$ from the problem. It can be shown readily that, after removing the modes $\Lambda e^{-r} < q < \Lambda$, the equation of motion for $\mathbf{v}^<$ can be written up to second order in λ_0 as

$$
\begin{aligned}
&(-i\omega + \nu_0 k^2)\, v_l^<(\hat{k}) \\
&\quad = f_l(\hat{k}) - \frac{i\lambda_0}{2} P_{lmn}(\mathbf{k}) \int G^0(\hat{q})\, G^0(\hat{k}-\hat{q})\, f_n^>(\hat{q})\, f_m^>(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\qquad - \frac{i\lambda_0}{2} P_{lmn}(\mathbf{k}) \int v_m^<(\hat{q})\, v_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\qquad + 4\left(\frac{i\lambda_0}{2}\right)^2 2D_0 P_{lmn}(\mathbf{k}) \int |G^0(\hat{q})|^2\, G^0(\hat{k}-\hat{q}) \\
&\qquad \times P_{n\mu\rho}(\mathbf{k}-\mathbf{q})\, P_{m\mu}(\mathbf{q})\, q^{-y} v_\rho^<(\hat{k}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\
&\qquad + O[(v^<)^3]
\end{aligned}
\tag{2.7}
$$

The second term on the right side of (2.7) is an induced random force, denoted by Δf_l, with zero mean if the forces are assumed to be statistically homogeneous.

Equation (2.7) is an approximation for $\mathbf{v}^<(\hat{k})$ that is valid in the limit $k \to 0$. It should be noted that, in addition to the terms accounted for in (2.7), the scale elimination procedure introduces terms like

$$
\tfrac{1}{2} i P_{lmn}(\mathbf{k}) \int G^0(\hat{q})\, f_m^>(\mathbf{q})\, v_n^<(\mathbf{k}-\mathbf{q})
$$

which we neglect, since they vanish after averaging over the force $\mathbf{f}^>$. It should be emphasized that the mean square of such terms does not vanish, but they go rapidly to zero when $k \to 0$. Another type of contribution generated by the scale elimination procedure and which is not taken into consideration in (2.7) is of the form

$$
\lambda_0^3\, \delta\Gamma\, P_{lmn}(\mathbf{k}) \int v_m^<(\hat{q})\, v_n^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^3}
$$

where $\delta\Gamma$ is a "vertex" correction associated with the nonlinear term. It has been shown by Forster *et al.* (1977) that Galilean invariance implies that $\delta\Gamma = 0$ in the limit $k \to 0$.

In this work our goal is to assess the effect of small (and fast) eddies on the large (and slow) turbulent eddies. Thus, we are interested in the asymptotic description of the modes $v^<(\mathbf{k}, \omega)$ in the limit $k \to 0$ and $\omega \to 0$. On the basis of previous applications of the RNG method to other physical systems, we may expect that the resulting analysis is still reasonably accurate for finite $\mathbf{k}$, ω in the inertial range.

To begin the analysis of (2.7), let us evaluate the last term on the right side:

$$R_l = -\lambda_0^2 \frac{2D_0}{(2\pi)^{d+1}} P_{lmn}(\mathbf{k}) \int^> |G^0(\hat{q})|^2 G^0(\hat{k}-\hat{q}) \times P_{n\mu\rho}(\mathbf{k}-\mathbf{q}) P_{m\mu}(\mathbf{q}) q^{-y} v_\rho^<(\hat{k})\, d\hat{q} \tag{2.8}$$

where the symbol $\int^>$ indicates integration over the band being removed

$$\int^> d\hat{q} \equiv \int_{\Lambda e^{-t} < q < \Lambda} d\mathbf{q} \int_{-\infty}^{\infty} d\Omega \tag{2.9}$$

where $\hat{q} = (\mathbf{q}, \Omega)$. Performing the frequency integration gives

$$R_l = -\lambda_0^2 \frac{2D_0\pi}{v_0(2\pi)^{d+1}} P_{lmn}(\mathbf{k}) \int^> \frac{P_{n\mu\rho}(\mathbf{k}-\mathbf{q}) P_{m\mu}(\mathbf{q}) q^{-y-2}\, d\mathbf{q}}{-i\omega + v_0 q^2 + v_0 |\mathbf{k}-\mathbf{q}|^2} v_\rho^<(\hat{k}) \tag{2.10}$$

We shall evaluate (2.10) in the limit $\omega \to 0$ and $k \to 0$. Changing the integration variable by replacing $\mathbf{q} \to \mathbf{q} + \frac{1}{2}\mathbf{k}$ gives

$$R_l = -\lambda_0^2 \frac{2D_0\pi}{(2\pi)^{d+1} v_0} P_{lmn}(\mathbf{k}) \times \int^> \frac{P_{n\mu\rho}(\tfrac{1}{2}\mathbf{k}-\mathbf{q}) P_{m\mu}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}) |\mathbf{q}+\tfrac{1}{2}\mathbf{k}|^{-y-2}\, d\mathbf{q}}{-i\omega + 2v_0 q^2 + v_0 k^2/2} v_\rho^<(\hat{k}) \tag{2.11}$$

Neglecting terms that are $O(k^2)$ as $k \to 0$ in the integrand on the right side of (2.11) gives

$$R_l = -\frac{\lambda_0^2 2D_0 \pi P_{lmn}(\mathbf{k})}{2(2\pi)^{d+1} v_0^2} \times \int^> q^{-2} |\mathbf{q}+\tfrac{1}{2}\mathbf{k}|^{-y-2} \{ [k_\mu P_{n\rho}(\mathbf{q}-\tfrac{1}{2}\mathbf{k}) - q_\rho P_{n\mu}(\mathbf{q}-\tfrac{1}{2}\mathbf{k})] \times P_{m\mu}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}) \}\, v_\rho^<(\hat{k})\, d\mathbf{q} \tag{2.12}$$

It is easy to check that, to $O(k^2)$,

$$P_{n\mu}(\mathbf{q}-\tfrac{1}{2}\mathbf{k}) P_{m\mu}(\mathbf{q}+\tfrac{1}{2}\mathbf{k}) \approx P_{mn}(\mathbf{q}) + \tfrac{1}{2}k_m q_n - \tfrac{1}{2} q_m k_n \tag{2.13}$$

Noting that $P_{lmn}(\mathbf{k}) = -P_{lmn}(-\mathbf{k})$, we conclude that the $O(k)$ terms on the right side of (2.13) do not contribute to leading order in the integral (2.12). Thus,

$$R_l = -\frac{\lambda_0^2 D_0 P_{lmn}(\mathbf{k})}{2\nu_0^2 (2\pi)^d} \times \int^{>} q^{-y-4} \left[k_\mu P_{n\rho}(\mathbf{q})\, P_{m\mu}(\mathbf{q}) + q_\rho P_{nm}(\mathbf{q}) \frac{y+2}{2} \frac{k_i q_i}{q^2} \right] d\mathbf{q}\, v_\rho^<(\hat{k}) \quad (2.14)$$

The angular integration in (2.14) is easily caried out using the well-known relations

$$\int q_\alpha q_\beta\, d^d q = \frac{S_d}{d} \delta_{\alpha\beta} \int q^{d+1}\, dq \quad (2.15)$$

and

$$\int q_\alpha q_\beta q_\gamma q_\delta\, d^d q = \frac{S_d}{d(d+2)} (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}) \int q^{d+3}\, dq \quad (2.16)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of a d-dimensional unit sphere. Using (2.15) and (2.16), we obtain

$$P_{lmn}(\mathbf{k})\, k_\mu v_\rho^<(\hat{k}) \int^{>} P_{n\rho}(\mathbf{q})\, P_{m\mu}(\mathbf{q})\, q^{-y-4}\, d\mathbf{q} = k^2 v_l^<(\hat{k}) \left[\frac{d-2}{d} + \frac{2}{d(d+2)} \right] \frac{e^{\varepsilon r} - 1}{\varepsilon} \quad (2.17)$$

where

$$\varepsilon = 4 + y - d \quad (2.18)$$

Also, we find

$$\frac{y+2}{2} P_{lmn}(\mathbf{k}) \int^{>} q^{-y-4} \frac{q_\rho q_i k_i}{q^2} P_{nm}(\mathbf{q})\, d\mathbf{q}\, v_\rho^<(\hat{k}) = -\frac{y+2}{d(d+2)} \frac{e^{\varepsilon r} - 1}{\varepsilon} S_d k^2 v_l^< \quad (2.19)$$

Combining (2.17)–(2.19), the result is

$$R_l = -\frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\varepsilon} \frac{S_d}{(2\pi)^3} \frac{d^2 - d - \varepsilon}{2(d+2)d} \frac{e^{\varepsilon r} - 1}{\varepsilon} k^2 v_l^< \quad (2.20)$$

This gives $R \approx -\Delta\nu(k)\, k^2 \mathbf{v}^<(\hat{k})$, so the effect of this term is to modify the viscous term on the left side of (2.7).

We conclude that, in the limit $k \to 0$, $\omega \to 0$, the correction to the viscosity is given by

$$\Delta\nu(0) = A_d \frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\varepsilon} \frac{e^{(4+y-d)r} - 1}{4 + y - d} \tag{2.21}$$

where

$$A_d = \tilde{A}_d \frac{S_d}{(2\pi)^d}; \qquad \tilde{A}_d = \frac{1}{2} \frac{d^2 - d - \varepsilon}{d(d+2)}; \qquad \varepsilon = 4 + y - d \tag{2.22}$$

Thus, the viscosity resulting from the elimination of the modes $v^>$ is

$$\nu_r = \nu_0 \left(1 + A_d \bar{\lambda}_0^2 \frac{e^{(4+y-d)r} - 1}{4 + y - d} \right) \tag{2.23}$$

where the dimensionless coupling constant $\bar{\lambda}_0$ is defined by

$$\bar{\lambda}_0 = \lambda_0 D_0^{1/2} / \nu_0^{3/2} \Lambda^{\varepsilon/2} \tag{2.24}$$

Substituting this result into (2.7) gives the intermediate-state Navier–Stokes equations (without rescaling)

$$\begin{aligned} v_l^<(\hat{k}) = {} & G_r(\hat{k})(f_l + \Delta f) \\ & - \frac{i\lambda_0}{2} G_r(\hat{k})\, P_{lmn}(k) \int v_m^<(\hat{q})\, v_n^<(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ & + O[(v^<)^3] \end{aligned} \tag{2.25}$$

where the intermediate-scale Green's function (propagator) is given by

$$G_r = (-i\omega + \nu_r k^2)^{-1}$$

with ν_r given by (2.23). Equation (2.25) is defined on the domain $0 < k < \Lambda e^{-r}$, unlike the original Navier–Stokes equation, which is defined on the larger interval $0 < k < \Lambda$.

2.3. Recursion Relations: Rescaling of the Variables

The next step of the RNG procedure, following Forster *et al.* (1977) and Ma and Mazenko (1975), consists in rescaling the variables according to

$$k' = ke^r; \qquad \omega' = \omega e^{z(r)}; \qquad v^<(\mathbf{k}, \omega) = \xi v'(\mathbf{k}', \omega') \tag{2.26}$$

Thus, the new variable $\mathbf{k}'$ is defined on the same interval $0 < k' < \Lambda$ as the wavevector $\mathbf{k}$ in the original Navier–Stokes equation. In terms of the new variables, given by (2.26), the intermediate Navier–Stokes equation is

$$v_l'(\hat{k}') = G_r(\hat{k}')\, f_l'(\hat{k}') - \frac{i\lambda(r)}{2} G_r(\hat{k}')\, P_{lmn}(\mathbf{k}') \times \int v_n'(\hat{q}')\, v_m'(\hat{k}' - \hat{q}') \frac{d\hat{q}'}{(2\pi)^{d+1}} + \cdots \tag{2.27}$$

where

$$G_r = [-i\omega + \nu(r)(k')^2]^{-1}$$

$$\mathbf{f}'(\hat{k}') = \mathbf{f}^{<}(\hat{k}) e^{\alpha(r)}\, \xi^{-1}(r) \tag{2.28}$$

$$\lambda(r) = \lambda_0 \xi(r) e^{-(d+1)r} \tag{2.29}$$

$$\nu(r) = \nu_r e^{\alpha(r) - 2r} \tag{2.30}$$

The correlation function characterizing the force $\mathbf{f}'(\hat{k}')$, given by expression (2.28), can be constructed readily using definition (2.1) and the new set of variables (2.26):

$$\langle f_i'(k, \omega)\, f_j'(k', \omega') \rangle = 2D'(2\pi)^{d+1} k^{-y} P_{ij}(\mathbf{k})\, \delta(\mathbf{k} + \mathbf{k}')\, \delta(\omega + \omega') \tag{2.31}$$

with

$$D' = \frac{D_0 \exp[3\alpha(r) + (d + y)r]}{\xi^2} \tag{2.32}$$

Noting that the elimination of small scales does not influence D_0, we choose the function ξ in such a way that $D' = D_0$ at each step of the RNG procedure:

$$\xi = \exp\left[\frac{3}{2}\alpha(r) + \frac{d + y}{2} r\right] \tag{2.33}$$

The procedure described so far is formally exact in the limit $r \to 0$. To eliminate a finite band of $\mathbf{k}$-space, one can iterate the above procedure by eliminating step by step infinitesimally narrow wavenumber bands. The coupling constants generated in this way depend on r and satisfy the following differential recursion relations, which follow from (2.22)–(2.24), (2.29), and (2.30):

$$\frac{dv}{dr} = v(r)[z - 2 + A_d \bar{\lambda}^2] \tag{2.34}$$

$$\frac{dD}{dr} = 0 \tag{2.35}$$

$$\frac{d\lambda}{dr} = \lambda(r)\left[\frac{3}{2}z - 1 - \frac{d-y}{2}\right] \tag{2.36}$$

Here we define z by

$$d\alpha/dr \equiv z$$

and the dimensionless expansion parameter $\bar{\lambda}$ is defined in terms of $\lambda(r)$, $v(r)$, and $D(r)$ as

$$\bar{\lambda}^2 = \lambda^2 D / v^3 A^{\varepsilon} \tag{2.37}$$

The recursion relation for $\bar{\lambda}$ can be derived readily from (2.34)–(2.36):

$$d\bar{\lambda}/dr = \tfrac{1}{2}\bar{\lambda}(\varepsilon - 3A_d\bar{\lambda}^2) \tag{2.38}$$

where $\varepsilon = 4 + y - d$.

