Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 201 (2008) 465–480
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

On Richardson extrapolation for fitted operator finite difference methods

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Abstract

Recently, there has been a great interest towards the higher order methods for singularly perturbed problems. As compared to their lower order counterparts, they provide better accuracy with fewer mesh points. Construction and/or implementation of direct higher order methods is usually very complicated. Thus a natural choice is to use some convergence acceleration techniques, e.g. Richardson extrapolation. However, as we see in this article, such techniques do not perform equally well on all type of methods. To investigate this, we consider two fitted operator finite difference methods (FOFDMs) developed by Patidar [K.C. Patidar, High order fitted operator numerical method for self-adjoint singular perturbation problems Appl. Math. Comput. 171(1) (2005) 547–566] and Lubuma and Patidar [J. Lubuma, K.C. Patidar, Uniformly convergent non-standard finite difference methods for self-adjoint singular perturbation problems, J. Comput. Appl. Math. 191 (2006) 228–238], referred to as FOFDM-I and FOFDM-II, respectively. The FOFDM-I is fourth and second order accurate for moderate and smaller values of ε, respectively. Unfortunately, Richardson extrapolation does not improve the order of this method. The FOFDM-II is second order uniformly convergent and we show that its order can be improved up to four by using Richardson extrapolation. Both the methods are analyzed for convergence and comparative numerical results supporting theoretical estimates are provided.

Keywords: Singular perturbations; Boundary value problems; Fitted operator methods; Richardson extrapolation

1. Introduction

The main aim of this paper is to investigate the performance of Richardson extrapolation when applied to various FOFDMs for Singular Perturbation Problems (SPPs).

It is known that the solutions of SPPs have large gradients when the singular perturbation parameter ε approaches to zero. In such limiting cases, boundary/interior layers are developed. The layer behavior of the solution lowers the order of convergence of the underlying numerical method. Standard methods have failed to resolve these problems unless a very fine mesh is considered, which unfortunately raises the

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[1] The research of this author was supported by the South African National Research Foundation.

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doi:10.1016/j.amc.2007.12.035

computational complexities. To overcome this difficulty, fitted mesh methods have been considered by various authors (see, e.g. [9,15,16,19]) since they provide reliable numerical results on a mesh with a reasonable number of grid points and hence make the method practically applicable. However, there are certain limitations of these fitted mesh methods (when one intend to design a direct higher order method) and therefore we consider in this paper the fitted operator type of methods.

Direct techniques to obtain high order methods for singularly perturbed problems are well documented. We provide here some of those works.

Gartland [3] examined a one-dimensional convection–diffusion problem where he constructed a scheme of order p ($p = 1, 2, 3, 4$) using collocation approach whereas a fourth order uniformly convergent scheme for a reaction–diffusion problem was presented in [7] where the Hermitian approximation of the second order derivative was used.

For a self-adjoint problem, O'Riordan and Stynes [17] gave a method using finite elements with uniform mesh. This method is second order accurate in L^∞-norm. In [18] a fitted operator finite difference method (FOFDM) was derived via Numerov's method and shown to be fourth order accurate for moderate value of ε and second order accurate for very small values of ε. On the other hand, in [19] a fitted mesh finite difference method (FMFDM) was shown to be fourth order ε-uniformly convergent.

On the other hand, Vulanovic presented a second order ε-uniformly convergent method in [24] for a non-linear problem. The same author gave a third order method for quasilinear problems in [25,26] whereas Wang [27] achieved third and fourth order convergence for a non-linear problem.

While none of the methods above is of order higher than four, there exist methods of arbitrary order (see, e.g. [5]) for certain class of problems.

Since the aim is to achieve a better accuracy, one would rather use a convergence acceleration strategy than any of the direct methods (which are tedious in most cases). Several methods for improving the accuracy have been designed in the past (see, e.g. [1,4,8,20–22] and the references therein). One of these convergence acceleration techniques (presented in [8]) was subsequently termed as Richardson extrapolation. It is a postprocessing procedure where a linear combination of two computed solutions approximating a particular quantity gives a third and better approximation ([15]). It was implemented in [10] for a system of first order linear ordinary differential equation, in [11,15] for a one-dimensional linear convection–diffusion problem, and in [22] for a quasilinear parabolic singularly perturbed convection–diffusion equations.

In this paper, we consider two FOFDMs for the solution of the self-adjoint problem

$$Ly \equiv -\varepsilon(a(x)y')' + b(x)y = f(x), \quad x \in [0,1], \quad y(0) = \eta_0, \quad y(1) = \eta_1, \tag{1.1}$$

where η_0 and η_1 are given constants and $\varepsilon \in (0,1]$. The functions $f(x)$, $a(x)$ and $b(x)$ are assumed to be sufficiently smooth that satisfy the conditions

$$a(x) \geqslant a > 0, \quad b(x) \geqslant b > 0.$$

The existence and uniqueness of solution of the above problem can be obtained by using the following two results (both of which are proved in Patidar [18]):

Lemma 1.1. *Let $\Psi(x)$ be any sufficiently smooth function that satisfies $\Psi(0) \geqslant 0$ and $\Psi(1) \geqslant 0$. Then $L\Psi(x) \geqslant 0$ for all $x \in (0,1)$ implies that $\Psi(x) \geqslant 0$ for all $x \in [0,1]$.*

Lemma 1.2. *Let $y(x)$ be the solution of the problem (1.1), then we have*

$$||y|| \leqslant b^{-1}||f|| + \max(\eta_0, \eta_1),$$

where $\|\cdot\|$ is the usual maximum norm.

The rest of the paper is organized as follows. We present two FOFDMs in Section 2 which are analyzed in Section 3. Comparative numerical results (before and after extrapolation) for these two methods are presented in Section 4. Finally, we conclude the paper in Section 5.

2. Numerical methods

Now, let n be a positive integer. Consider the following partition of the interval $[0,1]$:

$$x_0 = 0, \quad x_j = x_0 + jh, \quad j = 1(1)n, \quad h = x_j - x_{j-1}, \quad x_n = 1.$$

We denote the above mesh by μ_n whereas the mesh μ_{2n} is obtained by bisecting each mesh interval in μ_n, i.e.

$$\mu_{2n} = \{\tilde{x}_j\} \text{ with } \tilde{x}_0 = 0, \quad \tilde{x}_n = 1 \quad \text{and} \quad \tilde{x}_j - \tilde{x}_{j-1} = \tilde{h} = h/2, \quad j = 1(1)2n.$$

These two meshes will be used to derive the extrapolation formulae in the next section. Furthermore, we use the notations $V_j = V(x_j)$, $W_j = W(x_j)$ and $Z_j = Z(x_j)$ and we denote the approximations of V_j at the grid points x_j by the unknowns v_j.

Below we discuss two fitted operator finite difference methods (FOFDMs). To avoid notational complexities, we use the same notation A, v and F, for each of these two methods, to denote the matrix, the vector of unknowns and the right hand side vector, respectively.

