VIII. *The Deferred Approach to the Limit.*

*Part I.—Single Lattice.**—*By* LEWIS F. RICHARDSON, *F.R.S.*

*Part II.—Interpenetrating Lattices.**—*By* J. ARTHUR GAUNT, B.A.

PART I.

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* These phrases are explained in Part I, § 2, and Part II, § 1.

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§ 1. Introduction.

Various problems concerning infinitely many, infinitely small, parts, had been solved before the infinitesimal calculus was invented; for example, Archimedes on the circumference of the circle.* The essence of the invention of the calculus appears to be that the passage to the limit was thereby taken at the earliest possible stage, where diverse problems had operations like d/dx in common. Although the infinitesimal calculus has been a splendid success, yet there remain problems in which it is cumbrous or unworkable. When such difficulties are encountered it may be well to return to the manner in which they did things before the calculus was invented, postponing the passage to the limit until after the problem had been solved for a moderate number of moderately small differences.

For obtaining the solution of the difference-problem a variety of arithmetical processes are available. This memoir deals with central differences arranged in the simplest possible way, namely, that explained by the writer in the papers cited in the footnote.† Advancing differences are ignored, and so are the varieties of central-difference-process in which accuracy is gained by complicating the arithmetic at an early stage.

Confining attention to problems involving a single independent variable x, let h be the "step," that is to say, the difference of x which is used in the arithmetic, and let $\phi(x, h)$ be the solution of the problem in differences. Let $f(x)$ be the solution of the analogous problem in the infinitesimal calculus. It is $f(x)$ which we want to know, and $\phi(x, h)$ which is known for several values of h. A theory, published in 1910,‡ but too brief and vague, has suggested that, if the differences are "centred" then

$$\phi(x, h) = f(x) + h^2 f_2(x) + h^4 f_4(x) + h^6 f_6(x) \dots \text{to infinity} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

odd powers of h being absent. The functions $f_2(x), f_4(x), f_6(x)$ are usually unknown. Numerous arithmetical examples have confirmed the absence of odd powers, and have shown that it is often easy to perform the arithmetic with several values of h so small that $f(x) + h^2 f_2(x)$ is a good approximation to the sum to infinity of the series in (1).

* Many other arcs, areas and volumes are mentioned in 'Ency. Brit.,' IX Edn., vol. 13, pp. 5 to 8.

† For Differential equations—L. F. Richardson, 'Phil. Trans.,' A, vol. 210, pp. 307, 314, § 1, § 2. See especially a summary in the 'Mathematical Gazette' for July, 1925. For an Integral Equation, see 'Phil. Trans.,' A, vol. 223, p. 361.

‡ 'Phil. Trans.,' A., vol. 210, pp. 310, 311, § 1.2.

If generally true, this would be very useful, for it would mean that if we have found two solutions for unequal steps h_1, h_2, then by eliminating $f_2(x)$ we would obtain the desired $f(x)$ in the form

$$f(x) = \frac{h_2^2 \phi(x, h_1) - h_1^2 \phi(x, h_2)}{h_2^2 - h_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

This process represented by the formula (2) will be named the "h^2-extrapolation"

DEFN.

If the difference problem has been solved for three unequal values of h it is possible to write three equations of the type (1) for h_1, h_2, h_3, retaining the term $h^4 f_4(x)$. Then $f(x)$ is found by eliminating both $f_2(x)$ and $f_4(x)$. This process will be named the "h^4-extrapolation."

The purpose of this investigation is to examine the h^2-extrapolation anew, looking for exceptions and qualifications. Two questions pervade Part I:—

(1) Are the odd powers of h always absent?

(2) How small must h be in order that the h^2-extrapolation may make an improvement?

The analysis has been found to be complicated, in contrast with the simplicity of the arithmetical practice. The method employed is to obtain a formula showing $\phi(x, h)$ as a quite explicit function of h. An analogous formula is obtained for $f(x)$, under the restriction that $f(x)$ must have x-derivatives of all orders at every point of the range. The question whether $\operatorname{Limit}_{h \to 0} \phi(x, h)$ is equal or not to $f(x)$ is decided, usually in the affirmative, by comparing the two analogous formulæ, so that there is no need to bring in the method of LIPSCHITZ.*

Problems involving differential equations will be divided into the "marching" and "jury" varieties. These words† have been used in the following sense: A "marching" problem is one in which the arithmetical solution can be stepped out from one end of the range of x. A "jury" problem is one in which the solution must be determined by reference to both ends of the range considered together just as the verdict has to satisfy all the jurymen together.