Equation (2.38) implies that, if $\varepsilon < 0$, the effective coupling constant $\bar{\lambda} \to 0$ when $r \to \infty$. When $\varepsilon > 0$, $\bar{\lambda}$ tends to a fixed point $\bar{\lambda}^*$:

$$\bar{\lambda} \to \bar{\lambda}^* = (\varepsilon/3A_d)^{1/2} \qquad \text{as} \quad r \to \infty \tag{2.39}$$

according to the formula

$$\bar{\lambda}(r) = \bar{\lambda}_0 e^{\varepsilon r/2}\left[1 + \frac{3}{\varepsilon} A_d \bar{\lambda}_0^2 (e^{\varepsilon r} - 1)\right]^{-1/2} \tag{2.40}$$

At the fixed point $\bar{\lambda}^*$, the viscosity $v(r)$ becomes r-independent if

$$z = 2 - \varepsilon/3 \tag{2.41}$$

Treating ε as a small parameter, the value of $\bar{\lambda}$ should be evaluated in terms of the ε expansion with the parameter A_d calculated to the lowest order in ε. The accuracy and basis of this ε expansion will be discussed below.

2.4. Energy Spectrum

The expression (2.33) fully determines the scaling (2.26). Homogeneity relations can be constructed by demanding that the correlation functions

computed from the original and reduced (renormalized) equations of motion be the same for all $k < \Lambda e^{-r}$:

$$(2\pi)^{d+1} V_{ij}(\mathbf{k}, \omega) = \frac{\langle v_i(\mathbf{k}, \omega)\, v_j(\mathbf{k}', \omega')\rangle}{\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')}$$

$$= \xi^2 e^{-dr-\alpha(r)} \frac{\langle v_i(\mathbf{k}e^r, \omega e^\alpha)\, v_j(\mathbf{k}e^r, \omega e^\alpha)\rangle}{\delta(\mathbf{k}e^r+\mathbf{k}'e^r)\,\delta(\omega e^\alpha+\omega' e^\alpha)} \tag{2.42}$$

Noting (2.33) and that $\alpha = zr$ when z is constant gives the solution of (2.42) as

$$V_{ij}(k, \omega) = O[k^{-2z-y} V(\omega/k^z)] \tag{2.43}$$

The energy spectrum can be evaluated from (2.43) as

$$E(k) = \mathrm{Tr}\, k^{d-1} \int V_{ij}(k, \omega) \frac{d\omega}{2\pi}$$

$$= O(k^{-z-1+d-y}) = O(k^{-5/3+2/3(d-y)}) \tag{2.44}$$

where we use expression (2.41) for z.

The asymptotic solution (2.44) has been derived from the theory that takes into account only terms up to $\bar{\lambda}^2$. This is justified in terms of the ε expansion. A remaining problem is that the nonlinear terms generated by the renormalization procedure have been neglected. This problem is addressed next.

2.5. Irrelevant Nonlinear Terms

The typical nonlinear contribution in the Navier–Stokes equation after the elimination of small scales is

$$v_l^<(\hat{k}) = \mathrm{NS} + \mu(r)\, G(\hat{k})\, P_{lmn}(\mathbf{k}) \int P_{m\mu\nu}(\mathbf{q})\, G(\hat{q})$$

$$\times v_\mu^<(q_1)\, v_\nu^<(\hat{q}-\hat{q}_1)\, v_n^<(\hat{k}-\hat{q}-\hat{q}_1) \frac{d\hat{q}}{(2\pi)^{d+1}} \frac{d\hat{q}_1}{(2\pi)^{d+1}}$$

where NS symbolizes the terms in the Navier–Stokes equation taken into account in the RNG analysis given above. Performing the scale transformation (2.26), we find that

$$\mu(r) \approx \mu_0 \xi^2 e^{-(2d+2)r} = \mu_0 e^{-(d-y)r} \tag{2.45}$$

in the vicinity of the fixed point where relations (2.26)–(2.33) hold. It follows from (2.45) that, when $y < d$, the proportionality coefficient $\mu(r)$ tends exponentially to zero when $r \to \infty$, so that the new nonlinear contributions to the Navier–Stokes equation are *irrelevant*. This means in turn that the solution (2.43) is asymptotically exact in the limit $k \to 0$. On the other hand, if $y > d$, the theory diverges in the limit $k \to 0$, which is reflected in the growing importance of the nonlinearities generated by the elimination of small scales ($\mu \to \infty$ with $r \to \infty$). If $y = d$, it follows from (2.44) that

$$E(k) = O(k^{-5/3}) \tag{2.46}$$

The result (2.46) shows that the fluid driven by the random force (2.1) with $y = d$ generates velocity correlations described, in the limit $k \to 0$, by the Kolmogorov law (2.46). The only problem is that, as seen from (2.45), the nonlinear terms generated by the procedure are marginal, i.e., they approach a finite nonzero value. This means that formula (2.46) is not an exact solution of the problem in the limit $k \to 0$, but is at the edge of the region of convergence ($y \to d-$). One can hope that in this case the contributions from nonlinear operators are not too large, although the justification for this conclusion is weak. However, it is gratifying that the nonlinear operators with $y = d-$ do not grow to infinity, so one can hope that they lead only to logarithmic corrections to (2.46), as in the theory of critical phenomena.

2.6. Renormalized Equation of Motion

The RNG method has allowed us to develop the equation of motion for the velocity field modes with $k \to 0$ averaged over the small scales $q > \Lambda e^{-r}$. The equation of motion at the fixed point is

$$\frac{\partial \mathbf{v}^<}{\partial t} + (\mathbf{v}^< \cdot \nabla)\mathbf{v}^< = \mathbf{f}' - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v}^< \tag{2.47}$$

where $\nu(r)$ is the solution of the recursion reltion (2.34). It follows from (2.26) that, at the fixed point, the frequency scales as $\omega \approx k^z$. This implies, in turn, that the viscosity becomes k-dependent (since $z = 2/3$ when $y = d = 3$). Indeed,

$$\omega = O(k^{2/3}) \approx \nu(k) k^2 \tag{2.48}$$

so $\nu = O(k^{-4/3})$, which is a result well known from the theory of isotropic turbulence.

Note also that in the derivation of (2.47) we neglected the correction to the random force whose correlation function is proportional to k^2. In the limit $k \to 0$ this force is negligible in comparison with original forcing given by (2.1) with $y < -2$.

The major drawback of the RNG method in the formulation given here is that it does not provide the proportionality coefficients needed for simulations of real flows. Another problem with the method is that it deals with a fluid stirred by a random force with a given coefficient D_0. The latter problem is a major drawback, since in real flows the intensity of turbulent pulsations, which are proportional to D_0, has to be determined from the dynamics of the problem. In the next sections, we shall rework the RNG method described above to resolve some of these problems and to make the RNG technique suitable for the derivation of subgrid scale and transport turbulence models.

2.7. RNG Evaluation of the Kolmogorov Constant

It has been shown above that the elimination of the modes $\mathbf{v}^>(\hat{q})$ belonging to the band near the ultraviolet cutoff $\Lambda e^{-r} < q < \Lambda$ leads to the following corrections to viscosity at long space-time scales:

$$\Delta\nu(0) = A_d \frac{\lambda_0^2 D_0}{\nu_0^2 \Lambda^\varepsilon} \frac{e^{\varepsilon r} - 1}{\varepsilon} \tag{2.49}$$

where

$$A_d = \tilde{A}_d \frac{S_d}{(2\pi)^d}, \qquad \tilde{A}_d = \frac{1}{2} \frac{d^2 - d - \varepsilon}{d^2 + 2d}; \qquad \varepsilon = 4 + y - d$$

and $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere in d dimensions. From now on we consider only the case $y = d -$.

The elimination procedure described in the previous section is accurate in the limit $r \to 0$. We conclude that elimination of small scales with $\Lambda e^{-r} < q < \Lambda$ does not affect either the coupling constant λ_0 or the forcing amplitude D_0. The constancy of D_0 under this renormalization holds because, while the second term on the right side of Eq. (2.7) gives a zero-mean (averaged over $\mathbf{k}^>$) Gaussian random variable with correlation function proportional to k^2, this correction cannot be absorbed in the bare force (2.1), whose correlation function is proportional to k^{-y} ($y > -2$). Thus, $D = D_0$ and we must include a new random foce with correlation function proportional to k^2 in the renormalized Navier–Stokes equations. The fact that λ_0 is not renormalized is a consequence of Gaussian invariance (Forster *et al.*, 1977).

It is also possible to eliminate a finite band of modes $\Lambda e^{-r} < q < \Lambda$ by iterating the above procedure of eliminating an infinitesimally narrow band of modes but not performing the rescaling procedure (2.26). The goal of this unscaled iteration procedure is to generate a renormalized viscosity coefficient $\nu = \nu(r)$ and coupling constant $\bar{\lambda} = \bar{\lambda}(r)$ while $D(r) = D_0$ and $\lambda(r) = \lambda_0$ still hold. While the elimination of an infinitesimal band of modes is justified by the use of second-order perturbation solutions of the Navier–Stokes equations [or by comparing with the results of applying Kraichnan's (1959) direct-interaction approximation to this system], the result of the iteration procedure is no longer justifiable in this way. The nature of the errors incurred by the iteration procedure must be clarified later.

The functions $\nu(r)$ and $\bar{\lambda}(r)$ are most easily determined by taking the limit $r \to 0$ in (2.23) in order to obtain the differential equation

$$d\nu/dr = A_d \nu(r)\, \bar{\lambda}^2(r) \tag{2.50}$$

where

$$\bar{\lambda}^2(r) = \frac{\lambda_0^2 D_0}{\nu^3(r) \Lambda^4}\, e^{4r} \tag{2.51}$$

Since $\Lambda(r) = \Lambda e^{-r}$. Here we emphasize that the rescaling (2.26) is *not* done. The solution of (2.50)–(2.51) is

$$\nu(r) = \nu_0 [1 + \tfrac{3}{4} A_d \bar{\lambda}_0^2 (e^{4r} - 1)]^{1/3} \tag{2.52}$$

and

$$\bar{\lambda}(r) = \bar{\lambda}_0 e^{2r} [1 + \tfrac{3}{4} A_d \bar{\lambda}_0^2 (e^{4r} - 1)]^{-1/2} \tag{2.53}$$

which coincides with (2.40) when $y = d$. In the limit $r \to \infty$, the parameter $\bar{\lambda}$ given by (2.53) approaches the fixed point

$$\bar{\lambda}^* = (4/3A_d)^{1/2} \equiv (\varepsilon/3A_d)^{1/2} \tag{2.54}$$

It has been mentioned in the previous section that at the fixed point the coupling parameter (2.54) can be treated, from the point of view of the ε expansion, as a small parameter. Thus, in the zeroth order, neglecting the nonlinear term in the foced Navier–Stokes equation defined on the smaller domain $0 < k < \Lambda e^{-r}$, one has that the velocity field is determined by

$$v_i^<(\hat{k}) \approx G(\hat{k})\, f_i(\hat{k})$$

where the renormalized propagator $G(\hat{k})$ is given by

$$G(\hat{k}) = [-i\omega + \nu(r) k^2]^{-1} \tag{2.56}$$

If only modes with wavenumbers larger than $\Lambda(r)$ are removed by renormalization, then (2.45) gives a k-dependent viscosity in the limit $r \to \infty$:

$$\nu(k) = (\tfrac{3}{4} A_d D_0)^{1/2} k^{-4/3} \tag{2.57}$$

where we have set $\lambda_0 = 1$.

Equation (2.55) leads to the energy spectrum

$$\begin{aligned} E(k) &= \frac{1}{2} \frac{S_d k^2}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} \operatorname{Tr} V_{ij}(\mathbf{k}, \omega)\, d\omega \\ &= \frac{1}{2(\frac{3}{8}\tilde{A}_d)^{5/3}} \left(2D_0 \frac{S_d}{(2\pi)^d} \right)^{2/3} k^{-5/3} \\ &= 1.186 \left(2D_0 \frac{S_d}{(2\pi)^d} \right)^{2/3} k^{-5/3} \end{aligned} \tag{2.58}$$

Formula (2.58) has also been derived by Fournier and Frisch (1983). We remark that the numerical constants appearing in (2.58) and in later equations are evaluated at the fixed point to lowest order in ε. Thus, $\tilde{A}_d$ in (2.22) is evaluated at $\varepsilon = 0$ as $\tilde{A}_3 = 0.2$.

To derive the Kolmogorov constant for the inertial-range power spectrum we must relate D_0 in (2.58) to the mean rate of energy dissipation $\bar{\varepsilon}$. To do this we can use the solution (2.57) for $\nu(k)$ and the equation for energy balance following the calculation of Kraichnan (1971). Substituting the inertial-range spectral law

$$E(k) = C_K \bar{\varepsilon}^{2/3} k^{-5/3} \tag{2.59}$$

into the energy balance equation in the inertial range gives [Kraichnan, 1971, Eq. (3.1); see also Leslie, 1972]

$$\frac{(3/8\tilde{A}_d)^{1/2}}{C_K^2} \left(\frac{2D_0 S_d/(2\pi)^d}{\bar{\varepsilon}} \right)^{1/3} = 0.1904$$

so

$$C_K = 1.496 \left(\frac{2D_0 S_d/(2\pi)^d}{\bar{\varepsilon}} \right)^{1/6} \tag{2.60}$$

Consistency between (2.59) with (2.60) and (2.58) requires that $\bar{\varepsilon}$ and D_0 be proportional, namely,

$$2D_0 S_d/(2\pi)^d = 1.594\bar{\varepsilon} \tag{2.61}$$

Substituting (2.61) into (2.58) gives the energy spectrum

$$E_{\mathrm{RNG}}(k) = 1.617\bar{\varepsilon}^{2/3}k^{-5/3} \qquad (C_{\mathrm{K}} = 1.617) \tag{2.62}$$

The relation (2.61) will be used later in this paper to derive transport models.