The two FOFDMs, to be described below, are constructed by reducing the self-adjoint problem (1.1) into its normal form (note that, when the coefficient function $a(x)$ is constant, the two forms are the same), viz.

$$\widetilde{L}V \equiv -\varepsilon V'' + W(x)V = Z(x), \quad V(0) = \alpha_0\left(\equiv \frac{y(0)}{U(0)}\right), \quad V(1) = \alpha_1\left(\equiv \frac{y(1)}{U(1)}\right), \tag{2.1}$$

where

$$y(x) = U(x)V(x), \quad U(x) := \exp\left(-\frac{1}{2}\int_0^x P(\zeta)\mathrm{d}\zeta\right). \tag{2.2}$$

Here

$$P(x) = \frac{a'(x)}{a(x)}, \quad Q(x) = -\frac{b(x)}{\varepsilon a(x)}, \quad R(x) = -\frac{f(x)}{\varepsilon a(x)},$$
$$W(x) = -\varepsilon\left(Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}(P(x))^2\right) \quad \text{and} \quad Z(x) = -\varepsilon\left(R(x)\exp\left(\frac{1}{2}\int_0^x P(\zeta)\mathrm{d}\zeta\right)\right).$$

2.1. FOFDM-I

Using the theory of inverse monotone matrices, Patidar [18] designed a high order FOFDM to solve (1.1) via (2.1) and (2.2) as follows:

He defined the fitting comparison problem associated with (2.1) by

$$-\sigma(x,\varepsilon)V'' + W(x)V = Z(x), \quad V(0) = \alpha_0, \quad V(1) = \alpha_1, \tag{2.3}$$

where $\sigma(x,\varepsilon)$ is a fitting factor. Then the approximate solution of the problem (2.3) is sought by the Numerov's method:

$$-\left[\sigma_j^- - \frac{h^2}{12}W_{j-1}\right]v_{j-1} + \left[2\sigma_j^c + \frac{5h^2}{6}W_j\right]v_j - \left[\sigma_j^+ - \frac{h^2}{12}W_{j+1}\right]v_{j+1} = \frac{h^2}{12}[Z_{j-1} + 10Z_j + Z_{j+1}], \tag{2.4}$$

where $\sigma_j^{\pm}$ and σ_j^c are given by

$$\sigma_j^{\pm} = \frac{h^2 W_{j\pm1}}{12}\left(1 + \frac{3}{\sinh^2\left(\frac{\rho_j h}{2}\right)}\right) \quad \text{and} \quad \sigma_j^c = \frac{h^2 W_j}{12}\left(1 + \frac{3}{\sinh^2\left(\frac{\rho_j h}{2}\right)}\right). \tag{2.5}$$

In matrix notation, the scheme (2.4) can be written as the following tridiagonal system:

$$Av = F. \tag{2.6}$$

The entries corresponding to A and F in this case are

$$A_{ij} = r_j^-, \quad i = j+1;\ j = 1,2,\ldots,n-2;$$
$$A_{ij} = r_j^c, \quad i = j;\ j = 1,2,\ldots,n-1;$$
$$A_{ij} = r_j^+, \quad i = j-1;\ j = 2,3,\ldots,n-1;$$
$$F_1 = q_1^- Z_0 + q_1^c Z_1 + q_1^+ Z_2 - r_1^- \alpha_0,$$
$$F_j = q_j^- Z_{j-1} + q_j^c Z_j + q_j^+ + Z_{j+1}, \quad j = 2,3,\ldots,n-2,$$
$$F_{n-1} = q_{n-1}^- Z_{n-2} + q_{n-1}^c Z_j + q_{n-1}^+ Z_n - r_{n-1}^+ \alpha_1,$$

where

$$r_j^- = -\left[\sigma_j^- - \frac{h^2}{12} W_{j-1}\right], \quad r_j^c = \left[2\sigma_j^c + \frac{5h^2}{6} W_j\right], \quad r_j^+ = -\left[\sigma_j^+ - \frac{h^2}{12} W_{j+1}\right], \quad q_j^- = q_j^+ = \frac{h^2}{12},$$
$$q_j^c = \frac{5h^2}{6}; \quad j = 1,2,\ldots,n-1. \tag{2.7}$$

2.2. FOFDM-II

Subsequent to Patidar [18], Lubuma and Patidar [12] developed the following FOFDM (using the non-standard finite difference modeling rules of Mickens [14]) to solve (2.1):

$$-\varepsilon \frac{v_{j-1} - 2v_j + v_{j+1}}{\tilde{\phi}_j^2} + \widetilde{W}_j v_j = Z_j, \tag{2.8}$$

where

$$\widetilde{W}_j = \frac{W_{j-1} + W_j + W_{j+1}}{3}, \quad \tilde{\rho}_j = \sqrt{\frac{\widetilde{W}_j}{\varepsilon}} \quad \text{and} \quad \tilde{\phi}_j \equiv \frac{2}{\tilde{\rho}_j} \sinh\left(\frac{\tilde{\rho}_j h}{2}\right).$$

This leads to a tridiagonal system of linear equations

$$Av = F. \tag{2.9}$$

Corresponding entries of A and F in this case are

$$A_{ij} = r_j^-, \quad i = j+1;\ j = 1,2,\ldots,n-2,$$
$$A_{ij} = r_j^c, \quad i = j;\ j = 1,2,\ldots,n-1,$$
$$A_{ij} = r_j^+, \quad i = j-1;\ j = 2,3,\ldots,n-1,$$
$$F_1 = Z_1 - r_1^- \alpha_0, \quad F_{n-1} = Z_{n-1} - r_{n-1}^+ \alpha_1,$$
$$F_j = Z_j, \quad j = 2,3,\ldots,n-2,$$

where

$$r_j^- = -\frac{\varepsilon}{\tilde{\phi}_j^2}, \quad r_j^+ = -\frac{\varepsilon}{\tilde{\phi}_j^2} \quad \text{and} \quad r_j^c = \frac{2\varepsilon}{\tilde{\phi}_j^2} + \widetilde{W}_j. \tag{2.10}$$

Remark 2.1. In what follows, we will use the Richardson extrapolation procedure which is a convergence acceleration technique where a linear combination of two computed solutions approximating a particular quantity gives a third and better approximation. These two solutions are calculated on two different but nested meshes. As indicated in [22], this method is used to increase the accuracy of computed approximations of the solutions of classical boundary value problems (see [13]) and to improve the ε-uniform rates of convergence of computed solutions for linear singularly perturbed problems (see [6]).

We analyze some of the above FOFDMs in next section whereas the comparative numerical results obtained via these methods are presented in Section 4.

3. Analysis of the numerical methods

FOFDM-I was analyzed for convergence (before extrapolation) in [18]. Here we provide additional analysis, that is, the one after the extrapolation. Regarding FOFDM-II, we revisit the analysis (before extrapolation) presented in [12] and then present the analysis after extrapolation.

The analysis for each of the methods is divided into three parts. Firstly, we provide the error estimates where the approximate solution is the one obtained before extrapolation. These estimates are then used to derive the extrapolation formula. Finally, we provide the error estimates in which the approximate solution obtained after extrapolation is used.

