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ERROR EXPANSION FOR AN UPWIND SCHEME APPLIED TO A TWO-DIMENSIONAL CONVECTION-DIFFUSION PROBLEM*

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Abstract. We consider a singularly perturbed convection-diffusion problem in a rectangular domain. It is solved numerically using a first-order upwind finite-difference scheme on a tensor-product piecewise-uniform Shishkin mesh with $O(N)$ mesh points in each coordinate direction. It is known [G. I. Shishkin, *Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations*, Russian Academy of Sciences, Ural Branch, Ekaterinburg, Russia, 1992 (in Russian)] that the error is almost-first-order accurate in the maximum norm. We decompose the error into a sum of continuous almost-first-order terms and the almost-second-order residual under the assumption $\varepsilon \le CN^{-1}$, where ε is the singular perturbation parameter and C is a constant. This error expansion is applied to obtain maximum-norm error estimates for the Richardson extrapolation technique and derive bounds on the errors in approximating the derivatives of the true solution by divided differences of the computed solution. The analysis uses a decomposition of the true solution requiring fewer compatibility conditions than in earlier publications. Numerical results are presented that support our theoretical results.

Key words. convection-diffusion, upwind scheme, singular perturbation, error expansion, Richardson extrapolation, approximation of derivatives, Shishkin mesh

AMS subject classifications. 65N06, 65N15, 35C20

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1. Introduction.

The main result of this paper is a certain error expansion for the singularly perturbed two-dimensional convection-diffusion problem

$$
\begin{aligned}
Lu := -\varepsilon\triangle u + b_1 u_x + b_2 u_y + cu &= f \qquad \text{in } \Omega = (0,1)\times(0,1),\\
u &= 0 \qquad \text{on } \partial\Omega.
\end{aligned} \tag{1.1}
$$

Here ε is a small parameter that satisfies $0 < \varepsilon \ll 1$, while $b_1(x,y)$, $b_2(x,y)$, $c(x,y)$ are smooth functions with

$$
b_1(x,y) > \beta_1 > 0, \quad b_2(x,y) > \beta_2 > 0, \quad c(x,y) \ge 0 \quad \text{for all } (x,y) \in \bar{\Omega}, \tag{1.2a}
$$

where β_1, β_2 are positive constants. To simplify the presentation we assume that

$$
\beta_1 = \beta_2 = \beta > 0. \tag{1.2b}
$$

Note that all the results of this paper also hold true for the general case (1.2a); see Remarks 1.1 and 4.5.

The solution of problem (1.1) has exponential layers at the outflow boundaries $x = 1$ and $y = 1$ (see [8, 10]). We are interested in ε-uniform numerical methods that resolve the boundary layers. One approach is using layer-adapted highly nonuniform meshes.

Problem (1.1) is discretized using the standard first-order upwind scheme

$$
\begin{aligned}
L^N u^N := \left(-\varepsilon(\delta_x^2 + \delta_y^2) + b_{1,ij}D_x^- + b_{2,ij}D_y^- + c_{ij}\right)u_{ij}^N &= f_{ij} \qquad \text{in } \Omega^N,\\
u_{ij}^N &= 0 \qquad \text{on } \partial\Omega^N.
\end{aligned} \tag{1.3}
$$

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Here δ_x^2, δ_y^2, D_x^-, D_y^- are the standard finite-difference differentiation operators; see notation (2.1). Note from [11, 8, 10] that this scheme satisfies the maximum principle.

We discretize on the mesh $\bar\Omega^N = \bar\omega_{\sigma,N}\times\bar\omega_{\sigma,N} = \{(x_i,y_j)\in\bar\Omega:\ i,j=0,\dots,N\}$ that is the tensor-product of two equal piecewise-uniform meshes. Each of these one-dimensional meshes $\bar\omega_{\sigma,N}$ is constructed by dividing each of the subintervals $[0,1-\sigma]$ and $[1-\sigma,1]$ into $N/2$ equal subintervals of width H and h, respectively:

$$x_i = y_i = \begin{cases} iH & \text{for}\quad i=0,\dots,N/2, \quad \text{where}\quad H=2(1-\sigma)/N,\\ (1-2\sigma)+ih & \text{for}\quad i=N/2,\dots,N, \quad \text{where}\quad h=2\sigma/N.\end{cases} \tag{1.4}$$

Shishkin [13] was the first to suggest such piecewise-uniform meshes for problems like (1.1) with the mesh transition parameter $\sigma := \min\{(2/\beta)\,\varepsilon\ln N,\ 1/2\}$. For simplicity we assume that

$$\varepsilon \le CN^{-1}, \tag{1.5}$$

which is not a restriction in practical situations. This assumption implies that

$$\sigma = \frac{2}{\beta}\,\varepsilon\ln N. \tag{1.6}$$

Note also that $N^{-1} < H < 2N^{-1}$ and

$$\frac{h}{\varepsilon} = \frac{4}{\beta}\,N^{-1}\ln N, \qquad e^{-\beta(1-x_{N/2})/\varepsilon} = e^{-\beta(1-y_{N/2})/\varepsilon} = e^{-\beta\sigma/\varepsilon} = N^{-2}. \tag{1.7}$$

Further, let $\partial\Omega^N$ be the set of mesh points on the boundary, i.e., $\partial\Omega^N = \bar\Omega^N\cap\partial\Omega$, while $\Omega^N = \bar\Omega^N\backslash\partial\Omega^N$ is the set of the internal mesh points.

Thus the domain $\bar\Omega$ is dissected by the transition lines $x = 1-\sigma$ and $y = 1-\sigma$ into four parts

$$\begin{aligned} &\Omega_2 := [0,1-\sigma]\times(1-\sigma,1], &&\Omega_{12} := (1-\sigma,1]\times(1-\sigma,1],\\ &\bar\Omega_0 := [0,1-\sigma]\times[0,1-\sigma], &&\Omega_1\ := (1-\sigma,1]\times[0,1-\sigma]. \end{aligned}$$

The restriction of the mesh $\bar\Omega^N$ to each of them is a rectangular uniform mesh.

Remark 1.1. The analogue of $\bar\Omega^N$ for $\beta_1\ne\beta_2$ is the tensor-product rectangular mesh $\bar\omega_{\sigma_1,N}\times\bar\omega_{\sigma_2,N}$, where σ_k, H_k, h_k, $\bar\omega_{\sigma_k,N}$, for $k=1,2$, are defined similarly to σ, H, h, $\bar\omega_{\sigma,N}$ with β_k used instead of β; see, e.g., [8, p. 101].

The paper is organized as follows. Most of the notation is collected in section 2. In section 3 we analyze a decomposition of the true solution into an asymptotic expansion of order one and its residual. This decomposition and our estimates of its components require fewer compatibility conditions than in earlier publications [13, 7].

In section 4 we present a certain error expansion for the upwind scheme (1.3) on the Shishkin mesh (1.4). Shishkin [13] gave an ε-uniform almost-first-order estimate of the error in the discrete maximum norm, which was slightly improved in [5, Remark 3.3] to

$$\|u_{ij}^N - u(x_i,y_j)\| \le CN^{-1}\ln N.$$

We decompose the error into a sum of continuous almost-first-order terms and the almost-second-order residual (Theorem 4.1). This error expansion is applied in subsection 4.1 to obtain maximum-norm error estimates for the Richardson extrapolation technique, and in subsection 4.2 to derive bounds on the errors in approximating the derivatives. Section 5 is devoted to the proof of Theorem 4.1.

Similar error expansions were constructed in [9, 4] for one-dimensional convection-diffusion problems. These error expansions were used there to analyze the Richardson extrapolation technique. We mainly follow the analysis in [9], extending it to the two-dimensional problem. In subsection 4.2 we obtain a two-dimensional analogue of the one-dimensional estimates [1, 3, 4]. We follow the approach of [4], where, to analyze a defect correction method, the error expansion was also used to obtain bounds on the differences of the error in two adjoining nodes.

Richardson extrapolation applied to singularly perturbed problems was also studied in earlier publications of Shishkin [12, 14], where ε-uniform maximum-norm error estimates were obtained for a one-dimensional parabolic problem and a two-dimensional elliptic problem in an infinite strip.

Numerical results supporting our theory are presented in section 6.

2. Notation. Throughout the paper we use the following notation. Let k be a nonnegative integer and $\alpha \in (0, 1]$. The standard notation $C^k(\bar\Omega)$ is used for the space of functions whose derivatives up to order k are continuous on $\bar\Omega$, with the norm

$$\|v\|_k = \sum_{0\le l\le k} \sum_{i+j=l} \max_{(x,y)\in\bar\Omega} \left| \frac{\partial^{i+j}}{\partial x^i \partial y^j} v(x,y) \right|.$$

As usual, we simply write $C(\bar\Omega)$ and $\|v\|$ when $k = 0$. The notation $C^{k,\alpha}(\bar\Omega)$ is used for the space of Hölder continuous functions with the norm

$$\|v\|_{0,\alpha} = \sup_{x,x'\in\bar\Omega,\ x\neq x'} \frac{|v(x) - v(x')|}{\|x - x'\|_e^\alpha}, \qquad \|v\|_{k,\alpha} = \|v\|_k + \sum_{i+j=k} \left\| \frac{\partial^{i+j}}{\partial x^i \partial y^j} v \right\|_{0,\alpha},$$

where $\|\cdot\|_e$ is the Euclidean norm in R^2. Further, we shall use the notation $C^{1,1}(\bar\Omega)$ when $\alpha = 1$, and $C^{1,\alpha}(\bar\Omega)$ only when $\alpha \in (0, 1)$.

Let v be a discrete function defined on $\tilde\Omega^N \subset \bar\Omega^N$. By $\|v\|_{\tilde\Omega^N} = \max_{\tilde\Omega^N} |v_{ij}|$ we denote the discrete maximum norm of v on $\tilde\Omega^N$. Sometimes we shall simply write $\|v\|$ when $\tilde\Omega^N = \bar\Omega^N$.

The finite-difference operators are defined in a standard manner by

$$\begin{aligned} h_i &:= x_i - x_{i-1}, \qquad D_x^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{h_i}, \qquad \delta_x^2 v_{ij} := \frac{D_x^- v_{i+1,j} - D_x^- v_{ij}}{(h_i + h_{i+1})/2}, \\ h_j &:= y_j - y_{j-1}, \qquad D_y^- v_{ij} := \frac{v_{ij} - v_{i,j-1}}{h_j}, \qquad \delta_y^2 v_{ij} := \frac{D_y^- v_{i,j+1} - D_y^- v_{ij}}{(h_j + h_{j+1})/2}. \end{aligned} \tag{2.1}$$

Here v_{ij} is any discrete function. Note that when it is clear that $v(x, y)$ is a continuous function, we shall sometimes use the notation $v_{ij} := v(x_i, y_j)$, while when it is clear that v_{ij} is a discrete function, we shall sometimes use the notation $v(x_i, y_j) := v_{ij}$.