3. RNG SUBGRID SCALE TURBULENCE MODEL FOR LARGE-EDDY SIMULATIONS

Using (2.61), we can rewrite the result (2.52) of the RNG theory in terms of the total mean dissipation rate $\bar{\varepsilon}$ as

$$\nu_r = \nu_0[1 + a\bar{\varepsilon}/\nu_0^3\Lambda^4(e^{4r} - 1)]^{1/3} \tag{3.1}$$

where $a = 1.594(\frac{3}{8}\tilde{\Delta}_d) = 0.120$. The RNG subgrid scale turbulence model is derived as follows: Let $\tilde{\Delta}$ be the computational msh size and let $\Lambda_0 = \pi/\tilde{\Delta}$ be the wavevector corresponding to the scale $\tilde{\Delta}$. Our goal is to eliminate all scales $\Lambda_0 \leqslant k \leqslant \Lambda$ from the problem. The corresponding subgrid model is given by (3.1) with the wavevector $\Lambda_0 = \Lambda e^{-r}$ expressed through $\tilde{\Delta}$. However, it is customary to express the viscosity in terms of $\Delta = 2\tilde{\Delta}$. Here, Δ is the width of a suitably chosen Gaussian filter. It is known from the theory of isotropic turbulence that the dissipation cutoff $k_d \approx \Lambda$ is not an independent parameter, but obeys the relation

$$k_d = \Lambda = \gamma(\bar{\varepsilon}/\nu_0^3)^{1/4}$$

where $\gamma \approx 0.2$ according to experimental data. Thus, relation (3.1) becomes

$$\nu = \nu_0\left[1 + H\left(\frac{a\bar{\varepsilon}}{(2\pi)^4\nu_0^3}\Lambda^4 - C\right)\right]^{1/3} \tag{3.2}$$

where $C = a/\gamma^4$. Here, the Heaviside function, defined by $H(x) = x$ when $x \geqslant 0$ and $= 0$ otherwise, reflects the fact that $r > 0$ in (3.1).

Formula (3.2) express the renormalized viscosity in terms of $\bar{\varepsilon}$ and the filter length scale Δ. It is important that $\bar{\varepsilon}$ is a flow parameter that does not depend on the scale $\Lambda_0 = \Lambda e^{-r}$. This means that

$$\bar{\varepsilon} = \frac{\nu_0}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 = \frac{\nu(r)}{2}\left(\frac{\partial v_i^<}{\partial x_j} + \frac{\partial v_j^<}{\partial x_i}\right)$$

so that $\bar{\varepsilon}$ can be expressed entirely through the *resolvable field* $\mathbf{v}^<$. This makes it possible to use formula (3.2) for large-eddy simulations. Writing

$$\bar{\varepsilon} = \frac{\nu(r)}{2}\left(\frac{\partial v_i^<}{\partial x_j} + \frac{\partial v_j^<}{\partial x_i}\right)^2 \tag{3.3}$$

and substituting into (3.2), we obtain

$$\nu = \nu_0\left[1 + H\left(\frac{a\Delta^4}{2(2\pi)^4 \nu_0^3}\,\nu\left(\frac{\partial v_i^<}{\partial x_j} + \frac{\partial v_j^<}{\partial x_i}\right)^2 - C\right)\right]^{1/3} \tag{3.4}$$

It follows that, when $a\bar{\varepsilon}\Delta^4/2(2\pi)^4\nu_0^3 \gg C$,

$$\nu = c_s \Delta^2 \left|\frac{\partial v_i^<}{\partial x_j} + \frac{\partial v_j^<}{\partial x_i}\right| \tag{3.5}$$

where $c_s^2 = a/2(2\pi)^4$, so that

$$c_s = 0.0062 \qquad \text{(RNG)}$$

Formula (3.5) is the well-known Smagorinsky (1962) eddy viscosity, which has been widely used in large-eddy simulations. Deardorff (1970, 1971) was the first to use relation (3.5) for large-eddy simulations of shear flows. Deardorff (1971) argued that $c_s \approx 0.005$ worked best. Moin and Kim (1981) performed their simulations of wall-bounded shear flows with $c_s \approx 0.003$. However, in order to prevent the turbulence in the wall region from decaying, Moin and Kim redefined the average dissipation $\bar{\varepsilon}$ as the turbulent dissipation and separated effects of mean shear from the fluctuating shear as in a turbulence transport model. They defined $\bar{\varepsilon}$ as

$$\bar{\varepsilon} = \tfrac{1}{2}\nu_0\langle(s_{ij} - \langle s_{ij}\rangle)^2\rangle \tag{3.6}$$

where

$$s_{ij} = \frac{\partial v_i^<}{\partial x_j} + \frac{\partial v_j^<}{\partial x_i}$$

and $\langle\cdots\rangle$ stands for the horizontal average over all scales. Moin and Kim also neglected the effect of random forcing due to subgrid-scale motions. They pointed out that their calculated turbulent intensities were insensitive to variations of the constant in (3.5) by 40%. Thus, we conclude that the agreement between calculated and "experimental" data are rather good.

Although the renormalized equation of motion derived in this section is basically the same as that used far from the wall by Deardorff and others, it differs significantly in the wall region, where formula (3.5) is not valid. In the wall region the renormalized Eqs. (3.4) do not lead to a turbulent eddy viscosity proportional to Δ^2. Near the wall, the argument of $H(x)$ in (3.4) is negative, so $\nu = \nu_0$.

Relations (3.1) and (3.4) are, strictly speaking, valid at the fixed point or in the limit $r \to \infty$. However, we shall use these formulas for the calculation of turbulent flows in the entire interval $0 \leqslant r < \infty$. The nature of the errors incurrent can be illustrated by the limit $r \to 0$:

$$\nu_r = \nu_0 + A_d \bar{\lambda}_0^2 \frac{e^{\varepsilon r} - 1}{\varepsilon}$$

which is asymptotically accurate [see (2.23)]. Thus, the result is accurate in both limits $r \to \infty$ and $r \to 0$. Equations (3.1) and (3.4) describe the smooth transition between these two asymptotic solutions.

3.1. Role of the Random Force

Another important feature of finite systems is the role of the random force generated by the elimination of small scales. This force is a zero-mean Gaussian force given by the second term on the right side of (2.7). The analytic expression for the correlation function of this force is

$$\langle f_i(\bar{k}) f_k(\bar{k}') \rangle = D'(2\pi)^{d+1} k^2 P_{ij}(\mathbf{k})\, \delta(\bar{k} + \bar{k}') \tag{3.7}$$

where

$$D' = 2D_0^2 \lambda_0^2 \int \frac{d\hat{q}}{(2\pi)^{d+1}} P_{lmn}(\mathbf{k})\, P_{l\mu\nu}(\mathbf{k})\, P_{m\mu}(\mathbf{q})\, P_{n\nu}(\mathbf{k} - \mathbf{q})$$

$$\times q^{-d} |\mathbf{k} - \mathbf{q}|^{-d} |G^0(\hat{q})|^2 |G^0(\bar{k} - \hat{q})|^2 \tag{3.8}$$

The integrals in (3.8) are readily evaluated, giving

$$D' = B_d \frac{D_0^2 \lambda_0^2}{\nu_0^3 \Lambda^9} \frac{e^{9r} - 1}{9} \qquad (d = 3)$$

$$B_d = \frac{1}{2} \frac{d^2 - 2}{d(d+2)} \frac{S_d}{(2\pi)^d} \tag{3.9}$$

The recursion relation for $D'(r)$ is derived readily:

$$\frac{dD'}{dr} = B_d \frac{D_0}{\Lambda^5(r)} \bar{\lambda}^2(r) \tag{3.10}$$

For the approximate evaluation of (3.10) we take the coupling constant $\bar{\lambda} = \bar{\lambda}_*$ at the fixed point and find, using $y = d-$,

$$D' \approx \frac{4}{15} \frac{d^2-2}{d^2-d} \frac{2D_0 S_d/(2\pi)^d}{\Lambda_0^5} (e^{5r} - 1)$$

or, in other words, introducing the dissipation cutoff k_d and the cutoff corresponding to the largest eliminated scales k_c, we have

$$D' \approx 1.594 \frac{4}{15} \frac{d^2-2}{d^2-d} \frac{\bar{\varepsilon}}{k_d^5} \left[\left(\frac{k_d}{k_c} \right)^5 - 1 \right] \tag{3.11}$$

Far from the wall, where $k_d/k_c \gg 1$, the induced noise is smaller than the stirring force if

$$\frac{4}{15} \frac{d^2-2}{d^2-d} \frac{\bar{\varepsilon}}{k_c^5} k^2 \ll \frac{\bar{\varepsilon}}{k^3} \tag{3.12}$$

and thus

$$(k/k_c)^5 \ll 1 \tag{3.13}$$

This always holds when $k \ll k_c$. If, on the other hand, $k \approx k_c$, the induced noise is comparable with the stirring force and cannot be neglected. It is clear that the role of this noise is most important in the buffer region where $k_d \to k_c$. Indeed, setting $k_d/k_c = O(1)$, we conclude that in the buffer layer the bare and induced noises are of the same order.

4. TURBULENT HEAT AND MASS TRANSPORT

In this section, we apply the renormalization group method developed above to the problem of the distribution of a scalar advected by a turbulent fluid. The method is a combination of the ideas developed in the works of Forster *et al.* (1977) and the approach described in Sections 2 and 3. Despite the fact that modeling of flows coupled to a scalar field is of much practical importance in the description of heat and mass transfer, previous analytic theories have not led to much quantitative success. A passive scalar is governed by the equation of motion

$$\partial T/\partial t + (\mathbf{v} \cdot \nabla) T = \chi_0 \nabla^2 T \tag{4.1}$$

where the turbulent velocity $\mathbf{v}$ is the solution of the Navier–Stokes equation.

To analyze advection of a passive scalar governed by (4.1), we assume that the random temperature field, mixed by a turbulent fluid, is isotropic at small scales and is independent of the integral scale L. According to the Kolmogorov (1941) theory, the dynamics of the scalar field at scales much smaller than L are characterized by $\bar{\varepsilon}$, ν, and χ_0, and the rate of dissipation $\bar{N}$ of fluctuations of T:

$$\frac{\partial}{\partial t}\frac{1}{2}\int T^2(\mathbf{x}, t)\, d\mathbf{x} = -\chi_0 \int (\nabla T)^2\, d\mathbf{x} \equiv -\bar{N} \tag{4.2}$$

In the inertial range, transfer of T fluctuations dominates dissipation, so the spectrum depends only on $\bar{\varepsilon}$ and $\bar{N}$ (see Batchelor, 1959) and the inertial-range scalar spectrum is

$$E_T(k) = Ba\frac{\bar{N}}{\bar{\varepsilon}^{1/2}} k^{-5/3} \tag{4.3}$$

Here the constant Ba is called the Batchelor constant.

Another milestone of phenomenological modeling is the idea, proposed by Reynolds and extended by Prandtl and Colburn, that in the limit of large Reynolds number R the distributions of velocity and of passive scalar are similar. This leads to the inference that, at large R, the eddy viscosity ν and eddy diffusivity χ are similar, so that

$$\alpha \equiv P_t^{-1} = \chi/\nu \tag{4.4}$$

is nearly a constant. Here P_t is the turbulent Prandtl number. The near constancy of α in (4.4) has been confirmed experimentally in a variety of flows. The value $\alpha = 1.1$–1.4 $(P_t = 0.7$–$0.9)$ has been widely used in engineering studies.

Repeating the argument presented in Section 1, we model small scales by adding a random force to the right side of the Navier–Stokes equation. To derive the renormalized equation of motion, we Fourier transform (4.1) to obtain

$$T(\hat{k}) = -i\lambda_0' k_l g^0(\hat{k}) \int v_l(\hat{q})\, T(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{4.5}$$

together with the Navier–Stokes equation

$$v_l(\hat{k}) = G^0(\hat{k}) f_l(\hat{k}) - \frac{i\lambda_0}{2} P_{lmn}(\mathbf{k})\, G^0(\hat{k}) \int v_m(\hat{q})\, v_n(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{4.6}$$

Equations (4.5)–(4.6) are defined on the domain $\pi/L < k < k_d$, where L is the characteristic dimension of the system and k_d is the Kolmogorov's dissipation scale, $k_d \approx 0.2(\bar{\varepsilon}/\nu_0^3)^{1/4}$. The bare propagators $G^0(\mathbf{k}, \omega)$ and $g^0(\mathbf{k}, \omega)$ are defined by

$$G^0(\mathbf{k}, \omega) \equiv G^0(\hat{k}) = (-i\omega + \nu_0 k^2)^{-1} \tag{4.7}$$

$$g^0(\mathbf{k}, \omega) \equiv g^0(\hat{k}) = (-i\omega + \chi_0 k^2)^{-1} \tag{4.8}$$

Our goal is to eliminate modes $\mathbf{v}^>$ and $T^>$ belonging to the wavenumber strip $k_d e^{-r} \leqslant k \leqslant k_d$ and to derive an equation of motion for the modes $\mathbf{v}^<$ and $T^<$ belonging to the domain $\pi/L \leqslant k \leqslant k_d e^{-r}$.

It has been shown in Section 2 that the renormalized Navier–Stokes equation after elimination of small scales is (2.25) with modified viscosity ν_r given by (2.23) and the random foce $\Delta\mathbf{f}$ induced by the small-scale elimination procedure.

To develop the RNG procedure for Eq. (4.5), we rewrite it as

$$\begin{aligned} T(\hat{k}) = & -i\lambda_0' k_l g^0(\hat{k}) \int v_l^<(\hat{q})\, T^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ & - i\lambda_0' k_l g^0(\hat{k}) \int [v_l^>(\hat{q})\, T^<(\hat{k}-\hat{q}) \\ & + v_l^<(\hat{q})\, T^>(\hat{k}-\hat{q}) + v_l^>(\hat{q})\, T^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^{d+1}} \end{aligned} \tag{4.9}$$

To eliminate modes from the interval $k_d e^{-r} < k < k_d$, all terms $\mathbf{v}^>(\hat{k})$ and $T^>(\hat{k})$ should be removed as in Section 2. This introduces a formal expansion in powers of λ_0 and λ_0'. This procedure leaves the bare coupling constants λ_0 and λ_0' intact in accordance with the Galilean invariance of the equations of motion. Upon constructing the formal expansion for T in powers of λ_0 and λ_0' in which $\mathbf{v}^>$ and $T^>$ do not apear, one averages over the part of the random force $\mathbf{f}^>$ belonging to the strip $k_d e^{-r} < k < k_d$. This procedure formally eliminates the modes $\mathbf{v}^>$ and $T^>$ from the problem.

