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COMPLETED RICHARDSON EXTRAPOLATION IN SPACE AND TIME

SHANE A. RICHARDS*
Department of Applied Mathematics, The University of Adelaide, SA 5005, Australia

SUMMARY

The technique of Richardson extrapolation, which has previously been used on time-independent problems, is extended so that it can also be used on time-dependent problems. The technique presented is completed in the sense that the extrapolated solution is calculated at all spatial grid nodes which coincide with nodes of the finest grid considered. Numerical examples are presented when the technique is applied to the Lax–Wendroff and Crank–Nicholson finite difference schemes which are used to approximate solutions to the convection–diffusion equation. The examples show that extrapolation can be an easy and efficient way in which to produce accurate numerical solutions to time-dependent problems. © 1997 by John Wiley & Sons, Ltd.

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INTRODUCTION

Richardson extrapolation is a well-known simple technique which can be used to increase the rate of convergence of a number of numerical schemes. In particular, it can be applied to the class of numerical schemes which produce approximate solutions to either time-independent ordinary or partial differential equations on a grid. This class includes the method of finite differences, finite elements and finite volumes. Richardson extrapolation can be briefly described as follows. A numerical scheme is used to calculate the unknown true solution on two grids (typically uniform) of differing mesh size. A grid is referred to as the arrangement of points, called grid nodes, where the solution domain is spatially discretized and the true solution approximated. The finer grid is constructed by bisecting the coarser grid and hence all coarse grid nodes correspond in space with a fine grid node. Provided the coarser grid is initially 'fine' enough the solution calculated on the finer of the two grids will be a more accurate representation of the true solution. Where nodes of both grids coincide a solution which is more accurate than the fine grid solution is calculated. This is achieved by knowing the rate at which the numerical scheme used on both grids converges to the true solution and extrapolating a new solution from the coarse and fine grid solutions. Richardson extrapolation requires that the true solution be a smooth continuous function.

* Correspondence to: S. A. Richards, Institute of Marine Ecology, Building A11, University of Sydney, NSW 2006, Australia.

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The extrapolated solution is only calculated at the coarse grid nodes. Roache and Knupp[1] extended the technique so that the extrapolated solution could also be calculated at the intermediate fine grid nodes. This was then referred to as completed Richardson extrapolation. In this paper the technique is extended one step further where it can also be applied to time-dependent partial differential equations. The technique is also made more general by allowing the fine grid to be finer than just a bisection of the coarse grid. The extensions presented here are as simple as those presented in Roache and Knupp[1] and the result is a technique which is easy to implement.

METHOD

Here we consider problems involving one spatial dimension and denote the exact solution by $\phi(x, t)$. It is assumed that the problem is described by well-posed initial and boundary conditions. The exact solution is also assumed to be smooth and continuous and the solution domain defined by $0 \leqslant x \leqslant 1$ and $t \geqslant 0$. A uniform fine grid is constructed such that there are $I + 1$ nodes in the x-direction separated by $\Delta x = 1/I$. Nodes are separated in the t-direction by Δt. The (i, n) node is then set to correspond to the point $(i\Delta x, n\Delta t)$. A uniform coarse grid is constructed such that it overlaps the fine in the following way. Nodes are separated in x by $m\Delta x$ and in t by $m^{\gamma}\Delta t$, where m and γ are positive integers. The integer I is made a multiple of m so that the coarse grid exactly covers the spatial domain. The nodes of the coarse grid then coincide in space and time with the $(mi, m^{\gamma}n)$ nodes of the fine grid. The numerical approximation on the fine grid at node (i, n) is denoted $\phi^n_{f,i}$. The coarse grid approximations are denoted $\phi^n_{c,i}$, where the subscript i is a multiple of m and the superscript n a multiple of m^{γ}. An example of a coarse and fine grid defined by the parameters $I = 10$, $m = 2$ and $\gamma = 2$ is given in Figure 1.