For an arbitrary $\tilde\Omega \subset \bar\Omega$ and arbitrary constants a, b on $\bar\Omega^N$ define the function

$$\mathcal{E}_{ij}(a, \tilde\Omega;\ b) = \begin{cases} a & \text{for } (x_i, y_j) \in \tilde\Omega, \\ b & \text{for } (x_i, y_j) \in \bar\Omega^N \backslash \tilde\Omega. \end{cases}$$

We shall also use other similar notation, e.g., $\mathcal{E}_{ij}(a, i \le N/2;\ b)$.

Throughout the paper, C, sometimes subscripted, denotes a generic positive constant that is independent of ε and any mesh used.

3. Decomposition of the solution. In this section we decompose the solution into an asymptotic expansion of order one and its residual. We estimate the components of this decomposition and their derivatives.

THEOREM 3.1. *Let $\alpha \in (0,1)$, and β be from* (1.2). *Suppose that $f \in C^{3,1}(\bar\Omega)$ and satisfies the compatibility conditions*

$$f(0,0) = f(0,1) = f(1,1) = f(1,0) = 0, \tag{3.1a}$$

$$\left(\frac{f}{b_1}\right)_y (0,0) = \left(\frac{f}{b_2}\right)_x (0,0), \tag{3.1b}$$

$$\left(\frac{1}{b_1}\left(b_1\frac{\partial}{\partial x} - b_2\frac{\partial}{\partial y} - c\right)\left[\frac{f}{b_1}\right]\right)_y (0,0) = \left(\frac{f}{b_2}\right)_{xx} (0,0). \tag{3.1c}$$

Then the boundary-value problem (1.1) *has a classical solution $u \in C^{3,\alpha}(\bar\Omega)$, and this solution can be decomposed as*

$$u = (u_0 + v_0 + w_0 + z_0) + \varepsilon(u_1 + v_1 + w_1 + z_1) + \varepsilon^2 R,$$

where $u_0 \in C^{3,1}(\bar\Omega)$, $u_1 \in C^{1,1}(\bar\Omega)$, $\frac{\partial^k}{\partial x^k}v_1$, $\frac{\partial^k}{\partial y^k}w_1 \in C^{1,1}(\bar\Omega)$ for $k \ge 0$, $z_1 \in C^3(\bar\Omega)$, $R \in C^{1,1}(\bar\Omega)$, and

$$\|u_0\|_{3,1} + \|u_1\|_{1,1} \le C, \quad u_0(x,0) = u_0(0,y) = 0, \quad u_{0,xx}(0,0) = u_{0,yy}(0,0) = 0, \tag{3.2}$$

$$\begin{aligned} &v_0(x,y) = -u_0(1,y)e^{-b_1(1,y)(1-x)/\varepsilon}, \qquad w_0(x,y) = -u_0(x,1)e^{-b_2(x,1)(1-y)/\varepsilon},\\ &z_0(x,y) = u_0(1,1)e^{-b_1(1,1)(1-x)/\varepsilon - b_2(1,1)(1-y)/\varepsilon}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\left\|\frac{\partial^k}{\partial x^k}v_1(x,\cdot)\right\|_{1,1,[0,1]} \le C\varepsilon^{-k}e^{-\beta(1-x)/\varepsilon},\\ &\left\|\frac{\partial^k}{\partial y^k}w_1(\cdot,y)\right\|_{1,1,[0,1]} \le C\varepsilon^{-k}e^{-\beta(1-y)/\varepsilon} \qquad \textit{for } 0 \le k \le 3, \end{aligned} \tag{3.4a}$$

$$\left|\frac{\partial^{k+m}}{\partial x^k \partial y^m} z_1(x,y)\right| \le C\varepsilon^{-(k+m)}e^{-\beta((1-x)+(1-y))/\varepsilon} \qquad \textit{for } 0 \le k+m \le 3, \tag{3.4b}$$

$$\|R\| \le C, \qquad |LR(x,y)| \le C\big(1 + \varepsilon^{-1}e^{-\beta(1-x)/\varepsilon} + \varepsilon^{-1}e^{-\beta(1-y)/\varepsilon}\big). \tag{3.5}$$

Remark 3.1. In (3.4a) by $\|\frac{\partial^k}{\partial x^k}v_1(x,\cdot)\|_{1,1,[0,1]}$ we denote the norm of the function $\frac{\partial^k}{\partial x^k}v_1(x,y)$ as a function of the variable y in the space $C^{1,1}[0,1]$ of Hölder continuous functions. The second line in (3.4a) should be understood similarly.

Remark 3.2. Note that $C^{1,1}(\bar\Omega) = W^{2,\infty}(\Omega)$, and for any function in $C^{1,1}(\bar\Omega)$ its second partial derivatives exist almost everywhere [2, pp. 151, 154]. Hence, since $R \in C^{1,1}(\bar\Omega)$, in (3.5) the second inequality is to be understood in the sense that it holds true almost everywhere.

Remark 3.3. Shishkin [13, Theorem III.2.1] decomposed the solution into a smooth part and a layer part so that the layer part lay in the null space of L. A similar decomposition was constructed by Linß and Stynes [7]. They presented a full analysis and the explicit compatibility conditions. The solution was decomposed into an asymptotic expansion of order one and its residual. Then the residual was combined with the smooth part of the solution so that the layer part “almost” lay in the null space of L. Note that the hypotheses of our theorem are weaker than those of [7, Theorem 5.1]. In particular, since we do not combine the smooth part with the residual $\varepsilon^2 R$ and do not estimate the derivatives of the latter, our decomposition is useful only for small values of ε, e.g., under our assumption (1.5), but we require *fewer compatibility conditions* at the corner $(0, 0)$.

Proof. We mainly follow the proof and the notation of [7, Theorem 5.1], but omit certain parts of this proof that are unnecessary for our decomposition, and combine certain terms in a different manner.

By [7, Lemma 2.1], the compatibility conditions (3.1a) combined with $f \in C^{3,1}(\bar{\Omega})$ imply that $u \in C^{3,\alpha}(\bar{\Omega})$.

We decompose u as in [7]. Thus, u_0 and u_1 are the solutions of the reduced problems [7, (5.2)]. Note that the boundary conditions $u_0(x, 0) = u_0(0, y) = 0$ for u_0 yield $u_{0,xx}(0, 0) = u_{0,yy}(0, 0) = 0$ in (3.2), while the first estimate in (3.2) is obtained applying [7, Theorem 4.1] twice. First, $u_0 \in C^{3,1}(\bar{\Omega})$ since $f \in C^{3,1}(\bar{\Omega})$, while (3.1) implies the compatibility conditions [7, (4.8a), (4.8b), (4.8c)]. Second, $u_1 \in C^{1,1}(\bar{\Omega})$ since $\triangle u_0 \in C^{1,1}(\bar{\Omega})$, while the compatibility condition $\triangle u_0(0, 0) = 0$ corresponds to [7, (4.8a)].

Furthermore, v_1 and w_1 are given explicitly by [7, (5.11b), (5.15b)], while z_1 is the solution of the problem [7, (5.17b), (5.17c)]. By [7, Lemma 5.2], the compatibility condition $f(1, 1) = 0$ implies that there exists $z_1 \in C^3(\bar{\Omega})$ satisfying (3.4b).

Estimates (3.5) are derived similarly to [7, (5.31)] and the argument that follows it. Note that in [7] $R \in C^{2,\alpha}(\bar{\Omega})$, while we have $R \in C^{1,1}(\bar{\Omega})$; see Remark 3.2. The first estimate in (3.5) follows from the second by the maximum/comparison principle extended to functions in the Sobolev space $W^{1,2}(\Omega)$ (see [2, section 8.1]). ∎

4. Error expansion and its applications. In this section we present an expansion of the error of the upwind scheme (1.3) on the Shishkin mesh (1.4), (1.6) into a sum of continuous first-order terms and the second-order residual. This error expansion is applied in subsection 4.1 to obtain ε-uniform maximum-norm error estimates for the Richardson extrapolation technique, and in subsection 4.2 to derive bounds on the errors in approximating the derivatives.

THEOREM 4.1. *Suppose that* (1.5) *and the conditions of Theorem* 3.1 *are satisfied. Let u^N be the solution of the discrete problem* (1.3) *on the mesh* (1.4), (1.6). *Then*

$$u_{ij}^N - u(x_i, y_j) = H\Phi(x_i, y_j) + \left(\frac{h}{\varepsilon}\right)\Psi(x_i, y_j) + \mathcal{R}_{ij}^N, \tag{4.1}$$

where $\Phi(x, y)$ and $\Psi(x, y)$ are defined in terms of u_0, v_0, w_0, and z_0 from Theorem 3.1, *and $\varphi(x, y) \in C^{1,1}(\bar{\Omega})$ such that*

$$\|\varphi\|_{1,1} \le C, \tag{4.2}$$

as follows:

$$\begin{aligned}\Phi(x, y) = \varphi(x, y) - \varphi(1, y)e^{-b_1(1,y)(1-x)/\varepsilon} - \varphi(x, 1)e^{-b_2(x,1)(1-y)/\varepsilon}\\ + \varphi(1, 1)e^{-b_1(1,1)(1-x)/\varepsilon - b_2(1,1)(1-y)/\varepsilon},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Psi(x, y) &= \varepsilon^{-1}(1 - x)\frac{\left(b_1^2(1, y)v_0 + b_1^2(1, 1)z_0\right)}{2}\\ &+ \varepsilon^{-1}(1 - y)\frac{\left(b_2^2(x, 1)w_0 + b_2^2(1, 1)z_0\right)}{2},\end{aligned} \tag{4.4}$$

while the residual $\mathcal{R}_{ij}^N$ satisfies

$$|\mathcal{R}_{ij}^N| \le CN^{-2}\,\mathcal{E}_{ij}(1,\bar{\Omega}_0^N;\ln^2 N). \tag{4.5}$$

Proof. The whole of section 5 is devoted to the proof of this theorem; see also Remark 4.1. ∎

Remark 4.1. A careful inspection of the proof of Theorem 4.1 shows that

$$L^N(u^N-u) = H\frac{(b_1u_{0,xx}+b_2u_{0,yy})}{2} + \left(\frac{h}{\varepsilon}\right)\left[\frac{\varepsilon b_1(v_{0,xx}+z_{0,xx})}{2}+\frac{\varepsilon b_2(w_{0,yy}+z_{0,yy})}{2}\right]+\dots,$$

where ... denotes the terms whose contribution to the error is of almost-second order; see (5.1), (5.2), (5.7), (5.16), (5.18). The standard approach is to define the auxiliary continuous problems

$$L\bar{\Phi} = \frac{(b_1u_{0,xx}+b_2u_{0,yy})}{2} \ \text{ in } \Omega, \qquad \bar{\Phi} = 0 \text{ on } \partial\Omega, \tag{4.6a}$$

$$L\bar{\Psi}_1 = \frac{\varepsilon b_1(v_{0,xx}+z_{0,xx})}{2} \ \text{ in } \Omega, \qquad \bar{\Psi}_1 = 0 \text{ on } \partial\Omega, \tag{4.6b}$$

$$L\bar{\Psi}_2 = \frac{\varepsilon b_2(w_{0,yy}+z_{0,yy})}{2} \ \text{ in } \Omega, \qquad \bar{\Psi}_2 = 0 \text{ on } \partial\Omega, \tag{4.6c}$$

and derive the error expansion

$$u_{ij}^N - u(x_i,y_j) = H\bar{\Phi}(x_i,y_j) + \left(\frac{h}{\varepsilon}\right)[\bar{\Psi}_1(x_i,y_j)+\bar{\Psi}_2(x_i,y_j)]+\cdots,$$

where ... denotes almost-second-order terms; see, e.g., [9] for the one-dimensional case. Our proof mainly follows the analysis of [9], extending it to the two-dimensional case, but, as we shall see, the solutions of the two-dimensional auxiliary problems (4.6) are only in $C^{1,\alpha}(\bar{\Omega})$ since the first-order compatibility conditions are violated. Since the solutions of (4.6) do not exhibit enough smoothness for our analysis, our error expansion (4.1) uses their asymptotic expansions of order zero; see Remarks 4.2–4.4.