The third class of problem that will be treated is VOLTERRA's integral equation of the first kind. Various common interests such as the properties of μ and δ are treated first.

The deferred approach to the limit has also been considered by N. BOGOLOUBOFF and N. KRYLOFF in a recent paper,‡ in Russian.

* GOURSAT, 'Cours d'Analyse' (1925), tome 2, art. 391.

† See "Weather Prediction by Numerical Process" (Camb. Press), p. 3; also 'Math. Gazette,' July, 1925.

‡ N. BOGOLOUBOFF and N. KRYLOFF, "On the RAYLEIGH's principle in the theory of the differential equations of the mathematical physic and upon the EULER's method in the calculus of variations," 'Acad. des Sci. de l'Ukraine, Classe, Phys. Math.,' tome 3, fasc. 3 (1926).

§ 2. Replacement of Derivatives by Difference-Ratios.

If the problem is already expressed in the language of the calculus so as to determine a function $f(x)$, the first process is to replace every derivative of $f(x)$ by a difference-ratio of which it is the limit. The function whose differences are taken will be denoted by $\phi(x, h)$. The replacement could be done in a variety of ways, but the only one used in Part I will be that in which the derivative $f'(x)$ is replaced by the central difference ratio

$$\{\phi(x+\tfrac{1}{2}h) - \phi(x-\tfrac{1}{2}h)\}/h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The notation may be shortened by the use of the operators δ and μ defined, in accordance with Sheppard,* to be such that if $\phi(x)$ is any function of x, then

$$\delta\phi(x) = \phi(x+\tfrac{1}{2}h) - \phi(x-\tfrac{1}{2}h), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$\mu\phi(x) = \tfrac{1}{2}\{\phi(x+\tfrac{1}{2}h) + \phi(x-\tfrac{1}{2}h)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

A most important property of δ and μ proved by Sheppard is that they follow the ordinary laws of algebra, so that for instance

$$\mu\delta\phi(x) = \delta\mu\phi(x) \text{ and } \mu\{\phi(x) + \psi(x)\} = \mu\phi(x) + \mu\psi(x).$$

We must always think of $\phi(x)$ as a numerical table with step h in x. If $\phi(a)$ is a number in this table, then we cannot compute $\delta\phi(a)$ in any simple way, because $\delta\phi(a)$ involves $\phi(a+\frac{1}{2}h)$ and $\phi(a-\frac{1}{2}h)$ neither of which are tabulated.

Two possible ways of arranging the arithmetic need to be distinguished. Borrowing words from crystallography they may be named the methods of the single lattice and of the interpenetrating lattices.

Interpenetrating lattices is the name now proposed for the method, formerly called "stepover," in which there are two functions $\theta(x)$, $\phi(x)$ tabulated at alternate points thus :—

Table I.

$\phi(0)$			
$\theta(\frac{1}{2}h)$	$\delta\phi(\frac{1}{2}h)$		
$\phi(h)$	$\delta\theta(h)$	$\delta^2\phi(h)$	
$\theta(\frac{3}{2}h)$	$\delta\phi(\frac{3}{2}h)$	$\delta^2\theta(\frac{3}{2}h)$	$\delta^3\phi(\frac{3}{2}h)$
$\phi(2h)$	$\delta\theta(2h)$	$\delta^2\phi(2h)$	
$\theta(\frac{5}{2}h)$	$\delta\phi(\frac{5}{2}h)$		
$\phi(3h)$			

At any point, say, $x = \frac{3}{2}h$ there are both odd and even differences tabulated, so that when we have an equation connecting odd with even differences there is no need to use μ. The "lattice" consisting of ϕ and its differences is connected with the θ-lattice

* W. F. Sheppard, "Central Difference Formulæ," 'Lond. Math. Soc. Proc.,' vol. 31, p. 461 (1899), who refers to P. A. Hansen, 'Abhandlungen der kön. sächs. Ges.' (Leipzig), vol. xi, pp. 505–583 (1865) (vol. vii of 'Abh. der. math.-phys. Classe').

only by way of the range- and boundary-equations. The writer* has found interpenetrating lattices to be convenient in the arithmetic, but they need extra initial data, hitherto provided by various arbitrary devices, giving rise to oscillatory errors. Recently, however, Mr. J. A. GAUNT has discovered strict rules for taking the first step, which he explains in Part II.