Consider a numerical scheme with truncation error denoted $E_T(\phi, \Delta x, \Delta t)$. The truncation error is defined as the difference between the desired governing partial differential equation to which a solution is sought and the actual partial differential equation to which the numerical

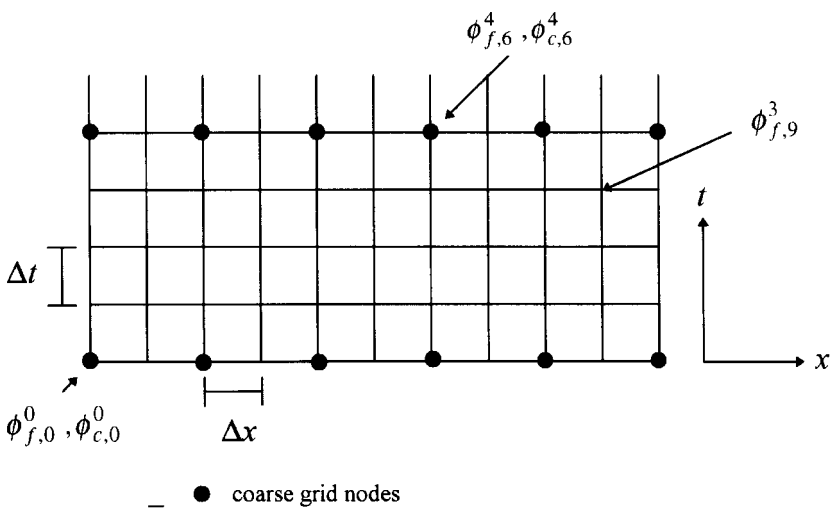

Figure 1. Example of a fine and coarse grid defined by $I = 10$, $m = 2$ and $\gamma = 2$

scheme provides a solution.[2] The truncation error is dependent on the unknown ϕ, prescribed coefficients of the governing partial differential equation (i.e. terms involving advection, diffusion, decay, sources and sinks) and the spatial and temporal spacings of the grid nodes.[2] The truncation error is assumed to have the following form:

$$E_T(\phi, \Delta x, \Delta t) = \mathrm{O}[(\Delta x)^{\alpha_x}, (\Delta t)^{\alpha_t}] = (\Delta x)^{\alpha_x}\Gamma_x + (\Delta t)^{\alpha_t}\Gamma_t + \mathrm{O}[(\Delta x)^{\beta_x}, (\Delta t)^{\beta_t}] \tag{1}$$

where α_x, α_t, β_x and β_t are positive integers such that $\beta_x > \alpha_x$ and $\beta_t > \alpha_t$. The terms Γ_x and Γ_t are generally complicated functions involving ϕ and its partial derivatives. One way to remove these two terms from the truncation error and hence produce a scheme with a higher rate of convergence is to calculate their form, often by intense algebraic manipulation, and then discretize them. This is tedious and will result in a cumbersome scheme with a stability condition which may be difficult to determine. An alternative and much simpler way to remove these two terms is to use an approach similar to Richardson extrapolation, which is now described in detail. It can be shown that the error produced by a numerical scheme over a time step of length Δt is of the form $\Delta t E_T$. Assume the exact solution is known at the nth time level; then the error in the calculation of ϕ at the $(i, n+1)$ fine grid node is

$$\phi(i\Delta x, (n+1)\Delta t) - \phi_{f,i}^{n+1} = \Delta t E_T = \Delta t(\Delta x)^{\alpha_x}\Gamma_{x,i}^n + (\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n + \mathrm{O}[\Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\beta_t+1}] \tag{2}$$

where the subscript i and superscript n indicate that the function is evaluated at the (i, n) fine grid node. If i is chosen to be a multiple of m then the error produced after one time step on the coarse grid is

$$\begin{aligned}\phi(i\Delta x, (n+m^\gamma)\Delta t) - \phi_{c,i}^{n+m^\gamma} &= m^\gamma \Delta t(m\Delta x)^{\alpha_x}\Gamma_{x,i}^n + (m^\gamma\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n + \mathrm{O}[\Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\beta_t+1}] \\ &= m^{\alpha_x+\gamma}\Delta t(\Delta x)^{\alpha_x}\Gamma_{x,i}^n + m^{\gamma(\alpha_t+1)}(\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n + \mathrm{O}[\Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\beta_t+1}]\end{aligned} \tag{3}$$