After removing the modes $k_d e^{-r} < k < k_d$, one can write the equation of motion for $T^<(\hat{k})$ up to the second order in λ_0' as

$$\begin{aligned} & (-i\omega + \chi_0 k^2)\, T^<(\hat{k}) \\ & = -i\lambda_0' k_l \int v_l^<(\hat{q})\, T^<(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \\ & \quad - 2(\lambda_0')^2 D_0 T^< k_l k_n \int |G^0(\hat{q})|^2\, g^0(\hat{k}-\hat{q})\, q^{-y} P_{ln}(\mathbf{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \end{aligned} \tag{4.10}$$

When the second term on the right side of (4.10) is evaluated in the limit $k \to 0$, $\omega \to 0$ it can be identified as a correction to the bare diffusivity, namely

$$\Delta\chi = \frac{k_l k_n}{k^2} 2(\lambda_0')^2 D_0 \int^{>} |G^0(\hat{q})|^2 g^0(\hat{k}-\hat{q}) P_{lm}(\mathbf{q}) q^{-y} \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{4.11}$$

As in Section 2, the integration in (4.11) is carried out over $\hat{q} = (\mathbf{q}, \omega)$, where $k_d e^{-r} < q < k_d$. For $k \to 0$, the result is

$$\Delta\chi = \frac{d-1}{d} K_d \frac{(\bar{\lambda}_0')^2 v_0^2}{\chi_0 + v_0} \frac{e^{\varepsilon r} - 1}{\varepsilon} \tag{4.12}$$

where $K_d = S_d/(2\pi)^d$ and the effective dimensionless coupling constant is

$$\bar{\lambda}_0' = \lambda_0' D_0^{1/2} / v_0^{3/2} k_d^2 \tag{4.13}$$

It may be shown that if $\lambda_0 = \lambda_0' = 1$, one can set $\bar{\lambda}(r) = \bar{\lambda}'(r)$ at each step of the renormalization procedure. By iteration, it is possible to eliminate modes from the finite band $k_d e^{-r} < k < k_d$ generating the renormalized viscosity $\nu = \nu(r)$ and coupling $\lambda = \lambda(r)$. Taking $r \to 0$, one derives the differential equation for the renormalized diffusivity (with $d = 3$)

$$\frac{d\chi}{dr} = \frac{2}{3} K_3 \frac{\bar{\lambda}^2 \nu^2}{\chi(r) + \nu(r)} \tag{4.14}$$

where

$$\bar{\lambda}^2 = \frac{\lambda_0^2 D_0}{k_d^4} \frac{e^{4r}}{\nu^3(r)} \tag{4.15}$$

Using (2.50) and (4.14) gives

$$\frac{d\alpha}{dr} = A_3 \bar{\lambda}^2 \left(\frac{2}{3\tilde{A}_3} \frac{1}{1+\alpha} - \alpha \right) \tag{4.16}$$

where $\alpha = \chi(r)/\nu(r)$ and $\tilde{A}_3 = 1/5$.

Equation (4.14) may be solved exactly using (2.52) and (2.53), with the result

$$\left| \frac{\alpha - a}{\alpha_0 - a} \right|^{(1+a)/(a+b)} \left| \frac{\alpha + b}{\alpha_0 + b} \right|^{(b-1)/(a+b)} = \frac{1}{[1 + \frac{3}{4} A_3 \bar{\lambda}_0^2 (e^{4r} - 1)]^{1/3}} \tag{4.17}$$

where

$$\begin{aligned} a &= \tfrac{1}{2}\{-1 + [1 + 4(d-1)/d\tilde{A}_3]^{1/2}\} = \tfrac{1}{2}[-1 + (43/3)^{1/2}] \approx 1.3929 \\ b &= \tfrac{1}{2}\{1 + [1 + 4(d-1)/d\tilde{A}_3]^{1/2}\} = \tfrac{1}{2}[1 + (43/3)^{1/2} \approx 2.3929 \end{aligned} \tag{4.18}$$

Thus, with $d=3$,

$$\left|\frac{\alpha-1.3929}{\alpha_0-1.3929}\right|^{0.6321}\left|\frac{\alpha+2.3929}{\alpha_0+2.3929}\right|^{0.3679}=\frac{\nu_0}{\nu} \tag{4.19}$$

In high-Reynolds-number, fully-developed turbulence where $\nu_0/\nu \to 0$, $\alpha \to 1.3929$ and the turbulent Prandtl number is $P_t=0.7179$ ($P_t \equiv \alpha^{-1}$).

The Batchelor constant Ba in (4.3) can be evaluated from the equations of motion at the fixed point using the modified Pao (1965, 1968) theory. The equation for the energy spectrum can be written in the vicinity of the fixed point as

$$\frac{\partial E(k,t)}{\partial t}=2D_0\frac{S_d}{(2\pi)^d}k^{-1}+T(k,t)-2\nu k^2E(k,t) \tag{4.20}$$

where now ν stands for the total viscosity derived using the RNG method. It is important to notice that, since $\bar{\varepsilon}$ is proportional to D_0, no new dimensional parameters are involved in (4.25). The rate of nonlinear energy transfer from wavenumbers less than k to wavenumbers greater than k is

$$W(k)=\int_k^\infty \hat{T}(k',t)\,dk' \tag{4.21}$$

where

$$\hat{T}'(k,t)=2D_0\frac{S_d}{(2\pi)^d}k^{-1}+T(k',t) \tag{4.22}$$

Following Pao (1965, 1968), we assume that the function $W(k)$ is k-independent in the inertial range and that the dimensionally correct prescription

$$W(k)=2\gamma_p\bar{\varepsilon}^{1/3}k^{5/3}E(k) \tag{4.23}$$

holds. In a statistically stationary state, Eq. (4.20) becomes

$$\frac{d}{dk}W(k)+2\nu k^2E(k)=0 \tag{4.24}$$

Substituting (4.23) into (4.24) gives a differential equation for $E(k)$. The solution of this equation satisfying the condition

$$2\nu\int k^2E(k)\,dk=\bar{\varepsilon}$$

is

$$E(k) = C_K k^{-5/3} \bar{\varepsilon}^{2/3} \exp\left(-\frac{3}{4\gamma_p} \bar{\varepsilon}^{-1/3} \nu k^{4/3} \right) \tag{4.25}$$

with $C_K = 1/(2\gamma_p)$. In the inertial range, $W(k) = \bar{\varepsilon}$.

The Batchelor constant may be derived using Pao's formulation if we introduce the scalar transfer function $W_T(k)$, which satisfies

$$\frac{dW_T(k)}{dk} + 2\alpha\nu E_T(k) = 0 \tag{4.26}$$

Equation (4.26) with $\alpha = 1.3929$ follows from the steady-state transport equation for the scalar. It follows from (4.24) and (4.26) that

$$\frac{dW(k)}{dW_T(k)} = \frac{E(k)}{\alpha E_T(k)} = \frac{C_K \bar{\varepsilon}}{\alpha \cdot Ba \cdot \bar{N}} \tag{4.27}$$

The differential Eq. (4.27) is solved by assuming that, in the inertial range, $\bar{\varepsilon}$ and $\bar{N}$ are constant, so that

$$W(k) = \frac{C_K \bar{\varepsilon}}{\alpha \cdot Ba \cdot \bar{N}} W_T(k) \tag{4.28}$$

Pao's theory in the inertial range gives

$$W_T(k) = \bar{N}$$

so that we must require that

$$Ba = C_K/\alpha = 1.16 \tag{4.29}$$

This number is in good agreement with experimental data, $Ba \approx 1.2$–1.4 (see Monin and Yaglom, 1975).

5. RNG-BASED TURBULENCE TRANSPORT APPROXIMATIONS

Turbulence transport approximations can be constructed using the RNG in several ways. In this section, we begin by deriving an RNG-based algebraic turbulence model. Let us assume that the integral scale $L = \pi/\Lambda_f$ corresponds to the largest fluctuating scales in the system. This means that $\mathbf{v}(\mathbf{k}) = \bar{\mathbf{v}}(\mathbf{k})$ is assumed to be nonfluctuating if $k < \Lambda_f$. Eliminating all

modes from the interval $\Lambda_f < k < \Lambda$, we obtain the equation of motion governing the man velocity $\bar{\mathbf{v}} = \mathbf{V}$. The (turbulent) viscosity can be obtained from the relation (3.1),

$$\nu = \nu_0 \left[1 + H\left(\frac{3}{8} \tilde{A}_d \frac{1.594 \bar{\varepsilon}}{\nu_0^3 \Lambda_f^4} - C \right) \right]^{1/3} \tag{5.1}$$

The result (5.1) is derived by systematic averaging over the small-scale isotropic fluctuating velocity components. It may be argued that formula (5.1) is valid only for the description of isotropic homogeneous turbulence, since it does not include the effects of strong anisotropy. However, it has been shown (Sivashinsky and Yakhot, 1985; Yakhot and Sivashinsky, 1986; Bayly and Yakhot, 1986) that in some cases strongly anisotropic small scales are effectively decoupled from the man velocity field $\mathbf{V}$, so that (5.1) may still hold. The integral scale $L = \pi/\Lambda_f$ in (5.1) should be viewed as corresponding to the largest scale within the inertial range. This point will be considered in detail in the next section.

It is convenient to express Λ_f in (5.1) through more familiar and more easily observable properties of the flow. To do this, we compute the isotropic part of the turbulent energy K,

$$\begin{aligned} K &= \int_{\Lambda_f}^{\infty} E(k)\, dk = \frac{3}{2} C_{\mathrm{K}} \frac{\bar{\varepsilon}^{2/3}}{\Lambda_f^{2/3}} \\ &= \frac{3}{2} 1.617 \left(\frac{3}{8} \tilde{A}_d 1.594 \right)^{1/3} \frac{\bar{\varepsilon}}{\nu \Lambda_f^2} = 1.195 \frac{\bar{\varepsilon}}{\nu \Lambda_f^2} \end{aligned} \tag{5.2}$$

where we use

$$\nu = (\tfrac{3}{8} \tilde{A}_d 1.594)^{1/3} (\bar{\varepsilon}/\Lambda_f^4)^{1/3} \tag{5.3}$$

which follows from (5.1). Eliminating Λ_f between (5.2) and (5.3) gives the turbulent viscosity expressed in terms of the energy K and dissipation rate $\bar{\varepsilon}$:

$$\nu = c_\nu K^2 / \bar{\varepsilon} \tag{5.4}$$

with $c_\nu = \frac{4}{3}(1.594 \frac{3}{8} \tilde{A}_d)^{1/3} / C_{\mathrm{K}}^2 \approx 0.0837$. This relation, which is usually called an algebraic K–$\bar{\varepsilon}$ model, has been widely used in turbulence modeling (Launder and Spalding, 1972; Reynolds, 1976; Launder *et al.*, 1975). The "experimentally" determined coefficient $c_\nu = 0.09$ is quite close to $c_\nu = 0.0837$ obtained here by the RNG method.

Relation (5.4) can be obtained by a somewhat different, although equivalent, calculation. Let us write the total turbulent energy density as

$$K=\frac{1}{2VT}\int v^2(\mathbf{x}, t)\, d^3x\, dt \tag{5.5}$$

where V and T are the (large) space and time extents of the flow. Introducing the Fourier decomposition (2.2) gives

$$K=\frac{1}{2}\int v_i(\hat{q})\, v_i(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4} \qquad (k\to 0, \omega\to 0) \tag{5.6}$$

Here $k\to 0$ stands for $k\to k_{\min}=\Lambda_f$, where Λ_f is the smallest wavevector at which the system dynamics is isotropic.

To evaluate (5.6), we rewrite the integral in terms of the decomposition into the $\mathbf{v}^>$ and $\mathbf{v}^<$ components:

$$2K=\int v_i^<(\hat{q})\, v_i^<(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4}+2\int v_i^>(\hat{q})\, v_i^<(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4}$$
$$+\int v_i^>(\hat{q})\, v_i^>(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4} \tag{5.7}$$

where the integration in the last term on the right is carried out over the interval $k_d e^{-r}\leqslant k\leqslant k_d\equiv\Lambda$. To leading order in λ_0, the expression (5.7) is

$$2K\approx\int v_i^<(\hat{q})\, v_i^<(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^4}+Q^>\equiv Q^<+Q^> \tag{5.8}$$

where

$$Q^>=2D_0\int |G^0(\hat{q})|^2\, P_{ii}(\mathbf{q})q^{-3}\frac{d\hat{q}}{(2\pi)^4} \qquad (y=d=3) \tag{5.9}$$

The integral $Q^>$ is evaluated as

$$Q^>=\frac{2D_0 S_d/(2\pi)^d}{\nu_0\Lambda^2}\frac{e^{2r}-1}{2} \tag{5.10}$$

It follows from (5.7)–(5.10) that the kinetic energy can be decomposed into the part due to components $v^<$ and an additional contribution $Q^>$ that takes into account eliminated modes from the interval $k_d e^{-r}<k<k_d$.

The result (5.10) can be iterated as done earlier in this paper. Replac-

ing v_0 and Λ by $v(r)$ and $\Lambda(r)$, we have that the differential relation for $Q(r)$ is simply

$$\frac{dQ}{dr} = \frac{2D_0}{v(r)\,\Lambda(r)^2}\frac{S_d}{(2\pi)^d}$$

$$= \frac{2D_0}{v_0\Lambda^2}\frac{S_d}{(2\pi)^d}\frac{e^{2r}}{[1+\frac{3}{4}\tilde{A}_d\bar{\lambda}_0^2(e^{4r}-1)]^{1/3}} \tag{5.11}$$

The recursion relation (5.11) is easily integrated in the limit of fully developed turbulence $\frac{3}{4}\tilde{A}_d\bar{\lambda}_0^2 e^{4r} \gg 1$. Integrating up to the scale Λ_f gives

$$Q = \frac{3}{2}\frac{2D_0 S_d/(2\pi)^d}{v\Lambda_f^2} \tag{5.12}$$

Substituting (5.12) into (5.8) and keeping in mind that $Q^< \to 0$ when $k \to \Lambda_f$, we obtain

$$K \approx \frac{3}{4}\frac{2D_0 S_d/(2\pi)^d}{v\Lambda_f^2} \tag{5.13}$$

so that

$$K = 1.195\bar{\varepsilon}/v\Lambda_f^2$$

which is identical to (5.2). Note that it follows by eliminating $\bar{\varepsilon}$ from (5.2) and (5.4) that

$$10v^2\Lambda_f^2 = K \tag{5.14}$$

The algebraic model (5.4) is valid only in the strongly turbulent regions of the flow. To account for the low-Reynolds-number parts of the flow, the recursion relation (45.11) must be integrated everywhere, including regions where $\frac{3}{4}\tilde{A}_d\bar{\lambda}_0^2 e^{4r} \approx O(1)$. This gives the differential transport model that is derived in the next section.