Single lattice, the only type discussed in Part I, is that in which $\phi(x)$ is tabulated at intervals h of x, exactly as in Table I omitting θ and its differences.

The differences $\delta^n\phi$ are centred at the values of x where ϕ is tabulated when n is even, and midway between those values of x when n is odd. If we have to form an equation involving both odd and even differences, and here ϕ itself would be reckoned as a difference of zeroth even order, then it is simplest if we connect together either

$$\phi, \quad \mu\delta\phi, \quad \delta^2\phi, \quad \mu\delta^3\phi \ldots,$$

or else

$$\mu\phi, \quad \delta\phi, \quad \mu\delta^2\phi, \quad \delta^3\phi \ldots,$$

but not a mixture. This convention that μ must precede either every odd power of δ or else every even power, but not both, rules throughout Part I. . . . . (4)

Differences arranged in either of these two sequences will be called "alternating differences," and the set ϕ, $\mu\delta\phi$, $\delta^2\phi$, $\mu\delta^3\phi$ will be said to be "centered with ϕ," the other set $\mu\phi$, $\delta\phi$, $\mu\delta^2\phi$ being "centered with $\mu\phi$."

The highest derivative in the given range-equation† *will be replaced by a difference which is not modified by* μ.—To see why this is desirable let us consider here marching and jury problems. Integral equations will be discussed in § 11.

Marching Problems.—Let the highest derivative in the given range-equation be $f^{(n)}(x)$. To make a marching problem we must have given at some value $x = a$ all lower derivatives as well as $f(a)$. That is to say $f^{(n-1)}(a), \ldots f''(a), f'(a), f(a)$. Suppose that we were to put $\mu\delta^n\phi(x, h)/h^n$ in place of $f^n(x)$ and $\delta^{n-1}\phi(x, h)/h^{n-1}$ in place of $f^{(n-1)}(x)$, so that μ occurred with the highest difference in the range-equation. Now let us make a table to show what tabular values of ϕ are involved in $\mu\delta^n\phi$.

TABLE II.

x.	ϕ.	$\delta\phi$.	$\mu\delta\phi$.	$\delta^2\phi$.	$\mu\delta^2\phi$.
a	e				
$a+\frac{1}{2}h$		$f-e$			
$a+h$	f		$\frac{1}{2}(g-e)$	$g-2f+e$	
$a+\frac{3}{2}h$		$g-f$			$\frac{1}{2}(j-g-f+e)$
$a+2h$	g		$\frac{1}{2}(j-f)$	$j-2g+f$	
$a+\frac{5}{2}h$		$j-g$			
$a+3h$	j				

* "Weather Prediction by Numerical Process" (Camb. Press), ch. 7/2.

† "Range-equation" means either the differential equation to be satisfied at every point of the range or else its analogue in differences.

From the mode of formation of these differences it is seen that $\mu\delta^n\phi$ spreads itself over $n+2$ tabular values of ϕ, among which the extreme outlying values of ϕ are always present. The highest difference given on the boundary is on this scheme $\delta^{n-1}\phi(a)$ which does not involve either of the aforesaid pair of extreme outlying values. Therefore when the boundary equations are substituted in the range equation we should have only one equation which is insufficient to determine these two outliers; and no further progress could be made. For example if the range equation be $\mu\delta\phi(x, h) = 1$ and the boundary equation be $\phi(0, h) = 0$ we cannot march out the solution.

On the contrary, if the highest derivative in the range equation be $\delta^n\phi$ it spreads itself over only $n+1$ tabular values of ϕ and we have sufficient equations to find all of them. For example $\delta\phi(x, h) = 1$ and $\mu\phi(0, h) = 0$ allow us to march.