Assume that for a few subsequent time steps on the fine grid the errors produced are of the same magnitude as the one produced from the nth to the $(n+1)$th time level. By this assumption the subsequent errors are approximated to $\mathrm{O}(\Delta t)$. The error contribution at each time step can be assumed to be approximately cumulative. The error on the fine grid at the $n + m^\gamma$ time level is

$$\begin{aligned}\phi(i\Delta x, (n+m^\gamma)\Delta t) - \phi_{f,i}^{n+m^\gamma} &= m^\gamma \Delta t(\Delta x)^{\alpha_x}\Gamma_{x,i}^n + m^\gamma(\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n \\ &\quad + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}]\end{aligned} \tag{4}$$

With two approximations to ϕ coinciding with the $(i, n+m^\gamma)$ node, $\Gamma_{x,i}^n$ and $\Gamma_{t,i}^n$ may be eliminated from (4) by considering

$$\begin{aligned}\phi_{R,i}^{n+m^\gamma} &= \frac{a\phi_{f,i}^{n+m^\gamma} - b\phi_{c,i}^{n+m^\gamma}}{a-b} = \phi(i\Delta x, (n+m^\gamma)\Delta t) + \frac{m^\gamma}{a-b}(bm^{\alpha_x} - a)\Delta t(\Delta x)^{\alpha_x}\Gamma_{x,i}^n \\ &\quad + \frac{m^\gamma}{a-b}(bm^{\gamma\alpha_t} - a)(\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}]\end{aligned} \tag{5}$$

where a and b are constants. If both leading error terms are to be removed, then the following must hold: $\alpha_x = \gamma\alpha_t$. Thus, extrapolation can be applied if the coarse and the fine grid are related by $\gamma = \alpha_x/\alpha_t$. The extrapolated solution is then calculated by

$$\begin{aligned}\phi_{R,i}^{n+m^\gamma} &= \frac{m^{\alpha_x}\phi_{f,i}^{n+m^\gamma} - \phi_{c,i}^{n+m^\gamma}}{m^{\alpha_x} - 1} \\ &= \phi(i\Delta x, (n+m^\gamma)\Delta t) + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}]\end{aligned} \tag{6}$$

The improved solution at the coarse grid nodes has a rate of convergence comparable to a scheme with a truncation error of at worst $\mathrm{O}[\Delta t(\Delta x)^{\alpha_x}, (\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+1}, (\Delta t)^{\beta_t}]$.

Thus far the technique of extrapolation has been used to produce a more accurate solution at coarse grid nodes only. The technique is now extended so that a solution with a high rate of convergence is calculated at the remaining fine grid nodes which lie on a coarse grid time level.

Let

$$\Psi_i^n = m^\gamma \Delta t(\Delta x)^{\alpha_x}\Gamma_{x,i}^n + m^\gamma(\Delta t)^{\alpha_t+1}\Gamma_{t,i}^n \tag{7}$$

The fine grid error at a coarse time level given by (4) can be written

$$\phi(i\Delta x, (n+m^\gamma)\Delta t) - \phi_{f,i}^{n+m^\gamma} = \Psi_i^n + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}] \tag{8}$$

The extrapolated solution at coarse grid nodes has been achieved by eliminating Ψ_i^n from (8), where i is a multiple of m. To derive an extrapolated solution at the interior fine grid nodes the unknown Ψ_i^n (i.e. i not a multiple of m) are approximated from the extrapolated approximations to Ψ_i^n at coarse grid nodes (i.e. i a multiple of m). Linear interpolation of Ψ in x gives

$$\Psi_{i+j}^n = \frac{1}{m}[(m-j)\Psi_i^n + j\Psi_{i+m}^n] + \mathrm{O}[\Delta t(\Delta x)^{\alpha_x+1}, (\Delta t)^{\alpha_t+1}\Delta x] \tag{9}$$