3.1. Analysis of FOFDM-I

3.1.1. Error estimates before extrapolation

The following estimates are obtained in [18]:

$$\max_{1\leqslant j\leqslant n-1}|V(x_j)-v_j|\leqslant\begin{cases}\frac{Mh^4}{\varepsilon}\left[1+\max\limits_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch\leqslant\varepsilon,\\ Mh^2\left[1+h^2\max\limits_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon}\right], & \text{when } Ch\geqslant\varepsilon,\end{cases}\tag{3.1}$$

where

$$E(x,\beta)=\left\{\exp\left(-x/\sqrt{\beta/\varepsilon}\right)+\exp\left(-(1-x)\sqrt{\beta/\varepsilon}\right)\right\}\quad\text{and}\quad 0<\beta\leqslant W(x).$$

Here and after, M and C denote positive constants which may take different values in different equations and inequalities but are always independent of h and ε.

3.1.2. Extrapolation formula

The FOFDM-I on the mesh μ_n satisfies (3.1). Denoting by $\tilde{v}$ the numerical solution computed on the mesh μ_{2n}, the estimate (3.1) reads

$$\max_{1\leqslant j\leqslant 2n-1}|V(\tilde{x}_j)-\tilde{v}_j|\leqslant\begin{cases}\frac{M}{\varepsilon}\left(\frac{h}{2}\right)^4\left[1+\max\limits_{1\leqslant j\leqslant 2n-1}\frac{E(\tilde{x}_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch\leqslant\varepsilon,\\ M\left(\frac{h}{2}\right)^2\left[1+\left(\frac{h}{2}\right)^2\max\limits_{1\leqslant j\leqslant 2n-1}\frac{E(\tilde{x}_j,\beta)}{\varepsilon}\right], & \text{when } Ch\geqslant\varepsilon.\end{cases}\tag{3.2}$$

To establish the suitable extrapolation formula, it is important to consider the two cases separately.

We start with the case in which $Ch\leqslant\varepsilon$.

It follows, from (3.1) and (3.2), that

$$V(x_j)-v_j=\frac{Mh^4}{\varepsilon}\left[1+\frac{E(x_j,\beta)}{\varepsilon^2}\right]+R_n(x_j),\quad 1\leqslant j\leqslant n-1$$

and

$$V(\tilde{x}_j)-\tilde{v}_j=\frac{M}{\varepsilon}\left(\frac{h}{2}\right)^4\left[1+\frac{E(\tilde{x}_j,\beta)}{\varepsilon^2}\right]+R_{2n}(\tilde{x}_j),\quad 1\leqslant j\leqslant 2n-1,$$

where both the remainders, $R_n(x_j)$ and $R_{2n}(\tilde{x}_j)$, are $O(h^4)$.

Therefore,

$$(V(x_j)-v_j)-16(V(x_j)-\tilde{v}_j)=R_n(x_j)-16R_{2n}(x_j)=O(h^4)\quad\forall x_j\in\mu_n.$$

Hence,

$$V(x_j) - \frac{16\tilde{v}_j - v_j}{15} = O(h^4) \quad \forall x_j \in \mu_n. \tag{3.3}$$

In the case when $Ch \geqslant \varepsilon$, estimates (3.1) and (3.2), respectively, give

$$V(x_j) - v_j = Mh^2\left[1 + h^2\frac{E(x_j,\beta)}{\varepsilon}\right] + R_n^*(x_j), \quad 1 \leqslant j \leqslant n-1$$

and

$$V(\tilde{x}_j) - \tilde{v}_j = M\left(\frac{h}{2}\right)^2\left[1 + \left(\frac{h}{2}\right)^2\frac{E(\tilde{x}_j,\beta)}{\varepsilon}\right] + R_{2n}^*(\tilde{x}_j), \quad 1 \leqslant j \leqslant 2n-1,$$

where both the remainders, $R_n^*(x_j)$ and $R_{2n}^*(\tilde{x}_j)$ are $O(h^2)$.

Thus,

$$(V(x_j) - v_j) - 16(V(x_j) - \tilde{v}_j) = O(h^2) \quad \forall x_j \in \mu_n$$

and consequently,

$$V(x_j) - \frac{16\tilde{v}_j - v_j}{15} = O(h^2) \quad \forall x_j \in \mu_n. \tag{3.4}$$

In view of equations (3.3) and (3.4), it is natural to use the formula

$$v_j^{\text{ext}} := \frac{16\tilde{v}_j - v_j}{15}, \quad j = 1(1)n-1 \tag{3.5}$$

in the extrapolation process, irrespective of the cases $Ch \leqslant \varepsilon$ or $Ch \geqslant \varepsilon$.

3.1.3. Error estimates after extrapolation

Unless indicated otherwise, in what follows, the functions with a symbol '$\sim$' means that they are evaluated at the mesh μ_{2n}. The only exceptions to this notation are with the denominator functions used in FOFDM-II and the functions W used in (3.38) where we use '$-$' with the denominator function evaluated at the mesh μ_{2n} whereas the one at the mesh μ_n has a '$\sim$' on top of it.

The local truncation error of the scheme (2.6) and (2.7) after extrapolation is

$$\left[\widetilde{L}^h\left(V - \frac{16\tilde{v} - v}{15}\right)\right]_j = \frac{16}{15}\left[\widetilde{A}(V - \tilde{v})\right]_j - \frac{1}{15}[A(V - v)]_j, \quad j = 1(1)n-1. \tag{3.6}$$

Here

$$(A(V-v))_j = (r_j^- - q_j^- W_{j-1})V_{j-1} + (r_j^c - q_j^c W_j)V_j + (r_j^+ - q_j^+ W_{j+1})V_{j+1} + \varepsilon(q_j^- V''_{j-1} + q_j^c V''_j + q_j^+ V''_{j+1}) \tag{3.7}$$

and

$$(\widetilde{A}(V-\tilde{v}))_j = (\tilde{r}_j^- - \tilde{q}_j^- W_{j-1})V_{j-1} + (\tilde{r}_j^c - \tilde{q}_j^c W_j)V_j + (\tilde{r}_j^+ - \tilde{q}_j^+ W_{j+1})V_{j+1} + \varepsilon(\tilde{q}_j^- V''_{j-1} + \tilde{q}_j^c V''_j + \tilde{q}_j^+ V''_{j+1}). \tag{3.8}$$

Using the Taylor series expansions, we obtain (when $Ch \leqslant \varepsilon$)

$$(A(V-v))_j = T_0 V_j + T_1 V'_j + T_2 V''_j + T_3 V'''_j + T_4 V^{(4)}(\xi_{1,j}) + T_4 V^{(4)}(\xi_{2,j}) \tag{3.9}$$

and

$$(\widetilde{A}(V-\tilde{v}))_j = \widetilde{T}_0 V_j + \widetilde{T}_1 V'_j + \widetilde{T}_2 V''_j + \widetilde{T}_3 V'''_j + \widetilde{T}_4 V^{(4)}(\tilde{\xi}_{1,j}) + \widetilde{T}_4 V^{(4)}(\tilde{\xi}_{2,j}), \tag{3.10}$$

where $\xi_{1,j} \in (x_{j-1}, x_j), \xi_{2,j} \in (x_j, x_{j+1})$ and $\tilde{\xi}_{1,j} \in \left(\frac{x_{j-1}+x_j}{2}, x_j\right), \tilde{\xi}_{2,j} \in \left(x_j, \frac{x_j+x_{j+1}}{2}\right)$.