Thus, if the highest derivative in the given range equation is of even order then the centering of that equation must be with ϕ, while if the highest derivative is odd the centering of the range equation must be with $\mu\phi$ . . . . (5)

Jury Problems.—In these the boundary conditions are not all at one point $x = a$; some are at $x = b$. There is no objection to the same rule being adopted for jury problems. And it is easy to show by examples that if μ were allowed to accompany the highest power δ^n in the range equation, difficulties would arise.

Thus, for $n = 3$ suppose that the boundary conditions were

$$f(a) = 0, \qquad f'(a) = 0, \qquad f(b) = 0.$$

The corresponding statement in the difference problem is

$$\phi(a + \tfrac{1}{2}h) = 0, \qquad \phi(a - \tfrac{1}{2}h) = 0, \qquad \phi(b) = 0.$$

TABLE III.

$x - a =$	$-\frac{1}{2}h$	0	$+\frac{1}{2}h$	h	$\frac{3}{2}h$	$2h$	$\frac{5}{2}h$	$3h$	$\frac{7}{2}h$	$4h$	$\frac{9}{2}h$
		Boundary									Boundary
$\phi =$	0		0		ϕ_1		ϕ_2		ϕ_3		0
				δ^3	$\mu\delta^3$	δ^3	$\mu\delta^3$	δ^3			

These zero values of ϕ are shown in position in Table III together with unknown "body-values" ϕ_1, ϕ_2, ϕ_3. The number of these is unimportant provided we know it. The marks δ^3 and $\mu\delta^3$ are placed in all the positions at which the corresponding range- (alias "body"-) equation can be centered. If δ^3 is used there are three body-equations to determine ϕ_1, ϕ_2, ϕ_3, that is just enough. If $\mu\delta^3$ is used, one necessary equation is lacking.

For these reasons the rule (5) will be adopted throughout Part I.

§ 3. Uncritical Successes.

§ 3.0. *Introduction.*

The h^2-extrapolation was discovered by a hint from theory followed by arithmetical experiments, which gave pleasing results.

The better theory of the following sections is complicated, and tends thereby to suggest that the practice may also be complicated; whereas it is really simple. Hence the reader, if not already familiar with arithmetical examples, is invited to attend to them before proceeding further.

§ 3.1. *An ancient problem retouched. To find the circumference of a circle of unit radius.*

Imagine that we are back in the time of Archimedes. As a first, and obviously very crude, approximation, take the perimeter of an inscribed square $= 4\sqrt{2} = 5\cdot6568$. As a second approximation, take the perimeter of an inscribed hexagon $= 6$ exactly. The errors of these two estimates should be to one another as $1/4^2 : 1/6^2$, that is as $9:4$, if the error is proportional to the square of the co-ordinate difference. Thus the extrapolated value is

$$6 + \tfrac{4}{5}(6 - 5\cdot6568) = 6\cdot2746.$$

The error in the extrapolated value is thus only $1/33$ of the error in the better of the two values from which it was derived; so that extrapolation seems a useful process. To get as good a result from a single inscribed regular polygon it would need to have 35 sides, and in the absence of any tables of sines, the calculation would take longer.

§ 3.2. *Napier's Exponential Base.*

Next suppose that we were living at a time before logarithms or Napier's base had been calculated, and that it was required to find

$$\operatorname*{Limit}_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = \operatorname*{Limit}_{n\to\infty}\ \phi_n \text{ say.}$$

If we put $-n$ in place of n the function $\left(\frac{2n+1}{2n-1}\right)^n$ is unchanged, and so, *if* an expansion like the following exists, valid for both signs of n

$$\left(\frac{2n+1}{2n-1}\right)^n - \operatorname*{Limit}_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = A_0 + \frac{A_1}{n} + \frac{A_2}{n^2} + \dots \frac{A_r}{n^r} \dots,$$

then the odd coefficients A_1, A_3 ... are necessarily zero. Also if the limit exists A_0 must vanish. And by making n sufficiently large the term A_2/n^2 will predominate.