for i a multiple of m and $j = 0, 1, \ldots, m$. Substituting (6) into (8) gives the coarse grid approximations for Ψ, namely

$$\Psi_i^n = \phi_{R,i}^{n+m^\gamma} - \phi_{f,i}^{n+m^\gamma} + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}] \tag{10}$$

where i is a multiple of m. By combining (8)–(10) an extrapolated solution at the fine grid nodes can be made,

$$\phi_{R,i+j}^{n+m^\gamma} = \phi_{f,i+j}^{n+m^\gamma} + \frac{1}{m}[(m-j)(\phi_{R,i}^{n+m^\gamma} - \phi_{f,i}^{n+m^\gamma}) + j(\phi_{R,i+m}^{n+m^\gamma} - \phi_{f,i+m}^{n+m^\gamma})] \tag{11}$$

The error in the above approximation has the form

$$\begin{aligned}\phi((i+j)\Delta x, (n+m^\gamma)\Delta t) = \phi_{R,i+j}^{n+m^\gamma} \\ + \mathrm{O}[(\Delta t)^2(\Delta x)^{\alpha_x}, \Delta t(\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+2}, (\Delta t)^{\beta_t+1}, \Delta t(\Delta x)^{\alpha_x+1}, (\Delta t)^{\alpha_t+1}\Delta x]\end{aligned} \tag{12}$$

Thus the new extrapolated solution, given by (11), has a rate of convergence comparable to a single grid scheme with a truncation error of at worst $\mathrm{O}[\Delta t(\Delta x)^{\alpha_x}, (\Delta x)^{\beta_x}, (\Delta t)^{\alpha_t+1}, (\Delta t)^{\beta_t}, (\Delta x)^{\alpha_x+1}, (\Delta t)^{\alpha_t}\Delta x]$.

One way to use the technique of extrapolation is to update both the coarse and fine grid solutions with the extrapolated solution after each coarse time step. Suppose a solution is required at $t = T$ (say) which for the numerical scheme being used can be calculated with N time steps on the coarse grid. An algorithm which implements the technique of extrapolation is given in Figure 2. It is not necessary to compare the coarse and fine grid solutions after each coarse grid time step to produce a higher-order convergent solution as the above reasoning is equally applicable when comparing solutions after any number of coarse time steps. For example, the extrapolated calculations may be performed just once after the N coarse time steps. The algorithm in Figure 2 would then be modified by moving the extrapolation calculations outside the coarse grid loop. One would expect, though, that the technique of extrapolation would be most effective at improving the accuracy of the numerical solution at time $t = T$ when it is applied after each coarse time step.

```
initialize the fine and coarse grid
for n_c = 1 to N do begin
    iterate one time step on the coarse grid
    for n_f = 1 to m^γ do
        iterate one time step on the fine grid
    for each coarse grid node do
        calculate the extrapolated solution using (6)
    for each fine grid node do
        calculate the extrapolated solution using (11)
    transfer the extrapolated solution to the coarse grid
    transfer the extrapolated solution to the fine grid
end
```

Figure 2. Algorithm describing the technique of completed Richardson extrapolation in space and time with extrapolation performed after each coarse grid time step

TEST EXAMPLES

The technique of extrapolation is now illustrated by considering the well-known convection–diffusion equation with time-dependent coefficients, namely

$$\frac{\partial \phi}{\partial t} + u(t)\frac{\partial \phi}{\partial x} = D(t)\frac{\partial^2 \phi}{\partial x^2} \tag{13}$$

A solution which satisfies the condition

$$\lim_{x \to \pm\infty} \phi(x, t) = 0 \tag{14}$$

for all $t \geqslant 0$ is

$$\phi(x, t) = \frac{\phi_0}{\sqrt{[4\pi T(t, t_0)]}} \exp\left\{-\frac{[x - U(t) - x_0]^2}{4T(t, t_0)}\right\} \tag{15}$$