Remark 4.2. $\varphi(x,y)$ used in Theorem 4.1 is the solution of the reduced problem

$$b_1\varphi_x + b_2\varphi_y + c\varphi = \frac{\big(b_1u_{0,xx}+b_2u_{0,yy}\big)}{2} \ \text{ in } \Omega, \qquad \varphi(x,y)=0 \ \text{ if } x=0 \text{ or } y=0, \tag{4.7}$$

where u_0 is from Theorem 3.1.

Remark 4.3. $\Phi(x,y)$ in (4.3) is an asymptotic expansion of order zero for the solution $\bar{\Phi}(x,y)$ of problem (4.6a). We chose to use $\Phi(x,y)$ instead of $\bar{\Phi}(x,y)$ since, as we shall prove in Lemma 5.7, $\Phi(x,y) \in C^{1,1}(\bar{\Omega})$, while by [7, Lemma 2.1], we have $\bar{\Phi}(x,y) \in C^{1,\alpha}(\bar{\Omega})$ for $\alpha \in (0,1)$. Note that generally $\bar{\Phi}(x,y) \notin C^{2,\alpha}(\bar{\Omega})$ for any $\alpha \in (0,1)$, since the right-hand side $(b_1u_{0,xx}+b_2u_{0,yy})/2$ does not generally vanish at $(1,1)$ and thus does not satisfy one of the compatibility conditions.

Remark 4.4. Decompose $\Psi(x,y)$ in (4.4) as $\Psi = (\Psi_1+\tilde{\Psi}_1)+(\Psi_2+\tilde{\Psi}_2)$; see (5.8) for details. Note that $\Psi_1(x,y)+\tilde{\Psi}_1(x,y)$ and $\Psi_2(x,y)+\tilde{\Psi}_2(x,y)$ are asymptotic expansions of order zero for the solutions $\bar{\Psi}_1(x,y)$ and $\bar{\Psi}_2(x,y)$ of problems (4.6b), (4.6c). By (3.3), one can easily check that Ψ_1, $\tilde{\Psi}_1$, Ψ_2, and $\tilde{\Psi}_2$ are chosen so that

$$-\varepsilon\Psi_{1,xx}+b_1(1,y)\Psi_{1,x} = \frac{b_1(1,y)\varepsilon v_{0,xx}}{2}, \qquad -\varepsilon\tilde{\Psi}_{1,xx}+b_1(1,1)\tilde{\Psi}_{1,x} = \frac{b_1(1,1)\varepsilon z_{0,xx}}{2},$$

$$-\varepsilon\Psi_{2,yy}+b_2(x,1)\Psi_{2,y} = \frac{b_2(x,1)\varepsilon v_{0,yy}}{2}, \qquad -\varepsilon\tilde{\Psi}_{2,yy}+b_2(1,1)\tilde{\Psi}_{2,y} = \frac{b_1(1,1)\varepsilon z_{0,yy}}{2}.$$

Remark 4.5. If $\beta_1 \neq \beta_2$ and the mesh $\bar\omega_{\sigma_1,N} \times \bar\omega_{\sigma_2,N}$ described in Remark 1.1 is used, then we have a slightly different error expansion:

$$u_{ij}^N - u(x_i, y_j) = H_1\Phi_1(x_i, y_j) + H_2\Phi_2(x_i, y_j)$$
$$+ \left(\frac{h_1}{\varepsilon}\right)[\Psi_1 + \tilde\Psi_1](x_i, y_j) + \left(\frac{h_2}{\varepsilon}\right)[\Psi_2 + \tilde\Psi_2](x_i, y_j) + \mathcal{R}_{ij}^N.$$

Here $\Psi_1 + \tilde\Psi_1$ and $\Psi_2 + \tilde\Psi_2$ are the first and the second terms on the right-hand side in (4.4)—see Remark 4.4 and (5.8)—while Φ_1 and Φ_2 are defined by (4.3) with Φ and φ replaced by Φ_k and φ_k for $k = 1, 2$. These functions φ_1 and φ_2 are the solutions of the reduced problem (4.7) with the right-hand side $(b_1 u_{0,xx} + b_2 u_{0,yy})/2$ replaced by $b_1 u_{0,xx}/2$ and $b_2 u_{0,yy}/2$, respectively.

4.1. Richardson extrapolation. Now we shall see that the error expansion given by Theorem 4.1 immediately implies ε-uniform maximum-norm error estimates for the Richardson extrapolation technique.

In this subsection for the mesh $\bar\Omega^N$ we shall use the slightly different notation $\bar\Omega_{\sigma,N} := \bar\Omega^N = \bar\omega_{\sigma,N} \times \bar\omega_{\sigma,N}$. We shall also use the tensor-product rectangular mesh $\bar\Omega_{\sigma,2N} := \bar\omega_{\sigma,2N} \times \bar\omega_{\sigma,2N} = \{(\tilde x_i, \tilde y_j) \in \bar\Omega : \ i, j = 0, \ldots, 2N\}$. Here $\bar\omega_{\sigma,2N}$ is a piecewise-uniform mesh with the meshsizes $h/2$ and $H/2$ obtained uniformly bisecting the original mesh $\bar\omega_{\sigma,N}$. Note that $\bar\omega_{\sigma,2N}$ is also described by (1.4) with the same mesh transition parameter σ (1.6) and N replaced by $2N$. The two rectangular meshes are nested; i.e., $\Omega_{\sigma,N} = \{(x_i, y_j)\} \subset \Omega_{\sigma,2N} = \{(\tilde x_i, \tilde y_j)\}$, and $(x_i, y_j) = (\tilde x_{2i}, \tilde y_{2j})$.

Let $\tilde u_{ij}^{2N} = \tilde u^{2N}(\tilde x_i, \tilde y_j)$ be the solution of the discrete problem (1.3) on the mesh $\Omega_{\sigma,2N}$. Then under the conditions of Theorem 4.1, in addition to (4.1) we have

$$\tilde u^{2N}(\tilde x_i, \tilde y_j) - u(\tilde x_i, \tilde y_j) = \frac{1}{2} H\Phi(\tilde x_i, \tilde y_j) \frac{1}{2}\left(\frac{h}{\varepsilon}\right)\Psi(\tilde x_i, \tilde y_j) + \tilde{\mathcal{R}}^{2N}(\tilde x_i, \tilde y_j).$$

Hence

$$[\,2\tilde u^{2N}(x_i, y_j) - u_{ij}^N\,] - u(x_i, y_j) = 2\tilde{\mathcal{R}}^{2N}(x_i, y_j) - \mathcal{R}_{ij}^N,$$

and we arrive at the following.

COROLLARY 4.2. *Under the conditions of Theorem* 4.1, *we have*

$$\left|[\,2\tilde u^{2N}(x_i, y_j) - u_{ij}^N\,] - u(x_i, y_j)\right| \le CN^{-2}\,\mathcal{E}_{ij}(1, \bar\Omega_0; \ln^2 N)$$
$$= C\begin{cases} N^{-2} & in \ \ \bar\Omega^N \cap \bar\Omega_0, \\ N^{-2}\ln^2 N & in \ \ \bar\Omega^N \backslash \bar\Omega_0. \end{cases}$$

Thus, while the two computed solutions u_{ij}^N and $\tilde u_{ij}^{2N}$ are almost-first-order accurate, their linear combination $[2\tilde u^{2N}(x_i, y_j) - u_{ij}^N]$ is almost-second-order accurate ε-uniformly.

4.2. Approximation of derivatives. In this subsection we apply the error expansion given by Theorem 4.1 to derive bounds on the errors in approximating the derivatives of the true solution by divided differences of the computed solution.

COROLLARY 4.3. *Under the conditions of Theorem* 4.1, *we have*

$$\text{(4.8a)} \quad \left|D_x^- e_{ij}^N\right| + \left|D_x^- u_{ij}^N - u_x(x_{i-1/2}, y_j)\right| \le C\begin{cases} N^{-1} & in \ \ \bar\Omega^N \cap \bar\Omega_0, \\ N^{-1}\ln^2 N & in \ \ \bar\Omega^N \cap \Omega_2, \\ N^{-1}\ln N/\varepsilon & in \ \ \bar\Omega^N \cap (\Omega_1 \cup \Omega_{12}), \end{cases}$$

$$\text{(4.8b)}\quad \left|D_y^- e_{ij}^N\right| + \left|D_y^- u_{ij}^N - u_y(x_i, y_{j-1/2})\right| \le C \begin{cases} N^{-1} & in\ \bar\Omega^N \cap \bar\Omega_0, \\ N^{-1}\ln^2 N & in\ \bar\Omega^N \cap \Omega_1, \\ N^{-1}\ln N/\varepsilon & in\ \bar\Omega^N \cap (\Omega_2 \cup \Omega_{12}), \end{cases}$$

where $e_{ij}^N = u_{ij}^N - u(x_i, y_j)$ *is the error, while* $x_{i-1/2}$ *and* $y_{j-1/2}$ *are the midpoints of the segments* $[x_{i-1}, x_i]$ *and* $[y_{j-1}, y_j]$.

Proof. Since (4.8a) and (4.8b) are similar, we shall prove only bound (4.8a). By Theorem 3.1 and (1.5), (2.1), (1.4), the second inequality (4.8a) follows from the bound on $|D_x^- e_{ij}^N|$, so we need prove only the first bound (4.8a).