In the above discussion, we have given the basic steps of the averaging procedure that will be used to derive transport models. The basic idea of the method is summarized as follows: To obtain the mean of any nonlinear term in the velocity field, say Y, we compute $Y(k)$ in the limit $k \to 0$. This is done by repetitive averaging over shells in wavevector space $\Lambda e^{-r} \leqslant k \leqslant \Lambda$ using the Navier–Stokes equation to remove unwanted modes. Eliminating all modes from the interval $\Lambda_f < k < \Lambda$ leads to the evaluation of Y.

To illustrate the procedure again, we shall compute the skewness factor $\bar{S}_3$, defined as

$$-\bar{S}_3 = \overline{(\partial v_1/\partial x_1)^3}/\overline{[(\partial v_1/\partial x_1)^2]}^{3/2} \equiv A/B^{3/2} \tag{5.15}$$

First we calculate

$$A = \overline{(\partial v_1/\partial x_1))^3}$$

$$= -i \int q_1 Q_1 (k-q-Q)_1 \, v_1(\hat{q}) \, v_1(\hat{Q}) \, v_1(\hat{k}-\hat{q}-\hat{Q}) \frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}} \tag{5.16}$$

in the limit $k \to 0$.

Decomposing the velocity field into $\mathbf{v}^<$ and $\mathbf{v}^>$ components, we rewrite (5.16) as

$$A = A^< - i \int q_1 Q_1 (k-q-Q)_1 (a+b+c+d+e+f+g) \frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}} \tag{5.17}$$

where the seven terms a–g are given by

$$\begin{aligned}
a &= v_1^>(\hat{q}) \, v_1^<(\hat{Q}) \, v_1^>(\hat{k}-\hat{q}-\hat{Q}) \\
b &= v_1^>(\hat{q}) \, v_1^>(\hat{Q}) \, v_1^<(\hat{k}-\hat{q}-\hat{Q}) \\
c &= v_1^<(\hat{q}) \, v_1^>(\hat{Q}) \, v_1^>(\hat{k}-\hat{q}-\hat{Q}) \\
d &= v_1^>(\hat{q}) \, v_1^<(\hat{Q}) \, v_1^<(\hat{k}-\hat{q}-\hat{Q}) \\
e &= v_1^<(\hat{q}) \, v_1^>(\hat{Q}) \, v_1^<(\hat{k}-\hat{q}-\hat{Q}) \\
f &= v_1^<(\hat{q}) \, v_1^<(\hat{Q}) \, v_1^>(\hat{k}-\hat{q}-\hat{Q}) \\
g &= v_1^>(\hat{q}) \, v_1^>(\hat{Q}) \, v_1^>(\hat{k}-\hat{q}-\hat{Q})
\end{aligned} \tag{5.18}$$

and

$$A^< = -i \int q_1 Q_1 (k-q-Q)_1 \, v_1^<(\hat{q}) \, v_1^<(Q) \, v_1^<(\hat{k}-\hat{q}-\hat{Q}) \frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}} \tag{5.19}$$

We have to evaluate (5.17), eliminating modes $\mathbf{v}^>$ from the problem using the Navier–Stokes equation. Averaging over the random force $\mathbf{f}^>$, we find the following contribution from expression a of (5.18) to the integral (5.17):

$$\begin{aligned}
A_a = & -2D_0 \int q_1 Q_1 (k-q-Q)_1 \\
& \times [|G^0(\hat{q}+\hat{Q})|^2 \, G^0(\hat{q}) \, P_{1mn}(\mathbf{q}) \, P_{1n}(\mathbf{q}+\mathbf{Q}) \\
& \times |\mathbf{q}+\mathbf{Q}|^{-y} \, v_1^<(\hat{Q}) \, v_m^<(\hat{k}-\hat{Q}) \\
& - |G^0(\hat{q})|^2 \, G^0(-\hat{q}-\hat{Q}) \, P_{1mn}(\mathbf{q}+\mathbf{Q}) \, P_{1n}(\mathbf{q}) \\
& \times q^{-y} \, v_1^<(\hat{Q}) \, v_m^<(\hat{k}-\hat{Q})] \frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}}
\end{aligned} \tag{5.20}$$

The frequency integration is performed readily:

$$A_a = -\frac{\pi}{v_0^2} 2D_0 \int \frac{q_1 Q_1 (k-q-Q)_1}{q^2+|\mathbf{q}+\mathbf{Q}|^2} [P_{1mn}(\mathbf{q})\, P_{1n}(\mathbf{q}+\mathbf{Q}) |\mathbf{q}+\mathbf{Q}|^{-y-2}$$

$$- P_{1mn}(\mathbf{q}+\mathbf{Q})\, P_{1n}(\mathbf{q}) q^{-y-2}]\, v_1^<(\hat{Q})\, v_m^<(\hat{k}+\hat{Q}) \frac{d\mathbf{q}\, d\hat{Q}}{(2\pi)^{2d+1}} \tag{5.21}$$

It is clear that the expression in the square brackets in the integrand of (5.21) goes to zero when $Q \to 0$. Thus, we have to expand the integrand of (5.21) in powers of the small ratio $Q/q < 1$ and retain the first nonvanishing contribution. This can be done conveniently if we shift the variables in (5.21) by replacing $\mathbf{q} \to \mathbf{q} - \mathbf{Q}/2$ and let $k \to 0$. Thus,

$$A_a = \frac{2\pi D_0}{v_0^2} \int \frac{(q-Q/2)_1\, Q_1 (q+Q/2)_1}{2q^2 + \frac{1}{2}Q^2}$$

$$\times \left[P_{1mn}\left(\mathbf{q}-\frac{\mathbf{Q}}{2}\right) P_{1n}\left(\mathbf{q}+\frac{\mathbf{Q}}{2}\right) \left|\mathbf{q}+\frac{\mathbf{Q}}{2}\right|^{-y-2} \right.$$

$$\left. - P_{1mn}\left(\mathbf{q}+\frac{\mathbf{Q}}{2}\right) P_{1n}\left(\mathbf{q}-\frac{\mathbf{Q}}{2}\right) \left|\mathbf{q}-\frac{\mathbf{Q}}{2}\right|^{-y-2} \right] v_1^<(\hat{Q})\, v_m^<(\bar{k}-\hat{Q})\, d\mathbf{q}\, d\hat{Q}$$

After simple algebra we obtain

$$A_a = -\frac{2\pi D_0}{v_0^2} \int q_1^2 Q_1 Q_n P_{1m}(q)\, P_{1n}(\mathbf{q}) q^{-y-4} v_1^<(\hat{Q})\, v_m^<(\hat{Q}) \frac{d^d q\, q\hat{Q}}{(2\pi)^{2d+1}} \tag{5.22}$$

The angular integration in (5.22) leads to $A_a = 0$ when $d = 2$ and, when $d = 3$,

$$A_a = -\frac{5}{210} \frac{2D_0 S_3}{(2\pi)^3} \frac{1}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2} \int Q_1^2 [v_1^<(\hat{Q})]^2 \frac{d\hat{Q}}{(2\pi)^{d+1}} \tag{5.23}$$

It can be shown that the contributions from the terms a, b, and c to the integral in (5.17) are all equal to (5.23) and that d, e, f, and g do not contribute to (5.17) in the lowest order of the ε expansion. Thus

$$A = A^< - \frac{15}{210} \frac{2D_0 S_3}{(2\pi)^3} \frac{1}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2} \int Q_1^2 [v_1^<(\hat{Q})]^2 \frac{d\hat{Q}}{(2\pi)^{d+1}} \tag{5.24}$$

The relation (5.24) can be easily iterated if we notice that, for $d = 3$,

$$v \int Q_1^2 [v_1^<(\hat{Q})]^2 \frac{d\hat{Q}}{(2\pi)^{d+1}} = \bar{\varepsilon}_1 = \frac{1}{15} \bar{\varepsilon} \tag{5.25}$$

is independent of r. The result for A is

$$A = A^{<}(r) - \frac{15}{210}\bar{\varepsilon}_1 \int_0^r \frac{2D_0 S_3 e^{2r}\, dr}{(2\pi)^3 A^2 \nu^3(r)} \tag{5.26}$$

This result for A is best evaluated by rewriting it as

$$A = A^{<}(r) - \frac{15}{210}\bar{\varepsilon}_1 \int_0^{\infty} \frac{2D_0 S_3}{(2\pi)^3} \frac{e^{2r}\, dr}{A^2 \nu^3(r)} + \frac{15}{210}\bar{\varepsilon}_1 \frac{2D_0 S_3}{(2\pi)^3 A_0^2} \int_r^{\infty} \frac{2D_0 S_3}{(2\pi)^3} \frac{e^{2r}\, dr}{A^2 \nu^3(r)} \tag{5.27}$$

It follows from (5.27) that, in high-Reynolds-number, fully-developed turbulence,

$$A^{<} = -\frac{15}{420} \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}_1}{\nu^3 A_f^2} = -\frac{1}{420} \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}}{\nu^3 A_f^2} \tag{5.28}$$

Next we compute

$$B = \int q_1^2 v_1(\hat{q})\, v_1(\hat{k} - \hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} \tag{5.29}$$

in the limit $k \to 0$. It is easy to show that, to the lowest order in λ_0,

$$B = B^{<} + \int q_1^2 |G^0(\hat{q})|^2 P_{11}(\mathbf{q})\, 2D_0 q^{-3} \frac{d\hat{q}}{(2\pi)^4} \tag{5.30}$$

The integration in (5.30) leads to the result

$$B = B^{<} + \frac{1}{15} \frac{2D_0 S_d}{(2\pi)^d} \frac{r}{\nu_0} \tag{5.31}$$

Iterating the procedure, we find that, in regions where $\nu \gg \nu_0$,

$$B^{<} = \frac{1}{20} \frac{2D_0 S_d/(2\pi)^d}{\nu} \tag{5.32}$$

Combining (2.61), (5.28), and (5.32) gives

$$\bar{S}_3^{<}(r) = -\frac{A^{<}}{(B^{<})^{3/2}} = 0.1336 \left(\frac{2D_0 S_d/(2\pi)^d}{\nu^3 A_f^4} \right)^{1/2} \tag{5.33}$$

At the fixed point, (2.51)–(2.54) give with $d=3$

$$\frac{2D_0}{v^3\Lambda_f^4}\frac{S_d}{(2\pi)^d}=\frac{8}{3\tilde{A}_d}=13.333 \tag{5.34}$$

so that

$$\bar{S}_3^<(r)=0.4878 \tag{5.35}$$

We see that $S_3^<(r)$ is independent of r, so that $\bar{S}_3^<(r)=\bar{S}_3=0.4878$ in the limit $r\to 0$. This result is in good agreement with the experimental values of the skewness factor $\bar{S}_3=0.4$–0.6 (see Section 7).

It should also be mentioned that the RNG result that $\bar{S}_3=0$ in two-dimensional isotropic turbulent flow is an exact result (Herring *et al.*, 1974).

5.1. Energy Equation

The result (5.4) shows that, within the framework of the present RNG theory, the total viscosity is entirely determined by two characteristics of turbulent flow: the kinetic energy K and the dissipation rate $\bar{\varepsilon}$. Now we apply the RNG method to derive the equation governing the kinetic energy K. The equation of motion for $K(\mathbf{x}, t)$ follows directly from the Navier–Stokes equation (Tennekes and Lumley, 1972; Monin and Yaglom, 1975)

$$\frac{\partial K}{\partial t}+(\bar{\mathbf{v}}\cdot\nabla)K=P-D-\frac{\partial}{\partial x_i}p\bar{v}_i+\nu_0\frac{\partial^2 K}{\partial x_i^2} \tag{5.36}$$

Here $\mathbf{v}=\bar{\mathbf{v}}+\mathbf{v}'$, where $\bar{\mathbf{v}}=\langle\mathbf{v}\rangle$ is the (local) average velocity, $K=\frac{1}{2}\langle v'^2\rangle$, the production term P is given in terms of the eddy viscosity ν_T as

$$P=\frac{\nu_T}{2}\left(\frac{\partial\bar{v}_i}{\partial x_j}+\frac{\partial\bar{v}_j}{\partial x_i}\right)^2\equiv 2\nu_T\bar{S}_{ij}^2 \tag{5.37}$$

and

$$D=\nu_0(\partial v_i'/\partial x_j)^2 \tag{5.38}$$

Our goal is to evaluate the mean value of $\overline{K(x,t)}$, defined as

$$\bar{K}=\int_0^T dt\int_V K(x,t)\,d^3x=K(\mathbf{k},\omega)\qquad(k\to 0,\ \omega\to 0) \tag{5.39}$$

To evaluate (5.39), we Fourier transform (5.36):

$$K(\hat{k}) = G^0(\hat{k})[P(\hat{k}) - D(\hat{k})] - i\lambda_0 k_i G^0(\hat{k}) \int v_i(\hat{q})\, K(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^4}$$

$$- i\lambda_0 k_i G^0(\hat{k}) \int p(\hat{q})\, v_i(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^4} \tag{5.40}$$

In (5.40), the expression for the Fourier transform $p(\hat{k})$ of the pressure is obtained easily from the Navier–Stokes equation and the incompressibility condition as

$$p(\hat{k}) = -\frac{k_l k_m}{k^2} \int v_l(\hat{q})\, v_m(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^4} \tag{5.41}$$

We observe that, except for the last term on the right side, (5.40) is precisely the equation for a passive scalar K with "molecular diffusivity" $\chi_0 = \nu_0$ and "force" $P - D$. The renormalization-group procedure for such an equation has already been developed in Section 4.

Our concern now is to evaluate the role of the pressure–velocity correlation in the turbulent diffusion of kinetic energy K. To do this we decompose modes into their $<$ and $>$ components and express the last contribution to the right side of (5.40) as

$$Y_K = -i\lambda_0 k_i G^0(\hat{k})$$

$$\times \int [p^<(\hat{q})\, v_i^<(\hat{k}-\hat{q}) + p^>(\hat{q})\, v_i^<(\hat{k}-\hat{q}) + p^<(\hat{q})\, v_i^>(\hat{k}-\hat{q})$$

$$+ p^>(\hat{q})\, v_i^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^4} \tag{5.42}$$

with

$$p(\hat{q}) = -\lambda_0 \frac{q_\alpha q_\beta}{q^2} \int [v_\alpha^<(\hat{Q})\, v_\beta^<(\hat{q}-\hat{Q}) + 2v_\alpha^>(\hat{Q})\, v_\beta^<(\hat{q}-\hat{Q})$$

$$+ v_\alpha^>(\hat{Q})\, v_\beta^>(\hat{q}-\hat{Q})] \frac{d\hat{Q}}{(2\pi)^4} \tag{5.43}$$

All contributions to (5.40) up to λ_0^2 can be obtained by substitution of (5.43) into (5.42). Elimination of the modes $v^>(k)$ from the interval $k_d e^{-r} < k < k_d$ is carried out using the zeroth-order solution of the Navier–Stokes equation with subsequent averaging over the part of the random force acting in the domain $k_d e^{-r} < k < k_d$.