$$T(t, t_0) = \int_{t_0}^{t} D(\tau)\, \mathrm{d}\tau \tag{16}$$

$$U(t) = \int_{0}^{t} u(\tau)\, \mathrm{d}\tau \tag{17}$$

where ϕ_0, x_0 and $t_0 < 0$ are constants. For the examples presented here the solution domain is defined by $0 \leqslant x \leqslant 1$ and $0 \leqslant t \leqslant 1$, with $\phi_0 = 1$, $x_0 = 1/4$, $t_0 = -1/10$ and

$$u(t) = \frac{1}{4} \exp t \tag{18}$$

$$D(t) = \frac{1}{100} \exp t \tag{19}$$

The initial condition and Dirichlet boundary conditions are provided by the exact solution. The solution to (13) is a Gaussian peak which moves at an exponentially increasing rate in the positive x-direction and diffuses at an exponentially increasing rate.

The Lax–Wendroff finite difference scheme can be used to approximate a solution to (13), and is described by

$$\phi_i^{n+1} = \frac{1}{2}[2s^n + (c^n)^2 + c^n]\phi_{i-1}^n + [1 - (c^n)^2 - 2s^n]\phi_i^n + \frac{1}{2}[2s^n + (c^n)^2 - c^n]\phi_{i+1}^n \tag{20}$$

where

$$c^n = \frac{\Delta t}{\Delta x} u(n\Delta t) \tag{21}$$

$$s^n = \frac{\Delta t}{(\Delta x)^2} D(n\Delta t) \tag{22}$$

This scheme is stable when

$$0 < s^n < \frac{1}{2}[1 - (c^n)^2] \tag{23}$$

holds true for all time steps. The truncation error can also be shown to be of $O[\Delta t, (\Delta x)^2]$;[3] hence $\alpha_x = 2$ and $\alpha_t = 1$. Thus to use the technique of extrapolation with the Lax–Wendroff scheme requires that the two grids be related by $\gamma = 2$. Note that the Lax–Wendroff scheme will be stable on the fine grid if it is also stable on the coarse grid.

The root mean square errors (E_{RMS}) at the final time level ($T = 1$) produced by the Lax–Wendroff and the extrapolated Lax–Wendroff scheme are presented in Table I for various grid node spacings. Also presented is the approximate number of times the nodal calculation (20) is performed to obtain the numerical solution. By examining the rate at which the error decreases with decreasing grid size the rate of convergence for both schemes can be estimated. The bracketed values presented in Table I indicate the factor by which E_{RMS} decreases when Δx is decreased by m and Δt by m^2. For example, the entry (3·76) in row 2 of Table I(i) is calculated from the error measures in rows 1 and 2, i.e. $(1{\cdot}72 \times 10^{-2}/4{\cdot}58 \times 10^{-3})$. The entry (3·95) in row 3 is calculated from $(4{\cdot}58 \times 10^{-3}/1{\cdot}16 \times 10^{-3})$ etc. For a scheme that is converging at a rate of $O[\Delta t, (\Delta x)^2]$ we would expect these ratios to be near m^2, and for a scheme that is converging at a rate of $O[(\Delta t)^2, (\Delta x)^4]$ the ratios will be closer to m^4. When the Lax–Wendroff scheme is used on a single grid the numerical solution converges to the true solution at a rate of $O[\Delta t, (\Delta x)^2]$. This is evident from the error ratios being close to 4 in Table I(i) and 9 in Table I(iii).

The error in the final solution when four variations of extrapolation are performed (denoted a–d) are shown in Tables I(ii) and I(iv). The variations are: (a) extrapolation is only performed

Table I. Error measures for the Lax–Wendroff and extrapolated Lax–Wendroff calculations when the coarse and fine grids are related by $\gamma = 2$. See the text for a description of the type of extrapolation, indicated by (a)–(d). Note $\Delta x = 1/I$ and $\Delta t = 1/N$. The bracketed terms indicate the factor by which the error measure has been reduced by reducing the mesh size

(i) No extrapolation performed ($m = 2$)

Grid spacing I	N	E_{RMS}	Node evaluations
20	40	$1{\cdot}72 \times 10^{-2}$	800
40	160	$4{\cdot}58 \times 10^{-3}$ (3·76)	6400
80	640	$1{\cdot}16 \times 10^{-3}$ (3·95)	51200
160	2560	$2{\cdot}92 \times 10^{-4}$ (3·97)	409600
Suggested rate of convergence		$O\{\Delta t, (\Delta x)^2\}$	