Also

$$T_0 = -\sigma_j^- + 2\sigma_j^c - \sigma_j^+, \quad T_1 = h(\sigma_j^- - \sigma_j^+), \quad T_2 = -h^2\left[\frac{1}{2}(\sigma_j^- + \sigma_j^+) - \varepsilon\right],$$
$$T_3 = \frac{h_j^3}{6}(\sigma_j^- - \sigma_j^+), \quad T_4 = -\frac{h_j^4}{24}[\sigma_j^- + \sigma_j^+ - 2\varepsilon]. \tag{3.11}$$

The different expressions for $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3$ and $\widetilde{T}_4$, are equivalently found as in (3.11) by replacing h by $\tilde{h}$ and σ by $\tilde{\sigma}$.

Some algebraic manipulations yield

$$\left.\begin{array}{l} |T_0| \leqslant \frac{Mh^6}{\varepsilon}, \quad |\widetilde{T}_0| \leqslant \frac{M\tilde{h}^6}{\varepsilon}; \quad |T_1| \leqslant \frac{Mh^6}{\varepsilon}, \quad |\widetilde{T}_1| \leqslant \frac{M\tilde{h}^6}{\varepsilon}; \quad |T_2| \leqslant \frac{Mh^6}{\varepsilon}, \quad |\widetilde{T}_2| \leqslant \frac{M\tilde{h}^6}{\varepsilon}; \\ \quad |T_3| \leqslant \frac{Mh^8}{\varepsilon}, \quad |\widetilde{T}_3| \leqslant \frac{M\tilde{h}^8}{\varepsilon}, \quad |T_4| \leqslant \frac{Mh^8}{\varepsilon}, \quad |\widetilde{T}_4| \leqslant \frac{M\tilde{h}^8}{\varepsilon}. \end{array}\right\} \tag{3.12}$$

On the other hand, the following lemma (proved in [16]) provides bounds on the derivatives of solution:

Lemma 3.1. *For all $k \in \{0,1,2,3,4\}$ and $x \in [0,1]$, the solution $V(x)$ of (2.1) satisfies*

$$|V^{(k)}(x)| \leqslant M[1 + \varepsilon^{-k/2}E(x,\beta)],$$

where

$$0 < \beta \leqslant W(x) \quad \textit{and} \quad E(x,\beta) = \left\{\exp\left(-x/\sqrt{\beta/\varepsilon}\right) + \exp\left(-(1-x)\sqrt{\beta/\varepsilon}\right)\right\}.$$

Using the above lemma, relations (3.9) and (3.10), and the fact that $\tilde{h} < h$, we obtain

$$\max_{1\leqslant j\leqslant n-1} |(A(V-v))_j| \leqslant \frac{Mh^6}{\varepsilon}\left[1 + \max_{1\leqslant j\leqslant n-1} \frac{E(x_j,\beta)}{\varepsilon^2}\right] \tag{3.13}$$

and

$$\max_{1\leqslant j\leqslant n-1} |(\widetilde{A}(V-\tilde{v}))_j| \leqslant \frac{Mh^6}{\varepsilon}\left[1 + \max_{1\leqslant j\leqslant n-1} \frac{E(x_j,\beta)}{\varepsilon^2}\right]. \tag{3.14}$$

Furthermore, the matrices A and $\widetilde{A}$ are diagonally dominant by rows, therefore, we have the following result (due to Varah [23]) to estimate the norm of the associated matrix:

$$||A^{-1}|| \leqslant \max_j\{|r_j^c| - (|r_j^-| + |r_j^+|)\}^{-1}. \tag{3.15}$$

Using (2.7), we get

$$\{|r_j^c| - (|r_j^-| + |r_j^+|)\} \geqslant Mh^2.$$

Hence from the inequality (3.15), we have

$$||A^{-1}|| \leqslant \frac{M}{h^2} \tag{3.16}$$

and similarly

$$\|\widetilde{A}^{-1}\| \leqslant \frac{M}{\tilde{h}^2}. \tag{3.17}$$

Now, using the inequality

$$\max_j \left|V(x_j) - \frac{16\tilde{v}_j - v_j}{15}\right| \leqslant \frac{16}{15}\max_j |V(x_j) - \tilde{v}_j| + \frac{1}{15}\max_j |V(x_j) - v_j|,$$

along with (3.13), (3.14) and (3.16), (3.17) into

$$\max_j |V_j - v_j| \leqslant \|A^{-1}\| \max_j |(A(V-v))_j| \tag{3.18}$$

and

$$\max_j |V_j - \tilde{v}_j| \leqslant \|\widetilde{A}^{-1}\| \max_j |(\widetilde{A}(V - \tilde{v}))_j|, \tag{3.19}$$

we obtain

$$\max_j |V(x_j) - v_j^{\text{ext}}| \leqslant \frac{Mh^4}{\varepsilon}\left(1 + \max_j \frac{E(x_j,\beta)}{\varepsilon^2}\right). \tag{3.20}$$

On the other hand, when $Ch \geqslant \varepsilon$, we introduce some new notations

$$\begin{aligned}
&r_j^- = r_j^-(W_{j-1}), \quad r_j^+ = r_j^+(W_{j+1}), \quad r_j^c = r_j^c(W_j),\\
&R_j^- = r_j^-(W_0), \quad R_j^+ = r_j^+(W_0), \quad R_j^c = r_j^c(W_0),\\
&\tilde{r}_j^- = \tilde{r}_j^-(W_{j-1}), \quad \tilde{r}_j^+ = \tilde{r}_j^+(W_{j+1}), \quad \tilde{r}_j^c = \tilde{r}_j^c(W_j),\\
&\widetilde{R}_j^- = \tilde{r}_j^-(W_0), \quad \widetilde{R}_j^+ = \tilde{r}_j^+(W_0), \quad \widetilde{R}_j^c = \tilde{r}_j^c(W_0),
\end{aligned}$$

and since q_j's and $\tilde{q}_j$'s are independent of W_j's, we will have $Q_j = q_j$, $\widetilde{Q}_j = \tilde{q}_j$, etc.

In this case, then we have

$$\begin{aligned}
(A(V-v))_j = \{&[(r_j^- - q_j^- W_{j-1})V_{j-1} + (r_j^c - q_j^c W_j)V_j + (r_j^+ - q_j^+ W_{j+1})V_{j+1} + \varepsilon(q_j^- V''_{j-1} + q_j^c V''_j + q_j^+ V''_{j+1})]\\
&- [(R_j^- - Q_j^- W_0)V_{j-1} + (R_j^c - Q_j^c W_0)V_j + (R_j^+ - Q_j^+ W_0)V_{j+1}\\
&+ \varepsilon(Q_j^- V''_{j-1} + Q_j^c V''_j + Q_j^+ V''_{j+1})]\}
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{A}(V-\tilde{v}))_j = \{&[(\tilde{r}_j^- - \tilde{q}_j^- W_{j-1})V_{j-1} + (\tilde{r}_j^c - \tilde{q}_j^c W_j)V_j + (\tilde{r}_j^+ - \tilde{q}_j^+ W_{j+1})V_{j+1} + \varepsilon(\tilde{q}_j^- V''_{j-1} + \tilde{q}_j^c V''_j + \tilde{q}_j^+ V''_{j+1})]\\
&- [(\widetilde{R}_j^- - \widetilde{Q}_j^- W_0)V_{j-1} + (\widetilde{R}_j^c - \widetilde{Q}_j^c W_0)V_j + (\widetilde{R}_j^+ - \widetilde{Q}_j^+ W_0)V_{j+1}\\
&+ \varepsilon(\widetilde{Q}_j^- V''_{j-1} + \widetilde{Q}_j^c V''_j + \widetilde{Q}_j^+ V''_{j+1})]\},
\end{aligned}$$