All contributions to the equation of motion (5.40) stemming from (5.42)–(5.43) can be classified into two types of terms:

$$Y_K^0 = \lambda_0^2 k_i \int \frac{q_\alpha q_\beta}{q^2} v_\alpha^>(\hat{Q})\, v_\beta^>(\hat{q}-\hat{Q})\, v_i^<(\hat{k}-\hat{q}) \frac{d\hat{Q}\, d\hat{q}}{(2\pi)^8}$$

$$\approx k_i \int \frac{q_\alpha q_\beta}{q^2} |G^0(\hat{Q})|^2 P_{\alpha\beta}(\mathbf{Q})\, \delta(\mathbf{q}) \frac{d\hat{q}}{(2\pi)^4} v_i^<(k) \equiv 0 \tag{5.44}$$

and

$$Y_K^1 = \lambda_0^2 k_i \int \frac{q_\alpha q_\beta}{q^2}\, v_\alpha^>(\hat{Q})\, v_i^>(\hat{k}-\hat{q})\, v_\beta^<(\hat{q}-\hat{Q}) \frac{d\hat{Q}\, d\hat{q}}{(2\pi)^8}$$

$$\approx \lambda_0^2 k_i \int \frac{q_\alpha q_\beta}{q^2}\, P_{\alpha i}(\mathbf{k}-\mathbf{q})\, |\mathbf{k}-\mathbf{q}|^{-3}\, |G^0(\hat{k}-\hat{q})|^2\, v_\beta(\hat{k}) = O(k^3) \tag{5.45}$$

and can be neglected in the limit $k \to 0$. Thus, the pressure–velocity correlations do not contribute to the equation of motion for K to second order in the coupling parameter, which is considered small in terms of the ε expansion.

Recalling the results of Section 4 for the RNG description of a passive scalar, it follows from (5.40) that the RNG equation for mean turbulent kinetic energy is

$$\frac{\partial K}{\partial t} + (\bar{\mathbf{v}} \cdot \nabla) K = P - D + \frac{\partial}{\partial x_i} \alpha_K \nu \frac{\partial K}{\partial x_i}$$

where the "turbulent Prandtl number" α_K is found from the algebraic relation

$$\left| \frac{\alpha_K - 1.3929}{0.3929} \right|^{0.63} \left| \frac{\alpha_K + 2.3929}{3.3929} \right|^{0.37} = \frac{\nu_0}{\nu_T} \tag{5.46}$$

which is just (4.19) for the case $\chi_0 = \nu_0$. Here the turbulent viscosity ν_T is given by (3.2)–(3.3).

At this level of approximation, the dissipation $D \approx \bar{\varepsilon}$. For the production term that involves the unknown Reynolds stress $\overline{v_i v_j}$, one can use different types of closures, including

$$P \approx 2\nu_T \bar{S}_{ij}^2 \tag{5.47}$$

$$P \approx 0.3K(2\bar{S}_{ij}^2)^{1/2} \tag{5.48}$$

Thus the K equation is

$$\frac{\partial K}{\partial t}+(\bar{\mathbf{v}}\cdot\nabla)K=\frac{\nu_T}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j}+\frac{\partial \bar{v}_j}{\partial x_i}\right)^2-\bar{\varepsilon}+\frac{\partial}{\partial x_i}\alpha_K\nu_T\frac{\partial K}{\partial x_i} \tag{5.49}$$

or

$$\frac{\partial K}{\partial t}+(\bar{\mathbf{v}}\cdot\nabla)K=0.3(2\bar{S}_{ij}^2)^{1/2}-\bar{\varepsilon}+\frac{\partial}{\partial x_i}\alpha_K\nu\frac{\partial K}{\partial x_i} \tag{5.50}$$

Next we need to derive an equation governing the turbulent dissipation $\bar{\varepsilon}$.

5.2. Equation for the Dissipation Rate

The mean dissipation rate is defined in general by

$$\bar{\varepsilon}=\tfrac{1}{2}\nu_0\overline{(\partial v_i/\partial x_j+\partial v_j/\partial x_i)^2}=\varepsilon(\hat{k}) \qquad (\hat{k}\to 0) \tag{5.51}$$

It can be argued that strongly anisotropic fluctuations of the velocity field do not contribute to turbulent diffusivity (Sivashinsky and Yakhot, 1985; Bayly and Yakhot, 1986; Yakhot and Sivashinsky, 1986). Thus, we are interested in evaluating the isotropic part of the dissipation, defined as

$$\bar{\varepsilon}=\nu_0\overline{(\partial v_i/\partial x_j)^2}=\varepsilon(\hat{k}) \qquad (\hat{k}\to 0) \tag{5.52}$$

To evaluate (5.52), we write the equation of motion for

$$\varepsilon(\mathbf{x},t)=\nu_0[\partial v_i(\mathbf{x},t)/\partial x_j]^2 \tag{5.53}$$

Taking the time derivative of (5.52) and using the Navier–Stokes equation, we find

$$\frac{\partial \varepsilon}{\partial t}=2\nu_0\frac{\partial v_i}{\partial x_j}\frac{\partial}{\partial x_j}\left[\left(-v_l\frac{\partial v_i}{\partial x_l}-\frac{\partial p}{\partial x_i}+\nu_0\frac{\partial^2 v_i}{\partial x_l^2}\right)\right] \tag{5.54}$$

or

$$\frac{\partial \varepsilon}{\partial t}=-2\nu_0\frac{\partial v_i}{\partial x_j}\frac{\partial v_l}{\partial x_j}\frac{\partial v_i}{\partial x_l}-v_l\frac{\partial \varepsilon}{\partial x_l}+\nu_0\frac{\partial^2\varepsilon}{\partial x_l^2}-2\nu_0^2\left(\frac{\partial^2 v_i}{\partial x_j\,\partial x_l}\right)^2 \tag{5.55}$$

To derive the equation of motion for $\bar{\varepsilon}(x,t)=\varepsilon(k,t)$ in the limit $k\to 0$, we take the Fourier transform of (5.55):

$$\varepsilon(\hat{k})=-ig_\varepsilon^0 k_l\int v_l(\hat{q})\,\varepsilon(\hat{k}-\hat{q})\frac{d\hat{q}}{(2\pi)^3}-Y_\varepsilon^1(\hat{k})-Y_\varepsilon^2(\hat{k})-Y_\varepsilon^3(\hat{k}) \tag{5.56}$$

where

$$Y_\varepsilon^1 = 2\nu_0^2 g_\varepsilon^0 \int q_j(k-q)_j\, q_l(k-q)_l\, v_i(\hat{q})\, v_i(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^4} \tag{5.57}$$

$$Y_\varepsilon^2 = -2i\nu_0 g_\varepsilon^0 k_i \int q_j(k-q)_j\, v_i(\hat{q})\, p(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^4} \tag{5.58}$$

$$Y_\varepsilon^3 = -2i\nu_0 g_\varepsilon^0 \int q_j Q_j(k-q-Q)_l\, v_l(\hat{q})\, v_l(\hat{Q})\, v_i(\hat{k}-\hat{q}-\hat{Q}) \frac{d\hat{Q}\, d\hat{q}}{(2\pi)^8} \tag{5.59}$$

and the bare propagator is

$$g_\varepsilon^0 = (-\omega + \chi_\varepsilon^0 k^2)^{-1} \tag{5.60}$$

Here $\chi_\varepsilon^0 = \nu_0$ is the bare diffusivity of the dissipation rate ε.

To eliminate small scales from the problem, we decompose the velocity $\mathbf{v}$ and the scalar field ε into the two components $\mathbf{v}^<$ and $\mathbf{v}^>$ and $\varepsilon^<$ and $\varepsilon^>$, respectively. Thus,

$$\begin{aligned}\varepsilon(k) = -ig_\varepsilon^0 k_l \int \Big[& v_l^<(\hat{q})\, \varepsilon^<(\hat{k}-\hat{q}) + v_l^<(\hat{q})\, \varepsilon^>(\hat{k}-\hat{q}) + v_l^>(\hat{q})\, \varepsilon^<(\hat{k}-\hat{q}) \\ & + v_l^>(\hat{q})\, \varepsilon^>(\hat{k}-\hat{q}) \Big] \frac{d\hat{q}}{(2\pi)^4} - Y_\varepsilon^1 - Y_\varepsilon^2 - Y_\varepsilon^3 \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} Y_\varepsilon^1 = 2\nu_0^2 g_\varepsilon^0 \int & q_j(k-q)_j\, q_l(k-q)_l \\ & \times [2v_i^<(\hat{q})\, v_i^>(\hat{k}-\hat{q}) + v_i^>(\hat{q})\, v_i^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^4} + (Y_\varepsilon^1)^< \end{aligned} \tag{5.62}$$

$$\begin{aligned} Y_\varepsilon^2 = -2i\nu_0 g_\varepsilon^0 k_i \int & q_j(k-q)_j \\ & \times [v_i^<(\hat{q})\, p^>(\hat{k}-\hat{q}) + v_i^>(\hat{q})\, p^<(\hat{k}-\hat{q}) \\ & + v_i^>(\hat{q})\, p^>(\hat{k}-\hat{q})] \frac{d\hat{q}}{(2\pi)^4} + (Y_\varepsilon^2)^< \end{aligned} \tag{5.63}$$

$$\begin{aligned} Y_\varepsilon^3 = -2i\nu_0 g_\varepsilon^0 \int & q_j Q_j(k-q-Q)_l \\ & \times (a+b+c+d+e+f+g) \frac{d\hat{Q}\, d\hat{q}}{(2\pi)^8} + (Y_\varepsilon^3)^< \end{aligned} \tag{5.64}$$

where we have introduced the notation $Y(v^<) \equiv Y^<$. The expression (5.64) for Y_ε^3 involves seven contributions:

$$
\begin{aligned}
a &= v_i^>(\hat{q})\, v_l^<(\hat{Q})\, v_i^>(\hat{k}-\hat{q}-\hat{Q}) \\
b &= v_i^>(\hat{q})\, v_l^>(\hat{Q})\, v_i^<(\hat{k}-\hat{q}-\hat{Q}) \\
c &= v_i^<(\hat{q})\, v_l^>(\hat{Q})\, v_i^>(\hat{k}-\hat{q}-\hat{Q}) \\
d &= v_i^>(\hat{q})\, v_l^<(\hat{Q})\, v_i^<(\hat{k}-\hat{q}-\hat{Q}) \\
e &= v_i^<(\hat{q})\, v_l^<(\hat{Q})\, v_i^>(\hat{k}-\hat{q}-\hat{Q}) \\
f &= v_i^<(\hat{q})\, v_l^>(\hat{Q})\, v_i^<(\hat{k}-\hat{q}-\hat{Q}) \\
g &= v_i^>(\hat{q})\, v_l^>(\hat{Q})\, v_i^>(\hat{k}-\hat{q}-\hat{Q})
\end{aligned}
\tag{5.65}
$$

The elimination of the modes $\mathbf{v}^>$ and $\varepsilon^>$ from (5.61)–(5.65) is carried out as above: all modes $\varepsilon^>$ in (5.62)–(5.65) are eliminated using (5.61) and the modes $\mathbf{v}^>$ are eliminated using the Navier–Stokes equation. This generates an infinite expansion in powers of the Reynolds number. Next, averages are taken over the random force for $k_d e^{-r} < k < k_d$. The resulting equation does not include the modes $\mathbf{v}^>$ and $\varepsilon^>$. The results are calculated to the second order in the coupling parameter $\bar{\lambda}$.

The integral term in (5.61) is similar to the equation for a passive scalar ε. The RNG scale elimination procedure for a passive scalar has been described above in detail. Thus the sole effect of the second, third, and fourth terms within the integral on the right side of (5.61) is to generate a correction to the bare diffusivity:

$$
\delta\chi_\varepsilon = \frac{d-1}{d}\,\frac{2D_0 S_d}{(2\pi)^d}\,\frac{1}{\nu_0(\chi_\varepsilon^0+\nu_0)\Lambda_0^4}\,\frac{e^{4r}-1}{4} \tag{5.66}
$$

It is easy to show that the contribution of the pressure–velocity correlation term (5.63) is equal to zero to second order in $\bar{\lambda}_0$. This agrees with the conclusion of Hanjalić and Launder (1972).