(ii) Extrapolation is performed on the solution ($m = 2$). The grid spacing of the finer of the two grids is indicated

Grid spacing I	N	E_{RMS} (a)	E_{RMS} (b)	E_{RMS} (c)	E_{RMS} (d)	Node evaluations
40	160	$4{\cdot}71 \times 10^{-4}$	$6{\cdot}00 \times 10^{-3}$	$5{\cdot}45 \times 10^{-4}$	$5{\cdot}03 \times 10^{-4}$	7200
80	640	$2{\cdot}96 \times 10^{-5}$ (15·91)	$1{\cdot}52 \times 10^{-3}$ (3·95)	$3{\cdot}51 \times 10^{-5}$ (15·53)	$3{\cdot}22 \times 10^{-5}$ (15·62)	57600
160	2560	$1{\cdot}85 \times 10^{-6}$ (16·00)	$3{\cdot}81 \times 10^{-4}$ (3·99)	$2{\cdot}21 \times 10^{-6}$ (15·88)	$2{\cdot}02 \times 10^{-6}$ (15·94)	460800
Suggested rate of convergence		$O\{(\Delta t)^2, (\Delta x)^4\}$	$O\{\Delta t, (\Delta x)^2\}$	$O\{(\Delta t)^2, (\Delta x)^4\}$	$O\{(\Delta t)^2, (\Delta x)^4\}$	

(iii) No extrapolation performed ($m = 3$)

Grid spacing I	N	E_{RMS}	Node evaluations
20	50	$2{\cdot}18 \times 10^{-2}$	1000
60	450	$2{\cdot}52 \times 10^{-3}$ (8·65)	27000
180	4050	$2{\cdot}81 \times 10^{-4}$ (8·97)	729000
540	36450	$3{\cdot}12 \times 10^{-5}$ (9·01)	19683000
Suggested rate of convergence		$O\{\Delta t, (\Delta x)^2\}$	

(iv) Extrapolation is performed on the solution ($m = 3$). The grid spacing of the finer of the two grids is indicated

Grid spacing I	N	E_{RMS} (a)	E_{RMS} (b)	E_{RMS} (c)	E_{RMS} (d)	Node evaluations
60	450	$1{\cdot}60 \times 10^{-4}$	$6{\cdot}15 \times 10^{-3}$	$2{\cdot}11 \times 10^{-4}$	$1{\cdot}75 \times 10^{-4}$	28000
180	4050	$2{\cdot}03 \times 10^{-6}$ (78·82)	$6{\cdot}95 \times 10^{-4}$ (8·85)	$2{\cdot}72 \times 10^{-6}$ (77·57)	$2{\cdot}22 \times 10^{-6}$ (78·83)	756000
540	36450	$2{\cdot}51 \times 10^{-8}$ (80·88)	$7{\cdot}74 \times 10^{-5}$ (8·98)	$3{\cdot}38 \times 10^{-8}$ (80·47)	$2{\cdot}75 \times 10^{-8}$ (80·73)	20412000
Suggested rate of convergence		$O\{(\Delta t)^2, (\Delta x)^4\}$	$O\{\Delta t, (\Delta x)^2\}$	$O\{(\Delta t)^2, (\Delta x)^4\}$	$O\{(\Delta t)^2, (\Delta x)^4\}$	

on the coarse grid nodes using (6) at the final time level; (b) as (a) but now the solution is also calculated at fine grid nodes by assuming a piece-wise linear form between coarse grid extrapolated values; (c) the completed extrapolation scheme described by (6) and (11) is applied once at the final coarse time level; (d) the completed extrapolation scheme is applied after every coarse time step (as described by the algorithm presented in Figure 2). The error ratios indicate that the extrapolated solutions (a), (c) and (d) converge to the exact solution at a rate of $O[(\Delta t)^2, (\Delta x)^4]$. Assuming a piece-wise linear form for the extrapolated solution when calculating fine grid values, i.e. scheme (b), does not preserve the overall high rate of convergence. Also, when applying the completed extrapolation scheme after every coarse time step rather than only at the final time step a more accurate solution is produced, although in both cases the solution converges at a comparable rate.