So
$$\phi_n - \operatorname*{Limit}_{n\to\infty}\phi_n = \frac{A_2}{n^2} + \dots.$$

On performing the multiplications it is found easily that the function runs as follows :—

$n =$	1	2	3	4	5
$\left(\frac{2n+1}{2n-1}\right)^n =$	3·00000	2·7777$\dot{7}$	2·74400	2·73261	2·72741

Now if the errors are proportional to n^{-2} the error of ϕ_4 is to the error of ϕ_5 as 25 is to 16. Therefore the extrapolated value is $\phi_5 + \frac{16}{9}(\phi_5 - \phi_4)$. This works out to 2·71817.

NAPIER showed that the correct result is 2·71828, so that the error of our extrapolation was $-$ 0·00011, whereas the error of ϕ_5 was $+$ 0·00913, that is 82 times greater. To get as accurate a result without extrapolation as we did with it we should have to calculate $\phi_{5\sqrt{82}}$ that is ϕ_{45}, a tedious process seeing that logarithms had not yet been tabulated. So extrapolation is a great economiser of toil.

The following table is intended to show how $n^2\{\phi_n - \text{Lt}\,\phi_n\}$ approaches its limit which we have called A_2

n	1	2	3	4	5	∞
$n^2\{\phi_n - \text{Lt}\,\phi_n\}$	0·2817	0·2380	0·2315	0·2295	0·2283	0·2265

To find the limit A_2 we have $1/n \log \phi = \log(1+1/2n) - \log(1-1/2n)$. On expanding the logarithms, rearranging, and taking antilogs it is found that

$$\phi_n - e = e/12n^2 + \text{terms in } n^{-4} \text{ and higher even powers.}$$

So $A_2 = e/12 = 0{\cdot}2265$, which is entered in the table under ∞.

§ 3.3. *Correction to the Second Moment of Grouped Statistical Data.*

We require the second moment

$$\int_{-\infty}^{+\infty} x^2 f(x)\,dx = m_2, \quad \text{say} \qquad (1)$$

but instead of $f(x)$ being given, the data have been grouped so that we only know for all integral values of n

$$\int_{nh}^{(n+1)h} f(x)\,.\,dx = \text{F}\,(n+\tfrac{1}{2})\,h, \quad \text{say} \qquad (2)$$

It is the custom to calculate

$$\Sigma x^2 \text{F}(x) = m_2' \text{ say} \qquad (3)$$

which is an approximation to the integral in (1) when Σ sums for all groups, and then to adjust m_2' for the grouping by applying " SHEPPARD's correction "* which asserts that

$$m_2 = m_2' - \tfrac{1}{12}h^2\,\Sigma\text{F}(x) \qquad (4)$$

provided that $f(x)$ and all its derivatives vanish as x goes off to both $+\infty$ and $-\infty$. (5)

* SHEPPARD, 'Proc. Lond. Math. Soc.,' vol. 29, p. 353 (1898); K. PEARSON, 'Biometrika,' vol. 3, pp. 308 to 312; R. A. FISHER, 'Phil. Trans.,' A, vol. 222, pp. 359 to 363; WHITTAKER and ROBINSON, 'Calculus of Observations,' Blackie & Son, pp. 194 to 196.

It is seen that this is a special case of our h^2-extrapolation. SHEPPARD's fact that the coefficient is $-\frac{1}{12}$ for curves of all shapes that satisfy (5) is indeed remarkable. If instead $f(x)=1$ when $-a<x<a$ and $f(x)=0$ elsewhere, then a theory, familiar in connection with moments of inertia, shows that the coefficient is still $\frac{1}{12}$ but with the positive sign.

As a detailed illustration in marked contrast with the frequency curves to which SHEPPARD's correction is applicable, we may take the frequency y defined by $y=x^2$ when $-1 \leq x \leq 1$ and $y=0$ elsewhere. Instead of high contact at the ends of the range it has discontinuities.