It remains to evaluate (5.62) and (5.64). After elimination of the modes from the interval $\Lambda e^{-r} < k < \Lambda$, the expression for Y_ε^1 can be written as

$$
Y_\varepsilon^1 = (Y_\varepsilon^1)^< + y_1 + y_2 + 2\,\frac{2D_0 S_d}{(2\pi)^d}\,\nu_0 \Lambda^2\,\frac{1-e^{-(2+d-y)r}}{2+d-y} \tag{5.67}
$$

where

$$
\begin{aligned}
y_1 = &-4\nu_0^2 D_0 \int q_l q_j (k-q)_l (k-q)_j\, G^0(\hat{q})\, G^0(\hat{k}-\hat{q})\, |G^0(\hat{q}-\hat{Q})|^2 \\
&\times P_{i\alpha\beta}(\mathbf{q})\, P_{i\gamma\delta}(\mathbf{k}-\mathbf{q})\, P_{\beta\delta}(\mathbf{q}-\mathbf{Q})\, |\mathbf{q}-\mathbf{Q}|^{-y}\, v_\alpha^<(\hat{Q})\, v_\gamma^<(\hat{k}-\hat{Q})\,\frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}}
\end{aligned}
\tag{5.68}
$$

and

$$y_2 = -8\nu_0^2 D_0 \int |G^0(\hat{k}-\hat{q})|^2\, G^0(\hat{q}-\hat{Q})\, G^0(\hat{q})[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2$$

$$\times P_{i\alpha\beta}(\mathbf{q})\, P_{\beta\gamma\delta}(\mathbf{q}-\mathbf{Q})\, P_{i\delta}(\mathbf{q})\, q^{-y} v_\alpha^<(\hat{Q})\, v_\gamma^<(\hat{k}-\hat{Q})\, \frac{d\hat{q}\, d\hat{Q}}{(2\pi)^{2d+2}} \tag{5.69}$$

Here $\Lambda e^{-r} < q < \Lambda$ and the wavevectors $\mathbf{Q}$ and $\mathbf{k}$ belong to the interval $\Lambda_f < k < \Lambda e^{-r}$, so that $Q/q < 1$ and $k/q < 1$. We are interested in the limit $k \to 0$. After the frequency integration is performed, (5.68) becomes

$$y_1 = -\frac{8\pi}{\nu_0} D_0$$

$$\times \int \frac{[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2\, P_{i\alpha\beta}(\mathbf{q})\, P_{i\gamma\delta}(\mathbf{k}-\mathbf{q})\, P_{\rho\delta}(\mathbf{q}-\mathbf{Q})|\mathbf{q}-\mathbf{Q}|^{-y-2}}{(q^2+|\mathbf{k}-\mathbf{q}|^2)(q^2+|\mathbf{q}-\mathbf{Q}|^2)}$$

$$\times v_\alpha^<(\hat{Q})\, v_\gamma^<(\hat{k}-\hat{Q})\, \frac{d\mathbf{q}}{(2\pi)^{d+1}}\, \frac{d\hat{Q}}{(2\pi)^{d+1}} \tag{5.70}$$

and

$$y_2 = -\frac{8\pi}{\nu_0} D_0$$

$$\times \int \frac{[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2\, P_{i\alpha\beta}(\mathbf{q})\, P_{\beta\gamma\delta}(\mathbf{q}-\mathbf{Q})\, P_{i\delta}(\mathbf{q}) q^{-y}}{|\mathbf{k}-\mathbf{q}|^2(|\mathbf{k}-\mathbf{q}|^2+|\mathbf{q}-\mathbf{Q}|^2)(|\mathbf{k}-\mathbf{q}|^2+q^2)}$$

$$\times v_\alpha^<(\hat{Q})\, v_\gamma^<(\hat{k}-\hat{Q})\, \frac{d_{\mathbf{q}}}{(2\pi)^{d+1}}\, \frac{d\hat{Q}}{(2\pi)^{d+1}} \tag{5.71}$$

The integrations over the wavevector $\mathbf{q}$ in (5.70) and (5.71) can be carried out if we expand the integrands in powers of $Q/q < 1$ and $k/q < 1$. Second-order terms in Q/q produce corrections proportional to $\int Q^2 v^2(\hat{Q})\, d\hat{Q} = O[\varepsilon(k)]$ in the limit $k \to 0$. The fourth-order terms proportional to $k^2 \int Q^2 v^2(\hat{Q})\, d\hat{Q} = O[k^2 \varepsilon(k)]$ give rise to additional corrections to diffusivity that *must* be taken into account. Thus, expanding the integrands in (5.70) and (5.71) in powers of Q/q and k/q, we obtain to second order in $\bar{\lambda}$ $(d=3)$

$$y_1 = 0.217 \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}}{\nu_0^2 \Lambda^2} \frac{e^{2r}-1}{2}$$

$$- \frac{d-1}{d} \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{K}}{\nu} r - 0.18 \frac{2D_0}{\nu_0^2 \Lambda^4} \frac{e^{4r}-1}{4} k^2 \bar{\varepsilon} \tag{5.72}$$

$$y_2 = -0.250 \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2}$$
$$+ \frac{d-1}{d} \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{K}}{\nu} r + 0.18 \frac{2D_0}{v_0^2 \Lambda^4} \frac{e^{4r}-1}{4} k^2 \bar{\varepsilon} \tag{5.73}$$

It is interesting that the correction to the diffusivity of ε from the term Y_ε^1 vanishes because the last terms in (5.72) and (5.73) cancel each other identically.

To eliminate small scales from (5.64)–(5.65) we use a procedure similar to the one developed to evaluate the skewness $\bar{S}_3$. It is easy to show that to second order in λ_0^2, contributions to (5.63) coming from the terms *c*, *d*, *e*, *f*, and *g* vanish. It is an elementary, although tedious, calculation to show that

$$Y_\varepsilon^3 = (Y_\varepsilon^3)^< + \frac{d-2}{d+2} \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2}$$
$$+ i \frac{d-2}{d(d+2)} \frac{2D_0 S_d}{(2\pi)^d} \frac{1}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2}$$
$$\times \int Q_j v_i^<(\hat{Q})\, v_j^<(\hat{q})\, v_i^<(\hat{k}-\hat{q}-\hat{Q}) \frac{d\hat{Q}\, d\hat{q}}{(2\pi)^{2d+1}} \tag{5.74}$$

In the case of isotropic, homogeneous turbulence, the last term in (5.74) vanishes. If $\nabla \bar{\mathbf{v}} \neq 0$, this term is responsible for the production of the dissipation rate $\bar{\varepsilon}$. For now, let us neglect this term and substitute (5.66), (5.72), and (5.74) into (5.60). The result is

$$\varepsilon(\hat{k}) = -ig_\varepsilon k_l \int v_l(\hat{q})\, \varepsilon(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}}$$
$$- Y_\varepsilon^1 - Y_\varepsilon^2 - Y_\varepsilon^3 + g_\varepsilon \left[-\left(0.217 + \frac{d-2}{d+2} - 0.250\right) \right.$$
$$\left. \times \frac{2D_0 S_d}{(2\pi)^d} \frac{\bar{\varepsilon}}{v_0^2 \Lambda^2} \frac{e^{2r}-1}{2} - 2 \frac{2D_0 S_d}{(2\pi)^d} \nu_0 \Lambda^2 \frac{(1-e^{-2r})}{2} + P \right] \tag{5.75}$$

where P stands for the production term, which will be considered below. The propagator g_ε in (5.75) is

$$g_\varepsilon = [-i\omega + (\chi_\varepsilon^0 + \delta\chi_\varepsilon)k^2]^{-1} \tag{5.76}$$

and $\delta\chi_\varepsilon$ is given by (5.65). Equation (5.75) is defined on the interval $\Lambda_f < k < \Lambda e^{-r}$.

The functions $Y_\varepsilon^{(1-3)}$ in (5.75) are those in (5.56)–(5.58) but with $\nu^<$, g_ε, and ν replacing ν, g_ε^0, and ν_0, respectively. This renormalization procedure can now be iterated. The result is

$$\bar{\varepsilon} = -ig_\varepsilon k_l \int \bar{v}_l(q)\, \varepsilon(\hat{k}-\hat{q}) \frac{d\hat{q}}{(2\pi)^{d+1}} - ag_\varepsilon \frac{2D_0 S_d}{(2\pi)^d} + bg_\varepsilon \frac{2D_0 S_d \bar{\varepsilon}}{(2\pi)^d} + g_\varepsilon \bar{P} \tag{5.77}$$

where the parameters a and b are determined from the recursion relations derived from (5.75):

$$\frac{da}{dr} = -2\nu(r)\, A^2(r) \tag{5.78}$$

$$\frac{db}{dr} = -\left(-0.033 + \frac{d-2}{d+2}\right) \frac{1}{\nu^2(r)\, A^2(r)} \tag{5.79}$$

and

$$g_\varepsilon = [-i\omega + \alpha_\varepsilon(r)\, \nu k^2]^{-1} \tag{5.80}$$

The inverse Prandtl number α_ε is defined by (4.19) with $\alpha_0 = 1$ [so $\alpha_\varepsilon \equiv \alpha_K$ given by (5.46)].

The recursion relations (5.78) and (5.79) can be solved in the limit of high Reynolds number when $r \to \infty$. Using (2.52), one obtains the result

$$a = 3\nu A^2 \tag{5.81}$$

and

$$b = \frac{3}{2}\left(-0.033 + \frac{d-2}{d+2}\right) \frac{1}{\nu^2 A^2} \tag{5.82}$$

Thus, the equation of motion governing the man dissipation rate $\bar{\varepsilon}$ is

$$\frac{D\varepsilon}{Dt} = \bar{P} - 3\nu A_f^2 \frac{2D_0 S_d}{(2\pi)^d} + 0.2505 \frac{\bar{\varepsilon}}{\nu^2 A_f^2} \frac{2D_0 S_d}{(2\pi)^d} + \frac{\partial}{\partial x_i} \alpha_\varepsilon \nu \frac{\partial \bar{\varepsilon}}{\partial x_i} \tag{5.83}$$

Using (2.61), (5.2), and (5.14) to eliminate D_0, A_f, and ν, we obtain the dissipation equation

$$\frac{D\bar{\varepsilon}}{Dt} = \bar{P} - 1.7215 \frac{\bar{\varepsilon}^2}{K} + \frac{\partial}{\partial x_i} \alpha_\varepsilon \nu \frac{\partial \bar{\varepsilon}}{\partial x_i} \tag{5.84}$$

For homogeneous turbulence, (5.84) with $\bar{P}=0$ and the energy equation

$$DK/Dt = -\bar{\varepsilon} \tag{5.85}$$

show that homogeneous isotropic turbulence decays at high Reynolds number like

$$K = K_1(t-t_0)^{-1/0.7215} = K_1(t-t_0)^{-1.3307} \tag{5.86}$$

for suitable constants t_0, K_1. This result agrees well with experimental data (Monin and Yaglom, 1975).

To evaluate the production term $\bar{P}$, let us consider the last term on the right side of (5.74). In the case of shear turbulence $\nabla\bar{v}\neq 0$, so the major contribution to the integral in (5.74) comes from wavevectors coresponding to the largest scales in the system. Examination of the structure of this Fourier integral shows that it is given by

$$\bar{P} = -\frac{d-2}{d(d+2)}\frac{2D_0 S_d}{(2\pi)^d}\frac{e^{2r}-1}{2}\frac{\tau_{ij}^{<}(\mathbf{k})}{v_0^2 \Lambda^2}\frac{\partial \bar{v}_i}{\partial x_j} \tag{5.87}$$

where $\tau_{ij}^{<}(\mathbf{k})$ is the contribution to the Reynolds stress $-\overline{v_i v_j}$ at wavenumber $\mathbf{k}$. This relation can be iterated if we note that a calculation similar to that used to derived (5.2) shows that $\tau_{ij}(\mathbf{k})^{<} \nu(r)\Lambda^2(r)$ is proportional to $\bar{\varepsilon}$, so it is independent of r in the inertial range. Thus,

$$\frac{d\bar{P}}{dr} = -\frac{d-2}{d(d+2)}\frac{1}{v^3\Lambda^4}\frac{2D_0 S_d}{(2\pi)^d}\nu\Lambda^2\tau_{ij}^{<}\frac{\partial \bar{v}_i}{\partial x_j} \tag{5.88}$$

Noting that, to leading order in λ_0, we may assume that $\tau_{ij}^{<}(\mathbf{k})$ is statistically sharp at Λ_f, we obtain

$$\bar{P} = -\frac{d-2}{d(d+2)}\frac{2D_0 S_d/(2\pi)^d}{v^2\Lambda_f^2}\bar{\tau}_{ij}\frac{\partial \bar{v}_i}{\partial x_j} \tag{5.89}$$

in the high-Reynolds-number limit. Using (2.61) and (5.14) to eliminate D_0, ν, and Λ_f, we conclude

$$\bar{P} = -1.063\frac{\bar{\varepsilon}}{K}\bar{\tau}_{ij}\frac{\partial \bar{v}_i}{\partial x_j} \tag{5.90}$$

Thus, the high-Reynolds-number version of the $\bar{\varepsilon}$ equation is

$$\frac{D\bar{\varepsilon}}{Dt} = -1.063\frac{\bar{\varepsilon}}{K}\bar{\tau}_{ij}\frac{\partial \bar{v}_i}{\partial x_j} - 1.7215\frac{\bar{\varepsilon}^2}{K} + \frac{\partial}{\partial x_i}\alpha_\varepsilon \nu\frac{\partial \bar{\varepsilon}}{\partial x_i} \tag{5.91}$$

As a consistency check, notice that it follows from (5.74) and (5.89) that

$$Y_\varepsilon^3(k \to 0) = 2\nu_0 \overline{(\partial v_{il}/\partial x_j)(\partial v_{jl}/\partial x_j)(\partial v_i/\partial x_l)}$$
$$= O(d-2)/(d+2)] = 0 \tag{5.92}$$

when $d=2$. It can be shown directly that $Y_\varepsilon^3 = 0$ when $d=2$, using the incompressibility condition $\nabla \cdot \mathbf{v} = 0$.

The result (5.91) allows us to calculate the von Karman constant. We express all parameters in wall coordinates:

$$y_+ = \frac{y v_*}{\nu_0}; \quad v_*^2 = \frac{\tau_w}{\rho}; \quad v_+ = \frac{v}{v_*}; \quad K_+ = \frac{K}{v_*^2}; \quad \bar{\varepsilon}_+ = \frac{\varepsilon v_*^4}{\nu_0}; \quad \nu_+ = \frac{\nu}{n_0}$$

where τ_w is the wall shear stress. Consider a boundary layer in which all parameters are functions of the distance to the wall only. We use the simple version of the closure:

$$\bar{\tau}_{ij} = -\nu \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \tag{5.93}$$

In a stationary state, Eqs. (5.49) and (5.91) in the region where K is constant (in wall coordinates) give

$$\nu_+ \left(\frac{\partial v_+}{\partial y_+} \right)^2 - \bar{\varepsilon}_+ = 0 \tag{5.94}$$

and

$$1.063 \frac{\bar{\varepsilon}_+}{K_+} \nu_+ \left(\frac{\partial \bar{v}_+}{\partial y_+} \right)^2 - 1.7215 \frac{\bar{\varepsilon}_+^2}{K_+} + \frac{\partial}{\partial y_+} \alpha \nu_+ \frac{\partial \bar{\varepsilon}_+}{\partial y_+} = 0 \tag{5.95}$$

Using (5.93) in wall coordinates, the Navier–Stokes equations are simply

$$1 - \frac{y_+}{R_*} = \nu_+ \frac{\partial v_+}{\partial y_+} \tag{5.96}$$

so $\nu_+ \, \partial v_+/\partial y_+ = 1$ when $y_+/R_* \ll 1$. Using this, we obtain

$$\bar{\varepsilon}_+ = 1/\nu_+ \tag{5.97}$$

so that, using (5.4),

$$K_+ = 3.4159 \tag{5.98}$$

Substituting (5.97) and (5.98) and $\alpha_\varepsilon = 1.3929$ into (5.95) gives

$$v = \kappa y_+ \tag{5.99}$$

with the von Karman constant

$$\kappa = \left(\frac{1.7215 - 1.063}{1.3929 \cdot 3.416}\right)^{1/2} \approx 0.372 \tag{5.100}$$

Thus it follows from (5.96) that

$$v_+ = 2.688 \ln y_+ + c \tag{5.101}$$

6. DIFFERENTIAL TRANSPORT MODEL

The high-Reynolds-number version of the algebraic K–ε model derived in this work is given by relations (5.4), (5.48), (5.49), and (5.83). It is clear from (5.83) that in low-Reynolds-number flow regions where $K \to 0$ the algebraic model is poor because of uncertainty of terms of the type $\bar{\varepsilon}/K$. To derive a model valid in both high- and low-Reynolds-number regions of the flow we must solve the RNG differential recursion relations introduced in Section 5. The results for this differential transport model presented here have been obtained with the collaboration of Dr. A. Yakhot.