By Theorem 4.1, we have

$$\text{(4.9)}\qquad D_x^- e_{ij}^N = H D_x^- \Phi(x_i, y_j) + \left(\frac{h}{\varepsilon}\right) D_x^- \Psi(x_i, y_j) + D_x^- \mathcal{R}_{ij}^N.$$

First, using (4.5), (2.1), (1.4), we get estimate (4.8a) for $|D_x^- \mathcal{R}_{ij}^N|$.

Further in this proof and later throughout the paper, we shall use the inequalities

$$\text{(4.10)}\qquad \begin{aligned} \varepsilon^{-1}(1-x)e^{-b_1(1,y)(1-x)/\varepsilon} &\le Ce^{-\beta(1-x)/\varepsilon}, \\ \varepsilon^{-1}(1-y)e^{-b_2(x,1)(1-y)/\varepsilon} &\le Ce^{-\beta(1-y)/\varepsilon}. \end{aligned}$$

Define

$$\hat\Phi(x,y) = \varphi(x,y) - \varphi(x,1)e^{-b_2(x,1)(1-y)/\varepsilon}, \qquad \hat\Psi(x,y) = \frac{\varepsilon^{-1}(1-y)b_2^2(x,1)w_0(x,y)}{2}.$$

Since $|D_x^- \tilde\Phi_{ij}| \le \max_{\bar\Omega} |\hat\Phi_x|$, then by (4.2) we have $|D_x^- \hat\Phi_{ij}| \le C$. This implies estimate (4.8a) also for $|H D_x^- \hat\Phi_{ij}|$.

Similarly, we get $|D_x^- \hat\Psi_{ij}| \le C$. Note that in $\bar\Omega_0 \cup \Omega_1$ we have the sharper estimate $|D_x^- \hat\Psi_{ij}| \le CN^{-2}$, since (3.3) and (1.7) imply that $\max_{\bar\Omega_0 \cup \Omega_1} |\hat\Psi_x| \le CN^{-2}$. Hence, $|(h/\varepsilon) D_x^- \hat\Phi_{ij}|$ also satisfies inequality (4.8a).

We proceed similarly with $D_x^-(\Phi - \hat\Phi)_{ij}$ and $D_x^-(\Psi - \hat\Psi)_{ij}$. Using (4.3), (4.4), (3.3), we obtain $|D_x^-(\Phi - \hat\Phi)_{ij}| + |D_x^-(\Psi - \hat\Psi)_{ij}| \le 1/\varepsilon$. However, in $\bar\Omega_0 \cup \Omega_2$ we need sharper estimates. By (2.1), we have $|D_x^-(\Phi - \hat\Phi)_{ij}| \le (2/H) \max_{\bar\Omega_0 \cup \Omega_2} |\Phi - \hat\Phi|$. Combining this with (3.3), (1.7), we get $|D_x^-(\Phi - \hat\Phi)_{ij}| \le CN^{-1}$ in $\bar\Omega_0 \cup \Omega_2$. Similarly, $|D_x^-(\Psi - \hat\Psi)_{ij}| \le CN^{-1}$ in $\bar\Omega_0 \cup \Omega_2$. Hence, $|H D_x^-(\Phi - \hat\Phi)_{ij}|$ and $|(h/\varepsilon) D_x^-(\Psi - \hat\Psi)_{ij}|$ also satisfy inequality (4.8a).

Combining the estimates that we derived for the right-hand terms in (4.9), we obtain the first bound (4.8a). This completes the proof. □

5. Proof of Theorem 4.1.

5.1. Discrete maximum/comparison principle and its corollaries. In this subsection we state the comparison lemmas that will be used to prove Theorem 4.1.

It is well known that the upwind scheme (1.3) satisfies the *discrete maximum/comparison principle*, which implies the following comparison lemma.

LEMMA 5.1. *Let* $\tilde\Omega^N$ *be a connected submesh of* Ω^N.

(i) *If* $|L^N v_{ij}| \le L^N B_{ij}$ *in* $\tilde\Omega^N$ *and* $|v_{ij}| \le B_{ij}$ *on* $\partial\tilde\Omega^N$, *then* $|v_{ij}| \le B_{ij}$ *in* $\tilde\Omega^N$.

(ii) *If* $v_{ij} = 0$ *on* $\partial\tilde\Omega^N$, *then* $\|v\|_{\tilde\Omega^N} \le \beta^{-1}\|L^N v\|_{\tilde\Omega^N}$.

(iii) *If* $L^N v_{ij} = 0$ *in* $\tilde\Omega^N$, *then* $\|v\|_{\tilde\Omega^N} \le \|v\|_{\partial\tilde\Omega^N}$.

Proof. See [11, Chapter IV] and [8, Chapter 13]. ∎

The following three lemmas follow from Lemma 5.1(i). We defer their proofs to Appendix A.

LEMMA 5.2. *If $L^N v_{ij} = 0$ in Ω^N and $|v_{ij}| \le e^{-\beta(1-x_i)/\varepsilon}$ on $\partial\Omega^N$, then $|v_{ij}| \le CN^{-2}$ for $i \le N/2$.*

LEMMA 5.3. (i) *If $|L^N v_{ij}| \le e^{-\beta(1-x_i)/\varepsilon}$ in Ω^N and $v_{ij} = 0$ on $\partial\Omega^N$, then $|v_{ij}| \le CN^{-1}$ in $\bar{\Omega}^N$, and $|v_{ij}| \le CN^{-2}$ for $i \le N/2$.*

(ii) *If $|L^N v_{ij}| \le \mathcal{E}_{ij}(\varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon}, i > N/2;\ 0)$ in Ω^N and $v_{ij} = 0$ on $\partial\Omega^N$, then $|v_{ij}| \le C\mathcal{E}_{ij}(N^{-1}, i \le N/2;\ 1)$.*

(iii) *Let $|L^N v_{ij}| \le \varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon}$ for $i > N/2$, where v_{ij} is defined for $i = N/2, \dots, N$, $j = 0, \dots, N$, and $v_{ij} = 0$ on the boundary of this submesh, i.e., if $i = N/2, N$ or $j = 0, N$. Then $|v_{ij}| \le C$ for $i \ge N/2$.*

Remark 5.1. Clearly, the analogues of Lemmas 5.2 and 5.3, with x, i replaced by y, j, also hold true.

LEMMA 5.4. *If $|L^N v_{ij}| \le \mathcal{E}_{ij}(0, \bar{\Omega}_0;\ 1)$ and $v_{ij} = 0$ on $\partial\Omega^N$, then $|v_{ij}| \le C\varepsilon\, \mathcal{E}_{ij}(1, \bar{\Omega}_0;\ \ln N) \le CN^{-1}\, \mathcal{E}_{ij}(1, \bar{\Omega}_0;\ \ln N)$.*

5.2. Error and truncation error. We shall derive a representation of the error

$$e_{ij}^N := u_{ij}^N - u(x_i, y_j).$$

One can easily check that $L^N e_{ij}^N = -L^N u_{ij} + (Lu)_{ij} =: r_{ij}[u]$. Here for the truncation error we have used the notation

$$r_{ij}[v] := -L^N v_{ij} + (Lv)_{ij}. \tag{5.1}$$

Recalling the decomposition of u given by Theorem 3.1, we have

$$L^N e^N = r[u_0 + v_0 + w_0 + z_0] + \varepsilon r[u_1 + v_1 + w_1 + z_1] + \varepsilon^2 r[R]. \tag{5.2}$$

Furthermore, we study the contributions to the error of each of the right-hand side terms separately.

In this section and the related appendices we shall use the *notation*

$$\begin{aligned} L_1 v &:= -\varepsilon v_{xx} + b_1(x,y)v_x, & L_2 v &:= -\varepsilon v_{yy} + b_2(x,y)v_y, \\ L_1^N v &:= -\varepsilon\delta_x^2 v + b_1(x,y)D_x^- v, & L_2^N v &:= -\varepsilon\delta_y^2 v + b_2(x,y)D_y^- v, \end{aligned} \tag{5.3}$$

$$r_{1,ij}[v] := -L_1^N v_{ij} + (L_1 v)_{ij}, \qquad r_{2,ij}[v] := -L_2^N v_{ij} + (L_2 v)_{ij}, \tag{5.4}$$

so that $L = L_1 + L_2 + c$, $L^N = L_1^N + L_2^N + c$, and $r[v] = r_1[v] + r_2[v]$.

5.3. Contribution of $\varepsilon^2 r[R]$ in the maximum norm. The contribution to the error of this component of the right-hand side in (5.2) is described by the following result.

LEMMA 5.5. *If $L^N w_{ij} = r_{ij}[R]$ in Ω^N, where R is from (3.5), and $w_{ij} = 0$ on $\partial\Omega^N$, then $\|w\| \le C\varepsilon^{-1}N^{-1}$.*

Proof. Obviously,

$$\|w\| \le \|w + R\| + \|R\|. \tag{5.5}$$

Since $r_{ij}[R] = -L^N R_{ij} + (LR)_{ij}$, we have $L^N[w + R]_{ij} = (LR)_{ij}$. Recalling (3.5) and applying Lemmas 5.1(ii), 5.3(i), and 5.1(iii), we get

$$\|w + R\| \le C(1 + \varepsilon^{-1}N^{-1}) + \max_{\partial\Omega^N} |R_{ij}|.$$

Combining this with (5.5), (3.5) and observing that (1.5) implies $1 \le C\varepsilon^{-1}N^{-1}$, we complete the proof. ∎

Remark 5.2. The proof of this lemma does not use any estimates of the derivatives of R and thus allows us to use a decomposition of the solution requiring fewer compatibility conditions; see Remark 3.3.

Now, by (1.5), we have the following.

COROLLARY 5.6. *If $L^N v_{ij} = \varepsilon^2 r_{ij}[R]$ in Ω^N, where R is from* (3.5), *and $v_{ij} = 0$ on $\partial\Omega^N$, then $|v| \le CN^{-2}$.*

5.4. Contribution of $r[u_0]$. The contribution to the error of this component of the right-hand side in (5.2) is described by the following two lemmas.

LEMMA 5.7. (i) *The reduced problem* (4.7) *has a solution $\varphi(x,y) \in C^{1,1}(\bar\Omega)$ such that $\|\varphi\|_{1,1} \le C$, and thus $\Phi(x,y)$ from* (4.3) *using this function φ is also in $C^{1,1}(\bar\Omega)$.*

(ii) *If w satisfies*

$$L^N w_{ij} = \frac{\big(b_1 u_{0,xx} + b_2 u_{0,yy}\big)_{ij}}{2} \ \text{in } \Omega^N, \quad w_{ij} = 0 \quad \text{on } \partial\Omega^N, \tag{5.6}$$

then

$$|w_{ij} - \Phi(x_i, y_j)| \le CN^{-1}\, \mathcal{E}_{ij}(1, \bar\Omega_0;\ \ln N).$$

Proof. (i) Note that φ is the solution of the reduced problem (4.7) with the right-hand side $\big(b_1 u_{0,xx} + b_2 u_{0,yy}\big)/2$, which, by (3.2), is in $C^{1,1}(\bar\Omega)$ and vanishes at the corner $(0,0)$, i.e., satisfies the compatibility condition [7, (4.8a)]. Hence, applying [7, Theorem 4.1], we have $\varphi(x,y) \in C^{1,1}(\bar\Omega)$. This implies that $\Phi(x,y) \in C^{1,1}(\bar\Omega)$.