which when simplified, reduce to

$$\begin{aligned}
(A(V-v))_j = {}&[(r_j^- - R_j^-) - q_j^-(W_{j-1} - W_0)]V_{j-1} + [(r_j^c - R_j^c) - q_j^c(W_j - W_0)]V_j\\
&+ [(r_j^+ - R_j^+) - q_j^+(W_{j+1} - W_0)]V_{j+1}
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
(\widetilde{A}(V-\tilde{v}))_j = {}&[(\tilde{r}_j^- - \widetilde{R}_j^-) - \tilde{q}_j^-(W_{j-1} - W_0)]V_{j-1} + [(\tilde{r}_j^c - \widetilde{R}_j^c) - \tilde{q}_j^c(W_j - W_0)]V_j + [(\tilde{r}_j^+ - \widetilde{R}_j^+)\\
&- \tilde{q}_j^+(W_{j+1} - W_0)]V_{j+1}, \quad j = 1(1)n-1.
\end{aligned} \tag{3.22}$$

Simplifying (3.21), we obtain

$$|[A(V-v)]_j| \leqslant Mh^4\left[3V_j + h^2 V''_j + \frac{h^4}{24}(V^{(4)}(\xi_{1,j}) + V^{(4)}(\xi_{2,j}))\right]. \tag{3.23}$$

Applying Lemma 3.1, we get

$$\max_{1\leqslant j\leqslant n-1} |[A(V-v)]_j\| \leqslant Mh^4\left[1 + h^2 \max_{1\leqslant j\leqslant n-1} \frac{E(x_j,\beta)}{\varepsilon}\right]. \tag{3.24}$$

Similarly, Eq. (3.22) yields

$$\max_{1\leqslant j\leqslant n-1} |[\widetilde{A}(V-\tilde{v})]_j| \leqslant Mh^4\left[1 + h^2 \max_{1\leqslant j\leqslant n-1} \frac{E(x_j,\beta)}{\varepsilon}\right]. \tag{3.25}$$

Hence, from (3.16)–(3.19) and (3.24) and (3.25), we obtain

$$\max_{0<j\leqslant n} |V_j - v_j^{\text{ext}}| \leqslant Mh^2\left[1 + h^2 \max_{0<j\leqslant n} \frac{E(x_j,\beta)}{\varepsilon}\right]. \tag{3.26}$$

We have therefore established that

$$\max_{1<j\leqslant n-1}|V_j-v_j^{\mathrm{ext}}|\leqslant\begin{cases}\frac{Mh^4}{\varepsilon}\left[1+\max\limits_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right], & \text{when } Ch\leqslant\varepsilon,\\ Mh^2\left[1+h^2\max\limits_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon}\right], & \text{when } Ch\geqslant\varepsilon.\end{cases}\tag{3.27}$$

which means that the Richardson extrapolation does not improve the order of convergence of FOFDM-I.

3.2. Error analysis for FOFDM-II

3.2.1. Error estimates before extrapolation

The local truncation error of the scheme (2.9) and (2.10) is given by

$$\tau_j(V)=T_0V_j+T_1V_j'+T_2V_j''+T_3V_j'''+T_4V^{(4)}(\xi_j);\quad \xi_j\in(x_{j-1},x_{j+1}),\tag{3.28}$$

where

$$T_0=r_j^-+r_j^c+r_j^+-\widetilde{W}_j,\quad T_1=h(r_j^+-r_j^-),\quad T_2=\frac{h^2}{2}(r_j^++r_j^-)+\varepsilon,$$
$$T_3=\frac{h^3}{6}(r_j^+-r_j^-)\quad\text{and}\quad T_4=\frac{h^4}{24}(r_j^++r_j^-).\tag{3.29}$$

Further simplifications yield

$$T_0=T_1=T_3=0,\quad |T_2|\leqslant Mh^2\quad\text{and}\quad |T_4|\leqslant Mh^2.\tag{3.30}$$

Finally using Lemma 3.1 we obtain

$$\max_{1\leqslant j\leqslant n-1}|\tau_j(V)|\leqslant Mh^2\left[1+\max_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right].\tag{3.31}$$

Now since A is diagonally dominant by rows, we can estimate $||A^{-1}||$ by the relation (3.15). Since, $\{|r_j^c|-(|r_j^-|+|r_j^+|)\}\geqslant M$, we conclude that,

$$\|A^{-1}\|\leqslant M.\tag{3.32}$$

But the relation

$$\tau_j(V)=(AV)_j-(\widetilde{L}V)_j=(A(V-v))_j\tag{3.33}$$

implies that

$$\max_j|V_j-v_j|\leqslant\|A\|^{-1}\max_j|(A(V-v))_j|.\tag{3.34}$$

Hence, using (3.31) and (3.32), we obtain

$$\max_{1\leqslant j\leqslant n-1}|V_j-v_j|\leqslant Mh^2\left[1+\max_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^2}\right].\tag{3.35}$$

Now using the lemma (see [18] for details) on exponential behavior of the solution, we find that

$$\sup_{0<\varepsilon\leqslant 1}\max_{1\leqslant j\leqslant n-1}|V_j-v_j|\leqslant Mh^2.\tag{3.36}$$

3.2.2. Extrapolation formula

In this case, v and $\tilde{v}$ denote the computed solutions of problem (2.1) by the scheme (2.9) and (2.10) on the meshes μ_n and μ_{2n}, respectively. This implies that

$$|V_j - v_j| \leqslant Mh^2\left[1+\frac{E(x_j,\beta)}{\varepsilon^2}\right], \quad j=1(1)n-1$$

and

$$|V_j - \tilde{v}_j| \leqslant M(h/2)^2\left[1+\frac{E(\tilde{x}_j,\beta)}{\varepsilon^2}\right], \quad j=1(1)2n-1.$$

Therefore,

$$V_j - v_j = Mh^2\left[1+\frac{E(x_j,\beta)}{\varepsilon^2}\right] + R_n(x_j) \quad \forall x_j \in \mu_n$$

and

$$V_j - \tilde{v}_j = M(h/2)^2\left[1+\frac{E(\tilde{x}_j,\beta)}{\varepsilon^2}\right] + R_{2n}(\tilde{x}_j) \quad \forall \tilde{x}_j \in \mu_{2n}.$$

where both the remainders, $R_n(x_j)$ and $R_{2n}(\tilde{x}_j)$ are $O(h^2)$.