Let us first solve Eq. (5.11) for the function $Q = 2K$:

$$2\frac{dK}{dr} = 1.594\frac{\bar{\varepsilon}}{v(r)\,\Lambda^2(r)} = \frac{\bar{\varepsilon}e^{2r}}{v(r)\,\Lambda^2} \tag{6.1}$$

or

$$\frac{dK}{d[1/\Lambda^2(r)]} = \frac{1.594}{4}\frac{\bar{\varepsilon}}{v} \tag{6.2}$$

From (2.52) and (5.2) we obtain

$$d\frac{1}{\Lambda^2} = \frac{3}{2Y}\frac{\hat{v}^2\,d\hat{v}}{(\frac{3}{8}\hat{A}_d 1.5994\bar{\varepsilon}/v_0^3)^{1/2}} \tag{6.3}$$

where

$$\hat{v} = v/v_0 \tag{6.4}$$

and

$$Y = (\hat{v}^3 + C - 1)^{1/2} \tag{6.5}$$

Substituting (6.3)–(6.5) into (6.2) gives

$$d\frac{K}{\bar{\varepsilon}^{1/2}} = 1.7288 v_0^{1/2} \frac{\hat{v}\, d\hat{v}}{Y} \tag{6.6}$$

The differential relation (6.6) expresses the total viscosity v in terms of the kinetic energy K and the mean dissipation rate $\bar{\varepsilon}$. When $v/v_0 \gg 1$, the solution to (6.6) is identical to the algebraic model (5.4).

The low-Reynolds-number modification of the equation for the man dissipation rate $\bar{\varepsilon}$ can be written as

$$\frac{D\bar{\varepsilon}}{Dt} = P - \hat{a} + \hat{b} + \frac{\partial}{\partial x_i} \alpha_\varepsilon v \frac{\partial \bar{\varepsilon}}{\partial x_i} \tag{6.7}$$

where the functions P, $\hat{a}$, and $\hat{b}$ are derived from the following relations:

$$P = -0.08889 \frac{K}{v} \bar{\tau}_{ij} \frac{\partial \bar{v}_i}{\partial x_j} \tag{6.8}$$

and

$$d\frac{\hat{a}}{\bar{\varepsilon}^{3/2}} = -0.8267 \frac{\hat{v}^3\, d\hat{v}}{v_0^{1/2} Y^3} \tag{6.9}$$

$$d\frac{\hat{b}}{\bar{\varepsilon}^{3/2}} = -0.5764 \frac{d\hat{v}}{v_0^{1/2} Y} \tag{6.10}$$

Equation (6.8) is a direct consequence of expressions (5.14) and (5.4). The differential relations (6.9) and (6.10) can be obtained readily from (5.78)–(5.79) using the procedure (6.1)–(6.6). Detailed derivations and applications of this differential model will be published elsewhere.

7. DISCUSSION

The RNG method developed here is based on a number of ideas. First, there is the *correspondence principle*, which can be stated as follows. A turbulent fluid characterized in the inertial range by scaling laws can be described *in this inertial range* by a corresponding Navier–Stokes equation in which a random force generates velocity fluctuations that obey the scaling of the inertial range of the original unforced system. The dynamical equation with the random force is the basis for the systematic elimination of small scales and calculation of the renormalized transport coefficients.

Second, the RNG procedure is, strictly speaking, valid only in the asymptotic limits of $\hat{k} \to 0$ and $R \to \infty$ in which the scaling relations are

derived in the vicinity of the fixed point. What are the limits of validity of the RNG procedure? It is known from experimental data that the properties of turbulent fluids are approximately independent of the width of the inertial range if the Reynolds number is large enough. We believe that the results of the RNG fixed-point calculations can be aplied to any fluid that demonstrates Kolmogorov-like scale-invariant behavior in some range of the wavevectors and frequencies. This situation resembles the theory of critical phenomena in the sense that the critical exponents computed at the fixed point are approximately valid in the vicinity of the critical point when $|(T-T_c)/T_c| \ll 1$ (where T_c is the critical temperature).

The major drawback of the theory presented here is that, according to (2.45), the higher nonlinearities generated by the RNG procedure are marginal, i.e., they do not exponentially go to zero when the iteration parameter $r \to \infty$. As discussed in Section 2, we can hope that these terms produce small logarithmic corrections to the results derived here. It is interesting that the same kind of problem arises in the derivation of the hydrodynamic equations from molecular dynamics by the small-scale elimination procedure. It is well known (Dorfman, 1975; Wood, 1975) that the so-called super-Burnett coefficients, which are neglected in the Navier–Stokes equation, are weakly divergent, while the molecular viscosity computed in terms of time-correlation functions is finite. These weakly divergent coefficients are not known to upset the results of classical hydrodynamics based on the Navier–Stokes equation. We can hope that the marginality of the nonlinear terms generated by the RNG method are unimportant. Of course, this hope does not have a solid theoretical basis, so the value of the RNG method should perhaps best be judged by comparison of its predictions with experiments.

The magnitude of the Kolmogorov constant calculated here, $C_K = 1.617$, agrees with experimental data. However, measurements of C_K do not allow an unambiguous interpretation of experimental data. The most widely accepted value for C_K is $C_K \approx 1.4$–1.7. Similarly, the value of the turbulent Prandtl number derived here, $P_t = 0.7179$, is close to $P_t = 0.7$–0.9 accepted in the engineering literature (Landau and Lifshitz, 1982; Monin and Yaglom, 1975).

The RNG calculation for the skewness factor $\bar{S}_3$ gives $\bar{S}_3 = 0.4878$. Experimental data on $\bar{S}_3$ are quite scattered. Frenkiel *et al.* (1979) reported $\bar{S}_3 = 0.47$–0.48 and $\bar{S}_3 = 0.41$–0.44 measured in water and wind tunnels, respectively. The measurements of Antonia *et al.* (1984) in a plane jet showed $\bar{S}_3 = 0.43$. It must be mentioned, however, that the masurements of Wyngaard and Tennekes (1970) in an atmospheric boundary layer showed that, as the Reynolds number increased, the skewness factor $\bar{S}_3$ grew from $\bar{S}_2 \approx 0.6$ to $\bar{S}_3 \approx 1$. It is possible that the Reynolds number dependence is

due to large-scale anisotropy effects, which are quite strong in the planetary boundary layer measurements. The role of the anisotropy on measurements of $\bar{S}_3$ has been discussed by Antonia *et al.* (1984).

The RNG derivation of the transport model given in Section 5 deserves more discussion. The RNG procedure is based on the elimination of small scales, which are assumed to be isotropic. The fact that such a grossly simplified picture of turbulence leads to a K–$\bar{\varepsilon}$ model with $\nu = c_\nu K^2/\bar{\varepsilon}$, $c_\nu = 0.0837$, and the von Karman constant $K = 0.372$ deserves comment. Two basic questions that may be asked are: First, why does turbulence modeling based on oversimplified models give reasonable results? Second, why is it widely found in the engineering literature that more sophisticated schemes, such as third-order closure models, do not lead to substantial improvement over existing K–$\bar{\varepsilon}$ models? The second question may be understood on the basis of the present theory: Higher order nonlinear contributions are asymptotically unimportant and lead to small corrections to the results based on the second-order closures.

The first question is much more difficult to answer. Existing schemes do not take into account the strongly anisotropic eddies that dominate the wall layer. It is well known from experimental data and direct numerical simulations that the turbulent energy distribution in a channel flow has a pronounced maximum very close to the wall, namely at $y_+ \approx 13$–18. In this region, the turbulent viscosity, defined as $\nu_t = -\overline{v_i v_j}/(\partial \bar{v}_i/\partial x_j)$, is very small. On the other hand, far from the wall, the turbulent intensity is much lower, but the turbulent viscosity is $\nu_t \approx 0.08 R_*$, which is many orders of magnitude larger than molecular viscosity. The inevitable conclusion is that not all turbulent eddies interact with mean flow and, consequently, not all eddies contribute to turbulent viscosity. The same effect is found in studies of the relaminarization of turbulent channel flow in a strong magnetic field. It can be shown that, if the magnetic field is large enough, turbulent channel flow becomes strongly anisotropic and the velocity profile and the friction coefficient approach those of laminar flow. The total turbulent intensity, however, remains quite high. It seems that strongly anisotropic scales may not interact with the mean flow.

The interaction of small-scale flows with large-scale perturbations has been studied analytically by Sivashinsky and Yakhot (1985), Yakhot and Sivashinsky (1986), and Bayly and Yakhot (1986). It has been shown that, if the small-scale flow is sufficiently anisotropic, it either decouples from the large-scale flow or gives its energy to the large eddies. Only when small scales are sufficiently isotropic do they increase the dissipation of large scales and thus give rise to a positive turbulent viscosity. The wall region of the channel or pipe flow is dominated by strongly anisotropic structures (streaks) which do not interact directly with the mean flow and thus do not

contribute to turbulent viscosity. Thus, it is possible for the maximum of turbulent kinetic energy to be located where the turbulent viscosity is close to zero. The weak coupling of strongly anisotropic scales to the mean motion may be the reason for the success of turbulence modeling based on the elimination of isotropic eddies from the inertial range dynamics and for the apparent success of RNG methods for turbulence.

ACKNOWLEDGMENTS

We are grateful to Dr. A. Yakhot for numerous suggestions which influenced the course of this work. We also acknowledge support by the Air Force Office of Scientific Research under contract F49620-85-C-0026, the Office of Naval Research under contracts N00014-82-C-0451 and N00014-85-K-0201, and the National Science Foundation under grants MSM-8514128 and ATM-8414410.

REFERENCES

Antonia, R. A., Chambers, A. J., and Anselmet, F. (1984). *Phys. Chem. Hydrodyn.* **5**, 368.
Batchelor, G. (1959). *J. Fluid Mech.* **5**, 113.
Bayly, B., and Yakhot, V. (1986). *Phys. Rev. A*, in press.
Deardorff, J. W. (1970). *J. Fluid Mech.* **41**, 453.
Deardorff, J. W. (1971). *J. Comp. Phys.* **7**, 120.
De Dominicis, C., and Martin, P. C. (1979). *Phys. Rev. A* **19**, 419.
Dorfman, R. (1975). In Cohen, E. G. D. (ed.), *Fundamental Problems in Statistical Mechanics*, North-Holland, Amsterdam.
Forster, D., Nelson, D., and Stephen, M. (1977). *Phys. Rev. A* **16**, 732.
Fournier, J. P., and Frisch, U. (1978). *Phys. Rev. A* **17**, 747.
Fournier, J. P., and Frisch, U. (1983). *Phys. Rev. A* **28**, 1000.
Frenkiel, F. N., Klebanoff, P., and Huang, T. T. (1979). *Phys. Fluids* **22**, 1606.
Hanjalić, K., and Launder, B. E. (1972). *J. Fluid Mech.* **52**, 609.
Herring, J. R., Orszag, S. A., Kraichnan, R. H., and Foxz, D. G. (1974). *J. Fluid Mech.* **66**, 417.
Hohenberg, P. C., and Halperin, B. I. (1977). *Rev. Mod. Phys.* **49**, 435.
Kolmogorov, A. N. (1941). *Dokl. Akad. Nauk SSSR* **30**, 299.
Kraichnan, R. H. (1959). *J. Fluid Mech.* **5**, 497.
Kraichnan, R. H. (1961). *J. Math. Phys.* **2**, 124.
Kraichnan, R. H. (1971). *J. Fluid Mech.* **47**, 525.
Landau, L., and Lifshitz, E. M. (1982). *Fluid Mechanics*, Pergamon, New York.
Launder, B. E., and Spalding, D. B. (1972). *Mathematical Models of Turbulence*, Academic Press, New York.
Launder, B. E., Reece, G. J., and Rodi, W. (1975). *J. Fluid Mech.* **68**, 537.
Leslie, D. C. (1972). *Developments in the Theory of Turbulence*, Clarendon Press, Oxford.
Ma, S. K., and Mazenko, G. (1975). *Phys. Rev. B* **11**, 4077.
Moin, P., and Kim, J. (1981). *J. Fluid Mech.* **118**, 341.

Monin, A. S., and Yaglom, A. M. (1975). *Statistical Fluid Mechanics*, Vol. 2, MIT Press, Cambridge, Massachusetts.
Orszag, S., and Patera, A. (1981). *Phys. Rev. Lett.* **47**, 832.
Pao, Y. H. (1965). *Phys. Fluids* **8**, 1063.
Pao, Y. H. (1968). *Phys. Fluids* **11**, 1371.
Reynolds, W. C. (1976). *Annu. Rev. Fluid Mech.* **8**, 183.
Sivashinsky, G., and Yakhot, V. (1985). *Phys. Fluids* **28**, 1040.
Smagorinsky, J. (1963). *Monthly Weather Rev.* **91**, 99.
Tennekes, H., and Lumley, J. L. (1972). *A First Course in Turbulence*, MIT Press, Cambridge, Massachusetts.
Wilson, K. G. (1971). *Phys. Rev. B* **4**, 3174.
Wilson, K. G., and Kogut, J. (1974). *Phys. Rev.* **12C**, 77.
Wood, W. W. (1975). In Cohen, E. G. D. (ed.), *Fundamental Problems in Statistical Mechanics*, North-Holland, Amsterdam.
Wyld, H. W. (1961). *Ann. Phys.* **14**, 143.
Wyngaard, J. C., and Tennekes, H. (1970). *Phys. Fluids* **13**, 1962.
Yakhot, V. (1981). *Phys. Rev. A* **23**, 1486.
Yakhot, V., and Sivashinsky, G. (1986). *Phys. Rev. A*, submitted.