(ii) This part of the proof is given in Appendix B. ∎

LEMMA 5.8. *If $L^N v_{ij} = r_{ij}[u_0]$ in Ω^N, where u_0 is from Theorem* 3.1, *and $v_{ij} = 0$ on $\partial\Omega^N$, then*

$$|v_{ij} - H\Phi(x_i, y_j)| \le CN^{-2}\, \mathcal{E}_{ij}(1, \bar\Omega_0;\ \ln N).$$

Proof. Recalling (5.1) and using Taylor series expansions and (3.2), we obtain

$$\big|r_{ij}[u_0] - (h_i b_1 u_{0,xx} + h_j b_2 u_{0,yy})_{ij}/2\big| \le C(\varepsilon N^{-1} + N^{-2})\|u_0\|_3 \le CN^{-2}.$$

Furthermore, since $h_i = h_j = H$ for $(x_i, y_j) \in \bar\Omega_0$, we have

$$\big|r_{ij}[u_0] - H\,(b_1 u_{0,xx} + b_2 u_{0,yy})_{ij}\big| \le C[N^{-1}\, \mathcal{E}_{ij}(0, \bar\Omega_0;\ 1) + N^{-2}]. \tag{5.7}$$

Combining this with $L^N(v_{ij} - Hw_{ij}) = r_{ij}[u_0] - H(b_1 u_{0,xx} + b_2 u_{0,yy})_{ij}$, where w_{ij} is from Lemma 5.7, we get

$$\big|L^N(v_{ij} - Hw_{ij})\big| \le C[N^{-1}\, \mathcal{E}_{ij}(0, \bar\Omega_0;\ 1) + N^{-2}].$$

Now, applying Lemmas 5.4 and 5.1(ii), we have

$$|v_{ij} - Hw_{ij}| \le CN^{-2}\, \mathcal{E}_{ij}(1, \bar\Omega_0;\ \ln N).$$

By Lemma 5.7, this yields the statement of the lemma. ∎

5.5. Contribution of $r[v_0 + w_0 + z_0]$. Now we shall study the contribution to the error of the component $r[v_0 + w_0 + z_0]$ of the right-hand side in (5.2).

The main result of this subsection is the following.

LEMMA 5.9. *If $L^N v_{ij} = r_{ij}[v_0 + w_0 + z_0]$ in Ω^N, where v_0, w_0, z_0 are from Theorem* 3.1 *and $v_{ij} = 0$ on $\partial\Omega^N$, then*

$$\left| v_{ij} - \left(\frac{h}{\varepsilon}\right) \Psi(x_i, y_j) \right| \le CN^{-2}\, \mathcal{E}_{ij}(1, \bar{\Omega}_0;\ \ln^2 N),$$

where Ψ is from (4.4).

The whole subsection is devoted to the *proof* of this lemma.

Decompose Ψ from (4.4) as $\Psi = \Psi_1 + \Psi_2 + \tilde{\Psi}_1 + \tilde{\Psi}_2$, where

$$\begin{aligned} \Psi_1(x,y) &:= \frac{\varepsilon^{-1}(1-x)b_1^2(1,y)v_0}{2}, \quad & \tilde{\Psi}_1(x,y) &:= \frac{\varepsilon^{-1}(1-x)b_1^2(1,1)z_0}{2}, \\ \Psi_2(x,y) &:= \frac{\varepsilon^{-1}(1-y)b_2^2(x,1)w_0}{2}, \quad & \tilde{\Psi}_2(x,y) &:= \frac{\varepsilon^{-1}(1-y)b_2^2(1,1)z_0}{2}. \end{aligned} \tag{5.8}$$

Regarding the components of this decomposition, see Remark 4.4.

Now decompose v from Lemma 5.9 as $v_{ij} = V_{ij} + W_{ij} + Z_{ij}$, where

$$L^N V = r[v_0] \text{ in } \Omega^N, \qquad \left| V - \left(\frac{h}{\varepsilon}\right) \Psi_1 \right| \le CN^{-2} \text{ on } \partial\Omega^N, \tag{5.9a}$$

$$L^N W = r[w_0] \text{ in } \Omega^N, \qquad \left| W - \left(\frac{h}{\varepsilon}\right) \Psi_2 \right| \le CN^{-2} \text{ on } \partial\Omega^N, \tag{5.9b}$$

$$L^N Z = r[z_0] \text{ in } \Omega^N, \qquad \left| Z - \left(\frac{h}{\varepsilon}\right) (\tilde{\Psi}_1 + \tilde{\Psi}_2) \right| \le CN^{-2} \text{ on } \partial\Omega^N. \tag{5.9c}$$

Note that such a decomposition of the boundary condition $v_{ij} = 0$ on $\partial\Omega^N$ is possible. Indeed, if $x = 1$ or $y = 1$, we have $\Psi_1(x,y) + \tilde{\Psi}_1(x,y) = \Psi_2(x,y) + \tilde{\Psi}_2(x,y) = 0$, while if $x = 0$ or $y = 0$, we have $|\Psi_1| + |\Psi_2| + |\tilde{\Psi}_1| + |\tilde{\Psi}_2| \le C\varepsilon^{-2} \le CN^{-2}$. Hence, $|\Psi_1 + \Psi_2 + \tilde{\Psi}_1 + \tilde{\Psi}_2| \le CN^{-2}$ on $\partial\Omega^N$.

Since $(x_i, y_j) \in \bar{\Omega}_0$ if both $i, j \le N/2$, Lemma 5.9 follows from (5.10):

$$\left| V_{ij} - \left(\frac{h}{\varepsilon}\right) \Psi_1(x_i, y_j) \right| \le CN^{-2}\, \mathcal{E}_{ij} \left(\ln^2 N, i > \frac{N}{2};\ 1 \right), \tag{5.10a}$$

$$\left| W_{ij} - \left(\frac{h}{\varepsilon}\right) \Psi_2(x_i, y_j) \right| \le CN^{-2}\, \mathcal{E}_{ij} \left(\ln^2 N, j > \frac{N}{2};\ 1 \right), \tag{5.10b}$$

$$\left| Z_{ij} - \left(\frac{h}{\varepsilon}\right) [\tilde{\Psi}_1(x_i, y_j) + \tilde{\Psi}_2(x_i, y_j)] \right| \le CN^{-2}\, \mathcal{E}_{ij]} \left(\ln^2 N,\ i, j > \frac{N}{2};\ 1 \right). \tag{5.10c}$$

Further, we shall prove that (5.10) follows from the following two lemmas.

LEMMA 5.10. *For V, W, Z from* (5.9) *we have*

$$|V_{ij}| \le CN^{-2} \quad \textit{for } i \le \frac{N}{2}, \tag{5.11a}$$

$$|W_{ij}| \le CN^{-2} \quad \textit{for } j \le \frac{N}{2}, \tag{5.11b}$$

$$|Z_{ij}| \le CN^{-2} \quad \textit{if} \quad i \le \frac{N}{2} \quad \textit{or} \quad j \le \frac{N}{2}. \tag{5.11c}$$

Proof. We defer the proof of this lemma to Appendix C. ∎

Define the auxiliary discrete functions $\psi_{1,ij}$ for $i = N/2, \dots, N,\ j = 0, \dots, N$; $\psi_{2,ij}$ for $i = 0, \dots, N,\ j = N/2, \dots, N$; and $\tilde\psi_{1,ij},\ \tilde\psi_{2,ij}$ for $i, j = N/2, \dots, N$ as follows. Let them satisfy the discrete equations

$$(L^N \psi_1)_{ij} = \frac{\varepsilon (b_1 v_{0,xx})_{ij}}{2} \quad \text{for } i = \frac{N}{2}+1, \dots, N-1,\ j = 1, \dots, N-1, \tag{5.12a}$$

$$(L^N \psi_2)_{ij} = \frac{\varepsilon (b_2 w_{0,yy})_{ij}}{2} \quad \text{for } i = 1, \dots, N-1,\ j = \frac{N}{2}+1, \dots, N-1, \tag{5.12b}$$

$$(L^N \tilde\psi_1)_{ij} = \frac{\varepsilon (b_1 z_{0,xx})_{ij}}{2} \quad \text{for } i, j = \frac{N}{2}+1, \dots, N-1, \tag{5.12c}$$

$$(L^N \tilde\psi_2)_{ij} = \frac{\varepsilon (b_2 z_{0,yy})_{ij}}{2} \quad \text{for } i, j = \frac{N}{2}+1, \dots, N-1, \tag{5.12d}$$

and the following conditions on the boundaries of the submeshes, where they are defined:

$$\psi_{1,ij} = \Psi_1(x_i, y_j) \quad \text{if } \ i = \frac{N}{2},\ N \quad \text{or } j = 0,\ N, \tag{5.13a}$$

$$\psi_{2,ij} = \Psi_2(x_i, y_j) \quad \text{if } \ i = 0,\ N \quad \text{or } j = \frac{N}{2},\ N, \tag{5.13b}$$

$$\tilde\psi_{k,ij} = \tilde\Psi_k(x_i, y_j) \quad \text{if } \ i = \frac{N}{2},\ N \quad \text{or } j = \frac{N}{2},\ N, \quad k = 1, 2. \tag{5.13c}$$

Lemma 5.11. *For $\psi_1,\ \psi_2,\ \tilde\psi_1,\ \tilde\psi_1$ defined by* (5.12), (5.13) *and $\Psi_1,\ \Psi_2,\ \tilde\Psi_1,\ \tilde\Psi_1$ from* (5.8) *we have*

$$|\psi_{1,ij} - \Psi_1(x_i, y_j)| \le C\left(\frac{h}{\varepsilon}\right) \quad \textit{for } i = \frac{N}{2}+1, \dots, N, \tag{5.14a}$$

$$|\psi_{2,ij} - \Psi_2(x_i, y_j)| \le C\left(\frac{h}{\varepsilon}\right) \quad \textit{for } j = \frac{N}{2}+1, \dots, N, \tag{5.14b}$$

$$|\tilde\psi_{k,ij} - \tilde\Psi_k(x_i, y_j)| \le C\left(\frac{h}{\varepsilon}\right) \quad \textit{for } i, j = \frac{N}{2}+1, \dots, N, \quad k = 1, 2. \tag{5.14c}$$

Proof. This lemma is proved in Appendix C. ∎

Lemma 5.12. *Estimates* (5.10) *follow from Lemmas* 5.10 *and* 5.11.