Hence,

$$(V_j - v_j) - 4(V_j - \tilde{v}_j) = R_n(x_j) - 4R_{2n}(x_j) = O(h^2), \quad \forall x_j \in \mu_n$$

indicates that in the extrapolation process, we should use the formula

$$v_j^{\text{ext}} := \frac{4\tilde{v}_j - v_j}{3}, \quad j=1(1)n-1. \tag{3.37}$$

3.2.3. Error estimates after extrapolation

An analogue of (3.6) implies that the local truncation error of the scheme (2.9) and (2.10) after extrapolation should be given by

$$\begin{aligned}(L_*^h(V-v^{\text{ext}}))_j &= \frac{4}{3}(L_*^{\tilde{h}}(V-\tilde{v}))_j - \frac{1}{3}(L_*^h(V-v))_j \\ &= \frac{4}{3}\left[\left(-\varepsilon\widetilde{V}_j'' + \overline{W}_j\widetilde{V}_j\right) - \left(-\varepsilon\frac{\widetilde{V}_{j+1}-2\widetilde{V}_j+\widetilde{V}_{j-1}}{\bar{\phi}_j^2} + \overline{W}_j\widetilde{V}_j\right)\right] \\ &\quad - \frac{1}{3}\left[\left(-\varepsilon V_j'' + \widetilde{W}_jV_j\right) - \left(-\varepsilon\frac{V_{j+1}-2V_j+V_{j-1}}{\tilde{\phi}_j^2} + \widetilde{W}_jV_j\right)\right],\end{aligned} \tag{3.38}$$

where L_*^h and $L_*^{\tilde{h}}$ denote the discrete operators associated with FOFDM-II (i.e., relations (2.9) and (2.10)) when considered on meshes μ_n and μ_{2n}, respectively. (Note that $\bar{\phi}_j$ is obtained from $\tilde{\phi}_j$ by replacing h by $\tilde{h}$).

Some algebraic manipulations yield

$$(L_*^h(V-v^{\text{ext}}))_j \leqslant Mh^4V^{(vi)}(\xi_j), \quad \xi_j \in (x_{j-1}, x_{j+1}).$$

Using Lemma 3.1, we obtain

$$\max_{1\leqslant j\leqslant n-1}|L_*^h(V-v^{\text{ext}})|_j \leqslant Mh^4\left[1+\max_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^3}\right]. \tag{3.39}$$

Now using the following lemma (proved in [19])

Lemma 3.2 (Uniform stability estimate). *If ζ_i is any mesh function such that $\zeta_0 = \zeta_n = 0$, then*

$$|\zeta_i| \leqslant \frac{1}{\beta}\max_{1\leqslant j\leqslant n-1}|\widetilde{L}^h\zeta_j| \quad \textit{for } 0\leqslant i\leqslant n$$

for the mesh function $(V-v^{\text{ext}})_j$, we obtain

$$\max_{1\leqslant j\leqslant n-1}|V_j - v_j^{\text{ext}}| \leqslant Mh^4\left[1+\max_{1\leqslant j\leqslant n-1}\frac{E(x_j,\beta)}{\varepsilon^3}\right]. \tag{3.40}$$

Finally, using the lemma (see [18] for details) on exponential behavior of the solution, we find that

$$\sup_{0<\varepsilon\leqslant 1}\max_{0<j\leqslant n}|V_j - v_j^{\text{ext}}| \leqslant Mh^4.$$

In summary, we have the following main result:

Theorem 3.3. *Let $W(x), Z(x)$ be sufficiently smooth so that $V(x) \in C^4[0,1]$. Let $v_j^{\text{ext}}, j = 0(1)n$ be the approximate solutions of (2.1) obtained after extrapolation, with $v_0 = v_0^{\text{ext}} = V(0)$, and $v_n = v_n^{\text{ext}} = V(1)$. Then, there is a constant M independent of ε and h such that*

$$\sup_{0<\varepsilon\leqslant 1}\max_{0<j\leqslant n}|V_j - v_j^{\text{ext}}| \leqslant Mh^2 \quad \textit{for } \text{FOFDM-I},$$
$$\sup_{0<\varepsilon\leqslant 1}\max_{0<j\leqslant n}|V_j - v_j^{\text{ext}}| \leqslant Mh^4 \quad \textit{for } \text{FOFDM-II}.$$

4. Numerical results

In this section, we present some comparative numerical results for two test problems considered in [18].

Example 4.1. Consider problem (1.1) with

$$a(x) = 1 + x^2, \quad b(x) = (\cos x)/(3 - x)^3, \quad f(x) = 4(3x^2 - 3x + 1)[(x - 1/2)^2 + 2];$$
$$y(0) = -1, \quad y(1) = 0.$$

The exact solution for this problem is not available.

Example 4.2. Consider problem (1.1) with

$$a(x) = 1, \quad b(x) = 1 + x(1 - x),$$
$$f(x) = 1 + x(1 - x) + \left[2\sqrt{\varepsilon} - x^2(1 - x)\right]\exp\left[-(1 - x)/\sqrt{\varepsilon}\right] + \left[2\sqrt{\varepsilon} - x(1 - x)^2\right]\exp\left[-x/\sqrt{\varepsilon}\right].$$

Its exact solution is given by

$$y(x) = 1 + (x - 1)\exp\left[-x/\sqrt{\varepsilon}\right] - x\exp\left[-(1 - x)/\sqrt{\varepsilon}\right].$$

Since the exact solution is available for Example 4.2, the maximum errors at all the mesh points are calculated using the formula

$$e_{\varepsilon,n} := \max_{0\leqslant j\leqslant n}|y(x_j) - v_j|, \quad \text{before extrapolation}$$

and

$$e_{\varepsilon,n}^{\text{ext}} := \max_{0\leqslant j\leqslant n}|y(x_j) - v_j^{\text{ext}}|, \quad \text{after extrapolation},$$

where v_j is the solution of (1.1) obtained by using (2.1) and (2.2), and v_j^{ext} is the solution after extrapolation.

For Example 4.1, the exact solution is not available and therefore we use the double mesh principle [2] to evaluate the maximum errors at all the mesh points:

$$e_{\varepsilon,n} := \max_{0\leqslant j\leqslant n}|v_j^n - v_{2j}^{2n}|, \quad \text{before extrapolation}$$

and

$$e_{\varepsilon,n}^{\text{ext}} := \max_{0\leqslant j\leqslant n}|v_j^{\text{ext}} - v_{2j}^{\text{ext}}|, \quad \text{after extrapolation},$$

where v_{2j}^{2n} is the numerical solution of (1.1) obtained by using (2.1) and (2.2) on the mesh μ_{2n}.

The numerical rates of convergence are computed using the formula [2]: $r_k \equiv r_{\varepsilon,k} := \log_2(\tilde{e}_{n_k}/\tilde{e}_{2n_k})$, $k = 1, 2, \ldots$ where $\tilde{e}$ stands for $e_{\varepsilon,n}$ and $e_{\varepsilon,n}^{\text{ext}}$, respectively.