If now the range between $x=-1$ and $x=+1$ be divided into n equal sub-ranges each of h, so that $h=2/n$, and the area under the curve in each range be treated as if concentrated at the midpoint (*not* at the centroid) of the sub-range of x, which is the procedure contemplated by SHEPPARD, then it is found that the second-moment works out as follows :—

n	1	2	3	4	6	∞
Second moment . .	0	$0\cdot1666\dot{6}$	0·285322	0·333333	0·36968	0·40000
Error	0·40000	0·23333	0·114678	0·066667	0·03032	zero
$n^2 \times$ (error) . . .	0·4000	0·9333	1·0421	1·0667	1·0915	

If there are $2m$ equal sub-ranges, the second moment derived from the concentrated areas may be shown to be

$$\frac{1}{6m^5}\sum_{s=0}^{s=m-1}\{12s^4+24s^3+19s^2+7s+1\}.$$

It is seen from the numerical table that the raw moment has to be corrected by the addition of about $\frac{1\cdot1}{4}h^2$. We have here an illustration of a general method of correcting moments, applicable when SHEPPARD's rule is not, *namely, work the moment for two values of h and extrapolate on the assumption that the error is proportional to h^2.* In practice it may be difficult to make h small enough in comparison with the irregularities of the observed frequency.

§ 3.4. *Corrections to Fourier coefficients when the Data are Grouped.*

When Fourier coefficients are calculated from hourly values, which represent 60-minute means, the resulting amplitudes are too small, and correction-factors have to be applied, which according to DARWIN (' B.A. Report for 1883,' p. 98) have the following values :—

Period of wave in hours $= T =$	24	12	8	6
Factor $= \zeta =$	1·00286	1·01152	1·02617	1·04720

Let us see whether the addition to unity is proportional to the square of step h, here of one hour, when expressed in terms of the wave-period as unit of time.

$(\zeta - 1)\,T^2$	1·67	1·66	1·68	1·70

Thus this is another example in which an h^2-extrapolation would be valid.

§ 3.5. *A Sixth Order Linear Jury Problem.*

The following problem was suggested to me by Dr. HAROLD JEFFREYS,* F.R.S., who met it while extending RAYLEIGH'S theory of the equilibrium of a viscous fluid when the higher temperature is on the under side.

Given the " body equation "

$$\frac{d^6V}{dz^6} - 3\frac{d^4V}{dz^4} + 3\frac{d^2V}{dz^2} - V(1 - \lambda) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ is independent of z and is unknown and has to be determined so as to make (1) consistent with the boundary conditions

$$V = 0, \quad \frac{d^2V}{dz^2} = 0, \quad \frac{d^3V}{dz^3} = 0 \qquad \text{at both } z = +1 \text{ and } z = -1. \quad . \quad . \quad (2)$$

Representation of the boundary conditions by finite differences.—Let $a\ b \mid c\ d$ be values of V at equally spaced values of z which increases towards the right. Take the boundary midway in z between the points where $V = b$, $V = c$.

Then $\quad d^3V/dz^3 = 0$ is represented by $d - 3c + 3b - a = 0 \quad . \quad . \quad . \quad . \quad (3)$

Also $d^2V/dz^2 = 0$ is represented by $\frac{1}{2}(d - 2c + b) + \frac{1}{2}(c - 2b + a) = 0$, that is

$$d - c - b + a = 0 \qquad (4)$$

Again $V = 0$ is represented by

$$b + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4) and (5) we have

$$d + a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (3), (5), (6) we have

$$a = 3b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus, as might be immediately evident, there is only one degree of freedom among the four boundary numbers which must run thus, b being arbitrary,

$$3b \quad b \mid -b \quad -3b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These four numbers lie on a linear function of z.

* JEFFREYS has combined finite differences with a variable parameter, see 'Phil. Mag.,' Oct., 1926.

Representation of the body-equation by finite differences.—Being of the 6th order the corresponding difference equation involves 7 values of V. Let these be in sequence*

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \varepsilon \quad \zeta \quad \eta \qquad \ldots \ldots \ldots \qquad (9)$$

equally spaced at intervals h of z. Then centering the body-equation at the same point as δ, it becomes

$$0 = h^{-6}(\alpha - 6\beta + 15\gamma - 20\delta + 15\varepsilon - 6\zeta + \eta) - 3h^{-4}(\beta - 4\gamma + 6\delta - 4\varepsilon + \zeta) + 3h^{-2}(\gamma - 2\delta + \varepsilon) - \delta(1 - \lambda). \qquad (10)$$