Proof. To get the statement of this Lemma, it suffices to prove that

(a) estimate (5.10a) follows from (5.11a) and (5.14a),
(b) estimate (5.10b) follows from (5.11b) and (5.14b),
(c) estimate (5.10c) follows from (5.11c) and (5.14c).

(a) By (5.8), (3.3), (4.10), (1.7), we have $|\Psi_1(x_i, y_j)| \le CN^{-2}$ for $i \le N/2$. Combining this with (5.11a), we get (5.10a) for $i \le N/2$. Since we have (5.14a), then to obtain (5.10a) for $i > N/2$, it suffices to prove that

$$\left| V_{ij} - \left(\frac{h}{\varepsilon}\right) \psi_{1,ij} \right| \le C\left(\frac{h}{\varepsilon}\right)^2 \quad \text{for } i > \frac{N}{2}, \tag{5.15}$$

where ψ_1 is defined by (5.12a), (5.13a). Recalling the notation (5.4) and using Taylor series expansions and (3.2), (3.3), for $i > N/2$ we get

$$\left| r_{1,ij}[v_0] - \frac{h(b_1 v_{0,xx})_{ij}}{2} \right| \le Ch^2\varepsilon^{-3}e^{-\beta(1-x_{i+1})/\varepsilon}, \tag{5.16}$$

$$\left| r_{2,ij}[v_0] \right| \le (2\varepsilon + b_{2,ij}N^{-1}) \max_{y\in[0,1]} |v_{0,yy}(x_i, y)| \le CN^{-1}e^{-\beta(1-x_i)/\varepsilon}.$$

Note that, by (1.7), (1.5), we have $N^{-1} \le Ch^2\varepsilon^{-3}$ and $e^{-\beta(1-x_{i+1})/\varepsilon} \le Ce^{-\beta(1-x_i)/\varepsilon}$, while $L^N[V - (h/\varepsilon)\psi_1] = r_1[v_0] + r_2[v_0] - hb_1v_{0,xx}/2$. Hence,

$$\left| L^N \left[V_{ij} - \left(\frac{h}{\varepsilon}\right) \psi_{1,ij} \right] \right| \le C \left(\frac{h}{\varepsilon}\right)^2 \varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon} \quad \text{for } i > \frac{N}{2}.$$

Note that (5.9a), (5.13a) imply $|V - (h/\varepsilon)\psi_1| \le CN^{-2}$ on $\partial\Omega^N$, while (5.10a), (5.13a) imply $|V_{ij} - (h/\varepsilon)\psi_{1,ij}| = |V_{ij} - (h/\varepsilon)\Psi_1(x_i, y_j)| \le CN^{-2}$ for $i = N/2$. Now, applying Lemmas 5.3(iii) and 5.1(iii), we obtain (5.15). This completes part (a) of the proof.

(b) This part of the proof is analogous to part (a).

(c) Since this part of the proof is similar to part (a), we skip certain details. By (5.8), (3.3), (4.10), (1.7), we have $|\tilde\Psi_1(x_i, y_j)| + |\tilde\Psi_2(x_i, y_j)| \le CN^{-2}$ if $i \le N/2$ or $j \le N/2$. Combining this with (5.11c), we get (5.10c) if $i \le N/2$ or $j \le N/2$. Since we have (5.14c), then, to obtain (5.10c) for $i, j > N/2$, it suffices to prove that

$$\left| Z_{ij} - \left(\frac{h}{\varepsilon}\right) (\tilde\psi_{1,ij} + \tilde\psi_{2,ij}) \right| \le C \left(\frac{h}{\varepsilon}\right)^2 \quad \text{for } i, j > \frac{N}{2}. \tag{5.17}$$

Note that $L^N[Z - (h/\varepsilon)(\tilde\psi_1 + \tilde\psi_2)] = (r_1[z_0] - hb_1z_{0,xx}/2) + (r_2[z_0] - hb_2z_{0,yy}/2)$. Hence, using Taylor series expansions and (3.2), (3.3), for $i, j > N/2$ we get

$$\left| \left(r_1[z_0] - \frac{hb_1z_{0,xx}}{2} + r_2[z_0] - \frac{hb_2z_{0,yy}}{2} \right)_{ij} \right| \le Ch^2\varepsilon^{-3}(e^{-\beta(1-x_i)/\varepsilon} + e^{-\beta(1-y_j)/\varepsilon}), \tag{5.18}$$

which yields

$$\left| L^N \left[Z_{ij} - \left(\frac{h}{\varepsilon}\right) (\tilde\psi_{1,ij} + \tilde\psi_{2,ij}) \right] \right| \le C \left(\frac{h}{\varepsilon}\right)^2 \varepsilon^{-1}(e^{-\beta(1-x_i)/\varepsilon} + e^{-\beta(1-y_j)/\varepsilon}).$$

Combining this with the boundary conditions from (5.9c), (5.10c), (5.13c) and applying Lemmas 5.3(iii) and 5.1(iii), we obtain (5.17). This completes the proof. ∎

Proof of Lemma 5.9. By Lemmas 5.10, 5.11, and 5.12, we have (5.10), which yields the statement of Lemma 5.9. ∎

5.6. Contribution of $\varepsilon r[u_1 + v_1 + w_1 + z_1]$. The contribution to the error of this component of the right-hand side in (5.2) is described by the following lemma.

LEMMA 5.13. *If $L^N v_{ij} = \varepsilon r_{ij}[u_1 + v_1 + w_1 + z_1]$ in Ω^N, where u_1, v_1, w_1, z_1 are from Theorem* 3.1, *and $v_{ij} = 0$ on $\partial\Omega^N$, then*

$$|v_{ij}| \le CN^{-2}\, \mathcal{E}_{ij}(1, \bar\Omega_0;\, \ln N).$$

Proof. Since this result is very close to the well-known theorem by Shishkin [13, Theorem 2.3], [8, Theorem 13.2], while the argument is standard, we shall only sketch the proof. Note that it simplifies the argument that the truncation error in the right-hand side is multiplied by ε. By (3.2), (1.5), we have $|\varepsilon r[u_1]| \le C\varepsilon N^{-1} \le CN^{-2}$. By (3.4), (1.7), we get $|\varepsilon r_{1,ij}[v_1 + z_1]| \le C(h/\varepsilon)e^{-\beta(1-x_i)/\varepsilon}$ for $i > N/2$, and $|\varepsilon r_{1,ij}[v_1 + z_1]| \le Ce^{-\beta(1-x_{i+1})/\varepsilon} \le CN^{-2}$ for $i \le N/2$. The term $\varepsilon r_2[w_1 + z_1]$ is estimated similarly. We have to be careful with Taylor series expansions of v_1 and w_1 since $\frac{\partial^k}{\partial x^k} v_1$ and $\frac{\partial^k}{\partial y^k} w_1$ are generally in $C^{1,1}(\bar\Omega)$. By (3.4a) and Remark 3.1, we estimate as follows:

$$\big|r_{2,ij}[v_1]\big| \le C\big\|v_1(x_i,\cdot)\big\|_{1,1,[0,1]} \le Ce^{-\beta(1-x_i)/\varepsilon},$$
$$\big|r_{1,ij}[w_1]\big| \le C\big\|w_1(\cdot,y_j)\big\|_{1,1,[0,1]} \le Ce^{-\beta(1-y_j)/\varepsilon}.$$

Combining our estimates of all the components of the right-hand side and applying Lemmas 5.1(ii) and 5.3(i),(ii), we get the statement of the lemma. ∎

5.7. Proof of Theorem 4.1. The statement of the theorem is obtained by recalling (5.1), (5.2) and combining Corollary 5.6 and Lemmas 5.8, 5.9, 5.13. ∎

6. Numerical results. In this section we present numerical results illustrating our estimates for the Richardson extrapolation technique (Corollary 4.2) and on the errors in approximating the derivatives (Corollary 4.3).

We study the performance of the upwind scheme and the Richardson extrapolation technique when applied to the test problem from [6] in which $b_1 = 2$, $b_2 = 3$, $c = 1$,

$$u(x,y) = 2\sin x\,(1 - e^{-2(1-x)/\varepsilon})\,y^2(1 - e^{-3(1-y)/\varepsilon}),$$

and the right-hand side f is chosen so that (1.1) is satisfied. This problem was solved numerically using the upwind scheme (1.3) on the tensor-product piecewise-uniform Shishkin mesh from Remark 1.1 with $\beta_1 = 1.9$, $\beta_2 = 2.9$.

In Table 6.1 we present the errors before and after the Richardson extrapolation. The odd rows contain the maximum nodal errors $e^N := \|u^N_{ij} - u(x_i, y_j)\|$ in the specified subdomains of $\bar\Omega$, while the even rows contain the rates of convergence computed by the standard formula $r(e^N) = \log_2(e^N/e^{2N})$. Clearly, the Richardson extrapolation technique decreases the nodal errors and increases the rates of convergence. Note that the errors are very similar for $\varepsilon = 10^{-6}$ and $\varepsilon = 10^{-8}$, which confirms that our estimates are ε-uniform. The rates of convergence are slightly worse than predicted by Corollary 4.2. However, since our rates of convergence are consistent with those for the analogous one-dimensional problems [9, 4], we expect the rates of convergence to increase as N increases, similarly to [9, 4].

TABLE 6.1
Maximum nodal errors before and after Richardson extrapolation.

	$\varepsilon = 10^{-6}$				$\varepsilon = 10^{-8}$			
	Before extrapolation		After extrapolation		Before extrapolation		After extrapolation	
N	$\bar\Omega_0$	$\bar\Omega\backslash\bar\Omega_0$	$\bar\Omega_0$	$\bar\Omega\backslash\bar\Omega_0$	$\bar\Omega_0$	$\bar\Omega\backslash\bar\Omega_0$	$\bar\Omega_0$	$\bar\Omega\backslash\bar\Omega_0$
32	4.944e-2	1.430e-1	1.069e-3	1.404e-2	4.944e-2	1.430e-1	1.069e-3	1.404e-2
	0.901	0.623	1.727	1.265	0.901	0.623	1.727	1.265
64	2.649e-2	9.288e-2	3.230e-4	5.842e-3	2.649e-2	9.288e-2	3.229e-4	5.842e-3
	0.944	0.690	1.782	1.412	0.944	0.690	1.782	1.412
128	1.377e-2	5.759e-2	9.388e-5	2.195e-3	1.377e-2	5.759e-2	9.391e-5	2.195e-3
	0.978	0.748	1.832	1.517	0.978	0.748	1.832	1.517
256	6.990e-3	3.429e-2	2.638e-5	7.669e-4	6.990e-3	3.429e-2	2.638e-5	7.669e-4
	0.991	0.790			0.991	0.790		
512	3.518e-3	1.984e-2			3.518e-3	1.984e-2		

Table 6.2
Maximum nodal errors in approximating the derivatives.