Table 1
Results for Example 4.1 before extrapolation (maximum errors using FOFDM-I)

ε	$n=20$	$n=40$	$n=80$	$n=160$	$n=320$	$n=640$	$n=1280$
1.0e−01	3.62e−06	2.26e−07	1.41e−08	8.84e−10	5.59e−11	1.55e−12	8.33e−12
1.0e−02	3.79e−05	2.37e−06	1.48e−07	9.23e−09	5.84e−10	2.24e−11	9.42e−11
1.0e−04	1.29e−02	8.94e−04	5.78e−05	3.67e−06	2.30e−07	1.44e−08	8.86e−10
1.0e−06	2.67e−01	3.56e−02	1.89e−02	3.17e−03	2.25e−04	1.48e−05	9.35e−07
1.0e−08	2.99e−01	7.79e−02	1.99e−02	4.77e−03	7.15e−04	2.56e−03	6.54e−04
1.0e−10	2.99e−01	7.79e−02	1.99e−02	5.02e−03	1.26e−03	3.16e−04	7.80e−05
1.0e−11	2.99e−01	7.79e−02	1.99e−02	5.02e−03	1.26e−03	3.16e−04	7.92e−05
1.0e−12	2.99e−01	7.79e−02	1.99e−02	5.02e−03	1.26e−03	3.16e−04	7.92e−05
e_n	2.99e−01	7.79e−02	1.99e−02	5.02e−03	1.26e−03	3.16e−04	7.92e−05

Table 2
Results for Example 4.1 after extrapolation (maximum errors using FOFDM-I)

ε	$n=20$	$n=40$	$n=80$	$n=160$	$n=320$	$n=640$	$n=1280$
1.0e−01	3.64e−10	5.86e−12	3.89e−13	7.31e−13	2.68e−12	8.81e−12	4.15e−11
1.0e−02	6.99e−09	1.10e−10	4.07e−12	7.99e−12	1.62e−11	9.91e−11	3.26e−10
1.0e−04	9.68e−05	1.64e−06	2.61e−08	4.10e−10	1.03e−11	3.68e−11	9.51e−11
1.0e−06	2.34e−02	2.91e−03	7.28e−04	2.91e−05	5.19e−07	8.49e−09	1.36e−10
1.0e−08	5.97e−02	1.56e−02	3.73e−03	4.97e−04	2.06e−04	1.38e−04	9.11e−06
1.0e−10	5.97e−02	1.56e−02	3.98e−03	1.00e−03	2.52e−04	6.21e−05	1.01e−05
1.0e−11	5.97e−02	1.56e−02	3.98e−03	1.00e−03	2.52e−04	6.33e−05	1.58e−05
1.0e−12	5.97e−02	1.56e−02	3.98e−03	1.00e−03	2.52e−04	6.33e−05	1.58e−05
e_n^{ext}	5.97e−02	1.56e−02	3.98e−03	1.00e−03	2.52e−04	6.33e−05	1.58e−05

Table 3
Results for Example 4.1 before extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1},\ k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e−01	4.00e+00	4.00e+00	4.00e+00	3.98e+00	5.17e+00
1.0e−02	4.00e+00	4.00e+00	4.00e+00	3.98e+00	4.70e+00
1.0e−04	3.85e+00	3.95e+00	3.98e+00	4.00e+00	4.00e+00
1.0e−06	2.91e+00	9.18e−01	2.57e+00	3.81e+00	3.93e+00
1.0e−08	1.94e+00	1.97e+00	2.06e+00	2.74e+00	−1.84e+00
1.0e−10	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e−11	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e−12	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
r_n	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00

Table 4
Results for Example 4.1 after extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1},\ k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e−01	5.96e+00	3.91e+00	−9.08e−01	−1.87e+00	−1.72e+00
1.0e−02	5.99e+00	4.75e+00	−9.71e−01	−1.02e+00	−2.62e+00
1.0e−04	5.89e+00	5.97e+00	5.99e+00	5.32e+00	−1.84e+00
1.0e−06	3.01e+00	2.00e+00	4.64e+00	5.81e+00	5.93e+00
1.0e−08	1.94e+00	2.06e+00	2.91e+00	1.27e+00	5.81e−01
1.0e−10	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.02e+00
1.0e−11	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
1.0e−12	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00
r_n^{ext}	1.94e+00	1.97e+00	1.98e+00	1.99e+00	2.00e+00

Table 5
Results for Example 4.2 before extrapolation (maximum errors using FOFDM-I)

ε	n = 20	n = 40	n = 80	n = 160	n = 320	n = 640	n = 1280
1.0e−01	5.19e−07	3.27e−08	2.05e−09	1.28e−10	8.00e−12	5.38e−13	1.14e−12
1.0e−02	2.14e−05	1.43e−06	8.99e−08	5.63e−09	3.52e−10	2.21e−11	1.32e−12
1.0e−04	1.08e−03	6.00e−04	9.44e−05	6.53e−06	4.31e−07	2.71e−08	1.70e−09
1.0e−06	3.82e−04	9.94e−05	1.39e−04	1.52e−04	9.33e−05	1.98e−05	1.67e−06
1.0e−08	3.82e−04	9.94e−05	2.54e−05	1.01e−05	1.50e−05	1.62e−05	1.63e−05
1.0e−10	3.96e−04	1.00e−04	2.54e−05	6.43e−06	1.62e−06	1.26e−06	1.56e−06
1.0e−11	3.97e−04	1.01e−04	2.54e−05	6.43e−06	1.62e−06	4.06e−07	4.25e−07
1.0e−12	3.98e−04	1.02e−04	2.56e−05	6.43e−06	1.62e−06	4.06e−07	1.02e−07
e_n	3.98e−04	1.02e−04	2.57e−05	6.43e−06	1.62e−06	4.06e−07	1.02e−07

Table 6
Results for Example 4.2 after extrapolation (maximum errors using FOFDM-I)

ε	n = 20	n = 40	n = 80	n = 160	n = 320	n = 640	n = 1280
1.0e−01	3.86e−11	6.13e−13	1.61e−14	2.50e−14	1.12e−13	1.25e−12	7.47e−13
1.0e−02	1.61e−08	2.79e−10	4.41e−12	7.47e−14	6.35e−14	4.34e−13	6.75e−13
1.0e−04	5.56e−05	3.11e−06	3.51e−07	7.25e−09	1.25e−10	2.43e−12	2.37e−12
1.0e−06	9.02e−05	3.12e−05	1.58e−05	8.13e−06	8.54e−08	9.81e−08	2.78e−09
1.0e−08	8.06e−05	2.14e−05	6.24e−06	2.40e−06	1.43e−06	1.20e−06	1.06e−06
1.0e−10	7.97e−05	2.04e−05	5.25e−06	1.40e−06	4.35e−07	1.92e−07	1.31e−07
1.0e−11	7.96e−05	2.04e−05	5.18e−06	1.33e−06	3.60e−07	1.16e−07	5.54e−08
1.0e−12	7.96e−05	2.03e−05	5.16e−06	1.31e−06	3.36e−07	9.23e−08	3.14e−08
e_n^{ext}	7.96e−05	2.03e−05	5.15e−06	1.30e−06	3.28e−07	8.48e−08	7.41e−08