The extrapolated solution is always more accurate than the single grid solution when a similar sized mesh is used. The efficiency of the completed extrapolation scheme can be seen by noting the approximate number of node evaluations required to compute a solution. For example, to produce a solution having a root mean square error of around 3×10^{-5} requires nearly 20 million node evaluations when a single grid is used, but the same error can be achieved with less than 60,000 node evaluations when two grids are used and their solutions combined. It should be noted that the approximate number of node evaluations required can only be used as a rough guide to the time taken to compute a solution. The time taken to evaluate a node with the extrapolated scheme is slightly longer as it also includes the time taken to perform the extrapolations, i.e. the extra calculations (6) and (11). However, in practice these extra extrapolation calculations are often quick when compared with the numerical scheme calculations, particularly when the governing problem contains coefficients which have complicated functional forms.

Alternatively the solution to (13) can be approximated by the implicit Crank–Nicolson finite difference scheme, namely

$$\begin{aligned} &-(2s^{n+1/2}+c^{n+1/2})\phi_{i-1}^{n+1}+4(1+s^{n+1/2})\phi_i^{n+1}-(2s^{n+1/2}-c^{n+1/2})\phi_{i+1}^{n+1}\\ &\quad=(2s^{n+1/2}+c^{n+1/2})\phi_{i-1}^{n}+4(1-s^{n+1/2})\phi_i^{n}+(2s^{n+1/2}-c^{n+1/2})\phi_{i+1}^{n} \end{aligned} \tag{24}$$

This scheme is unconditionally stable and diagonally dominant when

$$0<\frac{1}{2}(c^{n+1/2}-2)\leqslant s^{n+1/2} \tag{25}$$

The truncation error can be shown to be of $O[(\Delta t)^2, (\Delta x)^2]$.[3] If the technique of extrapolation is to be applied to the Crank–Nicolson scheme, then the coarse and fine grid must be related by $\gamma = 1$. Again, stability will be ensured on the fine grid if it is also stable on the coarse grid.

Numerical results are presented in Table II. In this example the bracketed values represent the factor by which E_{RMS} decreases when both Δx and Δt are decreased by a factor of m. For a scheme which converges at a rate of $O[(\Delta t)^2, (\Delta x)^2]$ these ratios will be near m^2 and for a rate of convergence of $O[(\Delta t)^4, (\Delta x)^4]$ near m^4. The error ratios indicate that the Crank–Nicolson scheme when used on a single grid converges at a rate of $O[(\Delta t)^2, (\Delta x)^2]$ as expected, and the extrapolated solutions (a), (c) and (d) converge at a rate of $O[(\Delta t)^4, (\Delta x)^4]$. These results are similar in form to those of the Lax–Wendroff example and again indicate that the technique of extrapolation can produce a higher-order convergent solution than the single grid scheme at relatively little extra computational cost.

Table II. Error measures for the Crank–Nicolson and extrapolated Crank–Nicolson calculations when the coarse and fine grids are related by $\gamma = 1$. See the text for a description of the type of extrapolation, indicated by (a)–(d). Note $\Delta x = 1/I$ and $\Delta t = 1/N$. The bracketed terms indicate the factor by which the error measure has been reduced by reducing the mesh size

(i) No extrapolation performed ($m = 2$)

Grid spacing I	N	E_{RMS}	Node evaluations
20	20	$4{\cdot}79 \times 10^{-2}$	400
40	40	$1{\cdot}19 \times 10^{-2}$ (4·03)	1600
80	80	$2{\cdot}96 \times 10^{-3}$ (4·02)	6400
160	160	$7{\cdot}39 \times 10^{-4}$ (4·01)	25600
Suggested rate of convergence		$O\{(\Delta t)^2, (\Delta x)^2\}$	