	$\|D_x^- u^N - u_x\|$				$\|D_y^- u^N - u_y\|$			
	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$	
N	$\bar{\Omega}_0$	Ω_2	$\bar{\Omega}_0$	Ω_2	$\bar{\Omega}_0$	Ω_1	$\bar{\Omega}_0$	Ω_1
64	3.841e-2	8.811e-2	3.841e-2	8.811e-2	5.199e-2	1.819e-1	5.199e-2	1.819e-1
	0.938	0.711	0.938	0.711	1.001	0.811	1.001	0.811
128	2.005e-2	5.384e-2	2.005e-2	5.384e-2	2.598e-2	1.037e-1	2.598e-2	1.037e-1
	0.961	0.764	0.961	0.764	0.991	0.824	0.991	0.824
256	1.030e-2	3.171e-2	1.030e-2	3.171e-2	1.307e-2	5.856e-2	1.307e-2	5.856e-2
	0.974	0.805	0.974	0.805	0.996	0.838	0.996	0.838
512	5.241e-3	1.815e-2	5.241e-3	1.815e-2	6.554e-3	3.277e-2	6.554e-3	3.277e-2

Table 6.3
Maximum nodal errors in approximating ε-weighted derivatives.

	$\varepsilon \|D_x^- u^N - u_x\|$				$\varepsilon\|D_y^- u^N - u_y\|$			
	$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$	
N	Ω_1	Ω_{12}	Ω_1	Ω_{12}	Ω_2	Ω_{12}	Ω_2	Ω_{12}
64	2.524e-1	3.115e-1	2.524e-1	3.115e-1	3.739e-1	4.661e-1	3.739e-1	4.661e-1
	0.475	0.517	0.475	0.517	0.479	0.521	0.479	0.521
128	1.816e-1	2.176e-1	1.816e-1	2.176e-1	2.684e-1	3.248e-1	2.684e-1	3.248e-1
	0.616	0.641	0.616	0.641	0.618	0.644	0.618	0.644
256	1.185e-1	1.395e-1	1.185e-1	1.395e-1	1.749e-1	2.079e-1	1.749e-1	2.079e-1
	0.712	0.728	0.713	0.728	0.714	0.729	0.714	0.729
512	7.233e-2	8.427e-2	7.233e-2	8.427e-2	1.066e-1	1.254e-1	1.066e-1	1.254e-1

Tables 6.2 and 6.3 are clear illustrations of Corollary 4.2. In these tables we present the maximum nodal errors in approximating the derivatives and their rates of convergence computed as in Table 6.1.

In summary, our numerical results confirm our theoretical results.

Appendix A. Proof of Lemmas 5.2, 5.3, and 5.4 from subsection 5.1. If the conditions of Lemma 5.1(i) are satisfied, we say that B_{ij} is a *barrier function* for v_{ij}. Define the auxiliary discrete functions

$$B_i := \begin{cases} 2\left(1+\dfrac{\alpha h}{\varepsilon}\right)^{-N/2}\left(1+\dfrac{\alpha H}{\varepsilon}\right)^{-(N/2-i)}, & i = 0,\dots,\dfrac{N}{2}, \\ \left(1+\dfrac{\alpha h}{\varepsilon}\right)^{-(N-i)} + \left(1+\dfrac{\alpha h}{\varepsilon}\right)^{-N/2}, & i = \dfrac{N}{2},\dots,N, \end{cases} \tag{A.1}$$

$$\bar{B}_i := \begin{cases} 2\left(\dfrac{\varepsilon}{\beta}\right)\left(1+\dfrac{\beta H}{\varepsilon}\right)^{-(N/2-i)}, & i = 0,\dots,\dfrac{N}{2}, \\ 2\left(\dfrac{\varepsilon}{\beta}\right) + \sigma - (N-i)h, & i = \dfrac{N}{2},\dots,N. \end{cases} \tag{A.2}$$

It is assumed here that $\{x_i\}_{i=0}^N$ are the nodes of the mesh (1.4), (1.6). Furthermore, we shall use B_i and $\bar{B}_i$ normalized in different manners as discrete barrier functions.

Lemma A.1. *For any positive α the discrete function B_i from* (A.1) *is such that* $e^{-\alpha(1-x_i)/\varepsilon} < B_i \le C\mathcal{E}(N^{-2\alpha/\beta}, i \le N/2;\ 1)$ *and* $(-\varepsilon\delta_x^2 + \alpha D_x^-)B_i \ge 0$.

Proof. The lower bound for B_i follows from the inequality $e^{-t} \le (1+t)^{-1}$, which holds true for $t \ge 0$, with $t := \alpha h_i/\varepsilon$. The upper bound for B_i is obvious for $i > N/2$. For $i \le N/2$, it follows from $(1+t)^{-1} \le e^{-t+t^2}$, which we have for $t > 0$. Setting $t := \alpha h/\varepsilon$, we get $B_i \le 2(1+\alpha h/\varepsilon)^{-N/2} \le 2e^{-\alpha\sigma/\varepsilon + (\alpha h/\varepsilon)^2 N/2}$. Further, (1.7) implies $e^{-\alpha\sigma/\varepsilon} \le N^{-2\alpha/\beta}$ and $e^{(\alpha h/\varepsilon)^2 N/2} \le e$. This proves the upper bound for B_i.

The second inequality is checked using (1.4) and (2.1). In fact, $(-\varepsilon\delta_x^2+\alpha D_x^-)B_i = 0$ for $i \neq N/2$ and $(-\varepsilon\delta_x^2+\alpha D_x^-)B_i > 0$ for $i = N/2$. ∎

Proof of Lemma 5.2. Use B_i from Lemma A.1 with $\alpha := \beta$ as a barrier function for v_{ij}. Note that $L^N B_i \geq (b_{1,ij}-\beta)D_x^- B_i \geq 0$. ∎

LEMMA A.2. *The discrete function B_i from* (A.1) *with $\alpha := \beta/2$ is such that $B_i \leq C\mathcal{E}(N^{-1}, i \leq N/2;\ 1)$ and $L^N B_i \geq Ce^{-\beta(1-x_i)/\varepsilon}\mathcal{E}(N, i \leq N/2;\ \varepsilon^{-1})$.*

Proof. This lemma follows from Lemma A.1. The first property is obvious. To prove the second, note that $L^N B_i \geq (b_{1,ij}-\beta/2)D_x^- B_i \geq (\beta/2)D_x^- B_i$. By (2.1), (1.5), (1.4), calculations show that $D_x^- B_i = (h_i + 2\varepsilon/\beta)^{-1}B_i$ and $(h_i+2\varepsilon/\beta)^{-1} \geq C\mathcal{E}(N, i \leq N/2;\ \varepsilon^{-1})$. Recalling that $B_i > e^{-(\beta/2)(1-x_i/\varepsilon)} \geq e^{-\beta(1-x_i/\varepsilon)}$, we complete the proof. ∎

Proof of Lemma 5.3. This lemma follows from Lemma A.2.

(i) By (1.5), use $CN^{-1}B_i$ as a barrier function for v_{ij}.

(ii), (iii) Use CB_i as a barrier function for v_{ij}. ∎

Proof of Lemma 5.4. By (1.4), (1.6), for the discrete function $\bar{B}_i$ defined in (A.2) we have $0 < \bar{B}_i \leq C\varepsilon\ \mathcal{E}_{ij}(1, i \leq N/2;\ \ln N)$. Combining this with the analogous estimate for $\bar{B}_j$ and (1.5), we get

$$0 < \bar{B}_i + \bar{B}_j \leq C\varepsilon\ \mathcal{E}_{ij}(1, \bar{\Omega}_0;\ \ln N) \leq CN^{-1}\ \mathcal{E}_{ij}(1, \bar{\Omega}_0;\ \ln N).$$

By (2.1), (1.4), calculations show that $D_x^-\bar{B}_i = 1$ for $i > N/2$, while $D_x^-\bar{B}_i \geq 0$ for $i \leq N/2$. In particular, $D_x^-\bar{B}_{N/2} = 2(1+\beta H/\varepsilon)^{-1}$. Further, $L^N\bar{B}_i \geq b_{1,ij} \geq \beta$ for $i > N/2$, while $L^N\bar{B}_i \geq (-\varepsilon\delta_x^2+\beta D_x^-)\bar{B}_i = 0$ for $i < N/2$. For $i = N/2$ we also have $L^N\bar{B}_i \geq 0$, which follows from

$$L^N\bar{B}_i \geq \left(-\varepsilon\delta_x^2+\beta D_x^-\right)\bar{B}_i = \left[\beta + 2\varepsilon(h+H)^{-1}\right]D_x^-\bar{B}_i - \left[2\varepsilon(h+H)^{-1}\right]D_x^-\bar{B}_{i+1},$$

where $i = N/2$. These imply that $L^N\bar{B}_i \geq \beta\ \mathcal{E}_{ij}(0, i \leq N/2;\ 1)$. Combining this estimate with its analogue for $L^N\bar{B}_j$, we obtain

$$L^N(\bar{B}_i + \bar{B}_j) \geq \beta\ \mathcal{E}_{ij}(0, \bar{\Omega}_0;\ 1).$$

Hence, $(B_i + B_j)/\beta$ is a barrier function for v_{ij}. ∎

Appendix B. Proof of Lemma 5.7(ii).

Proof. Note that (4.3) implies that $\Phi(x,y) = 0$ if $x = 1$ or $y = 1$. Further, $|\Phi(x,y)| \leq C\varepsilon \leq CN^{-1}$ on $\partial\Omega$. Hence,

$$|w_{ij} - \Phi_{ij}| \leq CN^{-1} \quad \text{on } \partial\Omega^N. \tag{B.1}$$

To study $L^N(w-\Phi)$, note that, by (4.7), (5.6), we have $L^N w_{ij} = (b_1\varphi_x + b_2\varphi_y + c\varphi)_{ij}$. Hence

$$L^N(w-\Phi) = (b_1\varphi_x + b_2\varphi_y + c\varphi - L^N\varphi) + L^N(\varphi - \Phi). \tag{B.2}$$

Using Taylor series expansions, (1.5), and (4.2), which was proved in Lemma 5.7(i), we obtain for the first term on the right-hand side that

$$|(b_1\varphi_x + b_2\varphi_y + c\varphi)_{ij} - L^N\varphi_{ij}| \leq C(N^{-1}+\varepsilon)\|\varphi\|_{1,1} \leq CN^{-1}. \tag{B.3}$$

To estimate $L^N(\varphi - \Phi)$, we define

$$\Phi_1(x,y) := \varphi(1,y)e^{-b_1(1,y)(1-x)/\varepsilon}, \qquad \Phi_2(x,y) := \varphi(x,1)e^{-b_2(x,1)(1-y)/\varepsilon},$$
$$\Phi_{12}(x,y) := \varphi(1,1)e^{-b_1(1,1)(1-x)/\varepsilon - b_2(1,1)(1-y)/\varepsilon},$$

so that $\varphi - \Phi = \Phi_1 + \Phi_2 - \Phi_{12}$. Thus, recalling the notation (5.3), we have

$$L^N(\varphi - \Phi) = L_1^N(\Phi_1 - \Phi_{12}) + L_2^N(\Phi_2 - \Phi_{12}) + L_2^N\Phi_1 + L_1^N\Phi_2 + c(\Phi_1 + \Phi_2 - \Phi_{12}). \tag{B.4}$$