Table 7
Results for Example 4.2 before extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1},\ k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e−01	3.99e+00	4.00e+00	4.00e+00	4.00e+00	3.89e+00
1.0e−02	3.90e+00	4.00e+00	4.00e+00	4.00e+00	4.00e+00
1.0e−04	8.49e−01	2.67e+00	3.85e+00	3.92e+00	3.99e+00
1.0e−06	1.94e+00	−4.82e−01	−1.33e−01	7.06e−01	2.23e+00
1.0e−08	1.94e+00	1.97e+00	1.33e+00	−5.71e−01	−1.15e−01
1.0e−10	1.99e+00	1.98e+00	1.98e+00	1.99e+00	3.63e−01
1.0e−11	1.97e+00	1.99e+00	1.98e+00	1.99e+00	2.00e+00
1.0e−12	1.97e+00	1.99e+00	1.99e+00	1.99e+00	2.00e+00
r_n	1.97e+00	1.99e+00	2.00e+00	1.99e+00	2.00e+00

Table 8
Results for Example 4.2 after extrapolation (rates of convergence using FOFDM-I) $n_k = 20 \times 2^{k-1},\ k = 1(1)5$

ε	r_1	r_2	r_3	r_4	r_5
1.0e−01	5.98e+00	5.25e+00	−6.34e−01	−2.16e+00	−3.48e+00
1.0e−02	5.85e+00	5.98e+00	5.88e+00	2.35e−01	−2.77e+00
1.0e−04	4.16e+00	3.15e+00	5.60e+00	5.86e+00	5.69e+00
1.0e−06	1.53e+00	9.84e−01	9.55e−01	6.57e+00	−1.99e−01
1.0e−08	1.91e+00	1.78e+00	1.38e+00	7.44e−01	2.55e−01
1.0e−10	1.96e+00	1.96e+00	1.90e+00	1.69e+00	1.18e+00
1.0e−11	1.97e+00	1.98e+00	1.96e+00	1.89e+00	1.63e+00
1.0e−12	1.97e+00	1.98e+00	1.98e+00	1.96e+00	1.86e+00
r_n^{ext}	1.97e+00	1.98e+00	1.99e+00	1.98e+00	1.95e+00

Table 9
Results for Example 4.2 before extrapolation (maximum errors using FOFDM-II)

ε	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
1.0e−01	2.66E−03	6.55E−04	1.63E−04	4.07E−05	1.02E−05	2.54E−06	6.36E−07
1.0e−02	8.15E−03	2.02E−03	5.04E−04	1.26E−04	3.15E−05	7.87E−06	1.97E−06
1.0e−04	9.48E−03	2.58E−03	8.41E−04	1.86E−04	7.21E−05	2.06E−05	5.49E−06
1.0e−06	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.69E−05	4.70E−05	1.10E−05
1.0e−08	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.04E−05	1.01E−05	2.54E−06
1.0e−10	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.04E−05	1.01E−05	2.54E−06
1.0e−11	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.04E−05	1.01E−05	2.54E−06
1.0e−12	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.04E−05	1.01E−05	2.54E−06
e_n	9.48E−03	2.47E−03	6.32E−04	1.60E−04	4.04E−05	1.01E−05	2.54E−06

Table 10
Results for Example 4.2 after extrapolation (maximum errors using FOFDM-II)

ε	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
1.0e−01	1.31E−05	8.39E−07	5.28E−08	3.30E−09	2.07E−10	1.33E−11	3.17E−12
1.0e−02	1.83E−04	1.53E−05	1.03E−06	6.58E−08	4.13E−09	2.58E−10	1.63E−11
1.0e−04	2.17E−05	2.47E−05	4.34E−05	3.21E−05	3.48E−06	2.35E−07	1.50E−08
1.0e−06	2.23E−05	1.52E−06	9.99E−08	6.24E−09	1.81E−06	9.90E−07	4.93E−06
1.0e−08	2.23E−05	1.52E−06	9.99E−08	6.42E−09	4.08E−10	2.57E−11	2.44E−11
1.0e−10	2.23E−05	1.52E−06	9.99E−08	6.42E−09	4.08E−10	2.57E−11	1.61E−12
1.0e−11	2.23E−05	1.52E−06	9.99E−08	6.42E−09	4.08E−10	2.57E−11	1.61E−12
1.0e−12	2.23E−05	1.52E−06	9.99E−08	6.42E−09	4.08E−10	2.57E−11	1.61E−12
e_n^{ext}	2.23E−05	1.52E−06	9.99E−08	6.42E−09	4.08E−10	2.57E−11	1.61E−12

Table 11
Results for Example 4.2 before extrapolation (rate of convergence using FOFDM-II), $n_k = 8 \times 2^{k-1},\ k = 1(1)6$

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0e−01	2.02E+00	2.01E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
1.0e−02	2.01E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00	2.00E+00
1.0e−04	1.88E+00	1.62E+00	2.17E+00	1.37E+00	1.80E+00	1.91E+00
1.0e−06	1.94E+00	1.96E+00	1.98E+00	1.77E+00	−3.53E−03	2.09E+00
1.0e−08	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e−10	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e−11	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
1.0e−12	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00
r_n	1.94E+00	1.96E+00	1.98E+00	1.99E+00	1.99E+00	2.00E+00

Table 12
Results for Example 4.2 after extrapolation (rate of convergence using FOFDM-II), $n_k = 8 \times 2^{k-1},\ k = 1(1)6$

ε	r_1	r_2	r_3	r_4	r_5	r_6
1.0e−01	3.96E+00	3.99E+00	4.00E+00	4.00E+00	3.96E+00	2.06E+00
1.0e−02	3.58E+00	3.89E+00	3.96E+00	3.99E+00	4.00E+00	3.98E+00
1.0e−04	−1.86E−01	−8.14E−01	4.38E−01	3.20E+00	3.89E+00	3.97E+00
1.0e−06	3.88E+00	3.93E+00	4.00E+00	−8.18E+00	8.74E−01	−2.32E+00
1.0e−08	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	7.26E−02
1.0e−10	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
1.0e−11	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
1.0e−12	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00
r_n^{ext}	3.88E+00	3.93E+00	3.96E+00	3.98E+00	3.99E+00	3.99E+00

Furthermore, we compute $e_n := \max_{0<\varepsilon\leqslant 1} e_{\varepsilon,n}$ and $e_n^{\text{ext}} = \max_{0<\varepsilon\leqslant 1} e_{\varepsilon,n}^{\text{ext}}$ whereas the numerical rate of uniform convergence is computed as $r_n := \log_2(e_n/e_{2n})$ and $r_n^{\text{ext}} := \log_2(e_n^{\text{ext}}/e_{2n}^{\text{ext}})$. (Note that the negative entries in some of the tables for rates of convergence are due to the fact that the round-off errors propagate which can be seen from the corresponding entries in the error tables.)

5. Conclusion

In this paper, we have investigated the performance of Richardson extrapolation on some fitted operator finite difference methods. We considered two FOFDMs referred to as FOFDM-I and FOFDM-II which were designed to solve a class of self-adjoint problems in [19,12], respectively. These methods are analyzed for convergence (where the solution before and after extrapolation is used to derive the error estimates).

Richardson extrapolation does not improve the convergence of FOFDM-I which is of order four and two for some moderate and smaller values of ε, respectively. In the case of FOFDM-II, its second order accuracy is improved up to four, irrespective of the value of ε. The observations (made through Tables 1–12) and the associated analysis show that the performance of Richardson extrapolation is scheme dependent.

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