(ii) Extrapolation is performed on the solution ($m = 2$). The grid spacing of the finer of the two grids is indicated

Grid spacing I	N	E_{RMS} (a)	E_{RMS} (b)	E_{RMS} (c)	E_{RMS} (d)	Node evaluations
40	40	$1{\cdot}63 \times 10^{-3}$	$5{\cdot}64 \times 10^{-3}$	$1{\cdot}63 \times 10^{-3}$	$5{\cdot}46 \times 10^{-4}$	2000
80	80	$9{\cdot}80 \times 10^{-5}$ (16·63)	$1{\cdot}49 \times 10^{-3}$ (3·79)	$1{\cdot}00 \times 10^{-4}$ (16·30)	$3{\cdot}57 \times 10^{-5}$ (15·29)	8000
160	160	$6{\cdot}04 \times 10^{-6}$ (16·23)	$3{\cdot}79 \times 10^{-4}$ (3·93)	$6{\cdot}21 \times 10^{-6}$ (16·10)	$2{\cdot}27 \times 10^{-6}$ (15·73)	32000
Suggested rate of convergence		$O\{(\Delta t)^4, (\Delta x)^4\}$	$O\{(\Delta t)^2, (\Delta x)^2\}$	$O\{(\Delta t)^4, (\Delta x)^4\}$	$O\{(\Delta t)^4, (\Delta x)^4\}$	

(iii) No extrapolation performed ($m = 3$)

Grid spacing I	N	E_{RMS}	Node evaluations
20	20	$4{\cdot}79 \times 10^{-2}$	400
60	60	$5{\cdot}26 \times 10^{-3}$ (9·11)	3600
180	180	$5{\cdot}84 \times 10^{-4}$ (9·01)	32400
540	540	$6{\cdot}48 \times 10^{-5}$ (9·01)	291600
Suggested rate of convergence		$O\{(\Delta t)^2, (\Delta x)^2\}$	

(iv) Extrapolation is performed on the solution ($m = 3$). The grid spacing of the finer of the two grids is indicated

Grid spacing I	N	E_{RMS} (a)	E_{RMS} (b)	E_{RMS} (c)	E_{RMS} (d)	Node evaluations
60	60	$7{\cdot}18 \times 10^{-4}$	$5{\cdot}95 \times 10^{-3}$	$7{\cdot}20 \times 10^{-4}$	$2{\cdot}45 \times 10^{-4}$	4000
180	180	$8{\cdot}51 \times 10^{-6}$ (84·37)	$6{\cdot}92 \times 10^{-4}$ (8·60)	$8{\cdot}75 \times 10^{-6}$ (82·29)	$3{\cdot}19 \times 10^{-6}$ (76·80)	36000
540	540	$1{\cdot}04 \times 10^{-7}$ (81·83)	$7{\cdot}73 \times 10^{-5}$ (8·95)	$1{\cdot}08 \times 10^{-7}$ (81·02)	$3{\cdot}98 \times 10^{-8}$ (80·15)	324000
Suggested rate of convergence		$O\{(\Delta t)^4, (\Delta x)^4\}$	$O\{(\Delta t)^2, (\Delta x)^2\}$	$O\{(\Delta t)^4, (\Delta x)^4\}$	$O\{(\Delta t)^4, (\Delta x)^4\}$	

CONCLUSION

The two test examples presented here illustrate that completed extrapolation can be an easy and efficient way in which to produce high-order convergent numerical solutions to time-dependent problems. This technique can be applied to schemes which produce a discretized solution. Knowledge of the rate of convergence of the leading error terms in the truncation error is required in order to set up an appropriate ratio of nodal spacings in space and time. If the rate of convergence is unknown, then it can often be deduced by observing the rate at which errors are reduced when the grid size is reduced for problems where an exact solution is known. Stability of the numerical scheme considered may not be guaranteed on the finer of the two grids even if it is stable on the coarse grid. In both examples presented here there was no problem with loss of stability, but this may not always be the case and so must be checked. The technique of extrapolation can be extended to spatially two-dimensional problems in a similar manner as presented here.

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