For $i \le N/2$, using (1.3), (2.1), (1.4), and (1.7), we obtain

$$|L_1^N(\Phi_1 - \Phi_{12})_{ij}| \le CNe^{-\beta(1-x_{i+1})/\varepsilon} \le CNe^{-\beta(\sigma-h)/\varepsilon} \le CN^{-1}. \tag{B.5}$$

Consider $i > N/2$. First note that

$$-\varepsilon\Phi_{1,xx} + b_1(1,y)\Phi_{1,x} = 0, \qquad -\varepsilon\Phi_{12,xx} + b_1(1,1)\Phi_{12,x} = 0,$$

while the left-hand sides here are slightly different from $L_1\Phi_1$ and $L_1\Phi_{12}$. Hence,

$$L_1^N(\Phi_1 - \Phi_{12})_{ij} = (L_1^N - L_1)(\Phi_1 - \Phi_{12})_{ij} + [b_1(x_i,y_j) - b_1(1,y_j)]\Phi_{1,x}(x_i,y_j) - [b_1(x_i,y_j) - b_1(1,1)]\Phi_{12,x}(x_i,y_j).$$

Using Taylor series expansions to estimate the first term on the right-hand side, and the inequalities $|b_1(x,y) - b_1(1,y)| \le C(1-x)$ and $|b_1(x,y) - b_1(1,1)| \le C[(1-x) + (1-y)]$ combined with (4.10) to estimate the other terms, we obtain

$$|L_1^N(\Phi_1 - \Phi_{12})| \le C(h\varepsilon^{-2}e^{-\beta(1-x_{i+1})/\varepsilon} + e^{-\beta(1-x_i)/\varepsilon}) \le C(h\varepsilon^{-2}e^{\beta h/\varepsilon} + 1)e^{-\beta(1-x_i)/\varepsilon}.$$

Combining this with (B.5) and noting that, by (1.7), (1.5), $h\,\varepsilon^{-2} \ge C$ and $e^{\beta h/\varepsilon} \le C$, we get

$$|L_1^N(\Phi_1 - \Phi_{12})_{ij}| \le C\left[\left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}\left(\varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon}, i > \frac{N}{2};\ 0\right) + N^{-1}\right]. \tag{B.6}$$

Furthermore, one can easily see that

$$|L_2^N\Phi_{1,ij}| \le C\|\varphi\|_{1,1}e^{-\beta(1-x_i)/\varepsilon} \le Ce^{-\beta(1-x_i)/\varepsilon}. \tag{B.7}$$

Combining (B.4) with (B.6), (B.7), and their analogues for $L_2^N(\Phi_2 - \Phi_{12})$ and $L_1^N\Phi_2$, and then with (B.2), (B.3), we finally get the estimate

$$\begin{aligned}|L^N(w - \Phi)_{ij}| \le C\Bigg[&\left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}\left(\varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon}, i > \frac{N}{2};\ 0\right)\\ &+ \left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}\left(\varepsilon^{-1}e^{-\beta(1-y_j)/\varepsilon}, j > \frac{N}{2};\ 0\right)\\ &+ e^{-\beta(1-x_i)/\varepsilon} + e^{-\beta(1-y_j)/\varepsilon} + N^{-1}\Bigg].\end{aligned}$$

Combining this with (B.1) and applying Lemmas 5.1(ii),(iii) and 5.3(i),(ii), we obtain

$$\begin{aligned}|w_{ij} - \Phi_{ij}| &\le C\left[\left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}\left(N^{-1}, i \le \frac{N}{2};\ 1\right) + \left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}\left(N^{-1}, j \le \frac{N}{2};\ 1\right) + N^{-1}\right]\\ &\le C\left[\left(\frac{h}{\varepsilon}\right)\mathcal{E}_{ij}(N^{-1}, \bar{\Omega}_0;\ 1) + N^{-1}\right].\end{aligned}$$

By (1.7), this yields the statement of Lemma 5.7(ii). ◻

Appendix C. Proof of Lemmas 5.10 and 5.11.

Proof of Lemma 5.10. (a) Obviously,

$$|V| \le |V + v_0| + |v_0|, \tag{C.1}$$

where v_0 is defined in (3.3). Since $r[v_0] = -L^N v_0 + Lv_0$, we have $L^N[V + v_0] = Lv_0$. One can easily check that $-\varepsilon v_{0,xx} + b_1(1, y)v_{0,x} = 0$ holds true and implies that $L_1 v_0 = [b_1(x, y) - b_1(1, y)]v_{0,x}$. Combining this with $|b_1(x, y) - b_1(1, y)| \le C(1 - x)$ and (4.10), we get $|L_1 v_0| \le Ce^{-\beta(1-x)/\varepsilon}$, while $|(L_2 + c)v_0| \le Ce^{-\beta(1-x)/\varepsilon}$. Hence, $|Lv_0| \le Ce^{-\beta(1-x)/\varepsilon}$, which yields

$$\left|L^N[V + v_0]_{ij}\right| \le Ce^{-\beta(1-x_i)/\varepsilon} \quad \text{in } \Omega^N.$$

Combining this with the boundary condition

$$|(V + v_0)_{ij}| \le \left(\frac{h}{\varepsilon}\right) |\Psi_{1,ij}| + CN^{-2} + |v_{0,ij}| \le C(e^{-\beta(1-x_i)/\varepsilon} + N^{-2}) \quad \text{on } \partial\Omega^N,$$

and applying Lemmas 5.1(ii), 5.2, 5.3(i), we get $|(V + v_0)_{ij}| \le CN^{-2}$ for $i \le N/2$. Combining this with (C.1), (3.3), and (1.7), we complete part (a) of the proof.

(b) This part of the proof is analogous to part (a).

(c) Since this part of the proof is similar to part (a), we skip certain details. Again, we have $|Z| \le |Z + z_0| + |z_0|$, where z_0 is defined in (3.3), which implies $L^N[Z + z_0] = Lz_0$. Further, $-\varepsilon z_{0,xx} + b_1(1, 1)z_{0,x} = 0$ and $-\varepsilon z_{0,yy} + b_2(1, 1)z_0 = 0$ imply $Lz_0 = [b_1(x, y) - b_1(1, 1)]z_{0,x} + [b_2(x, y) - b_2(1, 1)]z_{0,y} + cz_0$. By (3.3), this yields $|Lz_0| \le Ce^{-\beta[(1-x)+(1-y)]/\varepsilon}$. Hence, $\left|L^N[Z + z_0]_{ij}\right| \le Ce^{-\beta[(1-x_i)+(1-y_j)]/\varepsilon}$ in Ω^N, while $|(Z + z_0)_{ij}| \le C(e^{-\beta[(1-x_i)+(1-y_j)]/\varepsilon} + N^{-2})$ on $\partial\Omega^N$. Applying Lemmas 5.1(ii), 5.2, 5.3(i), we get $|(Z + z_0)_{ij}| \le CN^{-2}$ for $i \le N/2$, and $|(Z + z_0)_{ij}| \le CN^{-2}$ for $j \le N/2$. Combining these two estimates, we proceed similarly to part (a). □

Proof of Lemma 5.11. (a) By (5.13a), we have $\psi_1 - \Psi_1 = 0$ on the boundary of the submesh $\{(x_i, y_j) : i = N/2, \dots, N,\ j = 0, \dots, N\}$ where ψ_1 is defined.

In this part of the proof we consider only $i > N/2$. Recalling the notation (5.3), we introduce the following decomposition:

$$L^N(\psi_1 - \Psi_1) = (L^N\psi_1 - L_1\Psi_1) - (L_1^N\Psi_1 - L_1\Psi_1) - (L_2^N + c)\Psi_1.$$

Using Taylor series expansions and (5.8), (3.3), (4.10), we have

$$|L_1^N\Psi_1 - L_1\Psi_1| \le Ch\,\varepsilon^{-2}e^{-\beta(1-x_{i+1})/\varepsilon}, \qquad |(L_2^N + c)\Psi_1| \le Ce^{-\beta(1-x_i)/\varepsilon}.$$

In addition, we claim that

$$|L^N\psi_{1,ij} - (L_1\Psi_1)_{ij}| \le Ce^{-\beta(1-x_i)/\varepsilon}. \tag{C.2}$$

Since (1.5), (1.7) imply that $h\,\varepsilon^{-2} \ge C$ and $e^{-\beta(1-x_{i+1})/\varepsilon} \le Ce^{-\beta(1-x_i)/\varepsilon}$, we have

$$|L^N(\psi_1 - \Psi_1)_{ij}| \le C(h/\varepsilon)\,\varepsilon^{-1}e^{-\beta(1-x_i)/\varepsilon}.$$

Further, by Lemmas 5.1(iii) and 5.3(iii), we get $|\psi_{1,ij} - \Psi_1(x_i, y_j)| \le C(h/\varepsilon + N^{-2})$, which yields statement (a) of the lemma.

To prove our claim (C.2), it suffices to check that

$$\left|\frac{b_1(x, y)\varepsilon v_{0,xx}}{2} - L_1\Psi_1\right| \le Ce^{-\beta(1-x)/\varepsilon}. \tag{C.3}$$

By Remark 4.4, we have $-\varepsilon\Psi_{1,xx} + b_1(1,y)\Psi_{1,x} = b_1(1,y)\varepsilon v_{0,xx}/2$, which implies

$$L_1\Psi_1 = \frac{b_1(x,y)\varepsilon v_{0,xx}}{2} + [b_1(x,y) - b_1(1,y)]\left(\frac{\varepsilon v_{0,xx}}{2} - v_{0,x}\right).$$

Furthermore, using (3.3), (4.10), and $|b_1(x,y) - b_1(1,y)| \le C(1-x)$, we obtain (C.3) and thus complete part (a) of the proof.

(c) This part of the proof is slightly different from part (a); namely, we have to estimate $L_2^N\tilde{\Psi}_1$ more carefully. Note that we consider only $i, j > N/2$ in part (c). Using the notation (5.4), we have $L_2^N\tilde{\Psi}_1 = -r_2[\tilde{\Psi}_1] - L_2\tilde{\Psi}_1$. Further, (5.8), (3.3), (4.10) imply that $|L_2\tilde{\Psi}_1| \le Ce^{-\beta(1-x)/\varepsilon}$ and $|r_{2,ij}[\tilde{\Psi}_1]| \le Ch\,\varepsilon^{-2}e^{-\beta(1-y_{j+1})/\varepsilon}$. Combining these two estimates, we proceed as in part (a).

(b), (d) These parts of the proof are analogous to parts (a) and (c), respectively. □

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