The simplest problem in algebra, having one degree of freedom.—In the arrangement

$$3b \quad b \;|\; -b \quad -3b \quad -b \;|\; b \quad 3b, \ldots \ldots \qquad (11)$$

let $3b$, b, etc., represent values of V equally spaced at intervals h of z; and let the vertical lines indicate the positions of the two boundaries, which are located exactly midway between pairs of values of V. On referring to (8) it is seen that the boundary conditions are satisfied. There are also enough values of V to satisfy the body equation (10) at the central point. That is to say, we equate each member of the set (11) to the corresponding member of the set (9) and substitute these values into (10). When this is done it is found that b cancels, that is to say the "amplitude" of the wave is arbitrary, and we are left with a linear equation for λ, which has for solution

$$\lambda = 141\tfrac{5}{8}. \qquad (12)$$

There are 3 steps of length h between the boundaries, so $h = 2/3$.

Two degrees of freedom.—We now increase by one the number of values of V between the boundaries, so as to make $h = 2/4$. Instead of (11) we consider the set

$$3b \quad b \;|\; -b \quad -3b \quad 3c \quad c \;|\; -c \quad -3c \ldots \qquad (13)$$

in which the numbers near both boundaries are of the form (8). It is now possible to fit the set (9) on to the set (13) in two ways. We omit either the right hand or the left hand member of (13) and in both cases equate the components that remain to the corresponding components of (9); and then substitute in (10). This operation might be called "writing out the body-equation at the available centres." In this way we obtain

$$0 = (3375 - 3\lambda)\, b + 2996\, c. \qquad (14)$$

$$0 = 2996 b + (3375 - 3\lambda)\, c. \qquad (15)$$

These equations are to be mutually consistent and so

$$0 = \begin{vmatrix} (3375 - 3\lambda) & 2996 \\ 2996 & (3375 - 3\lambda) \end{vmatrix} \qquad (16)$$

* The operator δ is not used in § 3.5, hence δ is free to mean a value of V.

This is a quadratic in λ and has roots

$$\lambda = 126\tfrac{1}{3} \quad \text{or} \quad 2123\tfrac{2}{3}. \qquad (17)$$

If desired, we could then find b/c.

More degrees of freedom.—Keeping the boundary values always in the relation (8) we may make $n+2$ steps between the boundaries; then the body equation can be satisfied at n tabular points. These n equations are homogeneous and linear in the values of V and will be consistent only if the determinant of their coefficients vanishes. This is very like LAGRANGE'S determinant in the theory of oscillations of a system having n degrees of freedom. This determinant is an equation of the nth degree in λ. Very conveniently it happens, owing to symmetry, that the cubic splits into a linear and a quadratic equation, and the quartic splits into two quadratics.

COLLECTION OF ROOTS OF DETERMINANTAL EQUATIONS.

Degrees of freedom.	h	$\lambda_1(h)$	$\lambda_2(h)$	$\lambda_3(h)$	$\lambda_4(h)$
1	2/3	$141\frac{5}{8}$			
2	2/4	$126\frac{1}{3}$	$2,123\frac{2}{3}$		
3	2/5	119·99	2,457·4	10,963	
4	2/6	116·84	2,584·5	14,618	37,476

Extrapolation.

So far plain proofs have been indicated for all the statements in § 3.5. We now make an assumption, namely that the smallest roots $\lambda_1(h)$ of all the determinantal equations are approximations to an unknown $\lambda_1(0)$. In fig. 1 the smallest root is plotted against h^2 and it is seen that the graph is nearly straight, thus suggesting that

$$\lambda_1(h) = \lambda_1(0) + A_2 h^2$$

nearly. If we assume that more accurately

$$\lambda_1(h) = \lambda_1(0) + A_2 h^2 + A_4 h^4,$$

and substitute for the three smallest values of h in the above table we have

$$126\tfrac{1}{3} = \lambda_1(0) + A_2(2/4)^2 + A_4(2/4)^4$$
$$119{\cdot}99 = \lambda_1(0) + A_2(2/5)^2 + A_4(2/5)^4$$
$$116{\cdot}84 = \lambda_1(0) + A_2(2/6)^2 + A_4(2/6)^4$$

from which set of equations, on eliminating A_2 and A_4, it is found that $\lambda_1(0) = 110{\cdot}4_5$.

Dr. JEFFREYS by an entirely different process also found $\lambda_1(0) = 110$. So here again we see that the extrapolation method is a convenient way of obtaining numerical results.

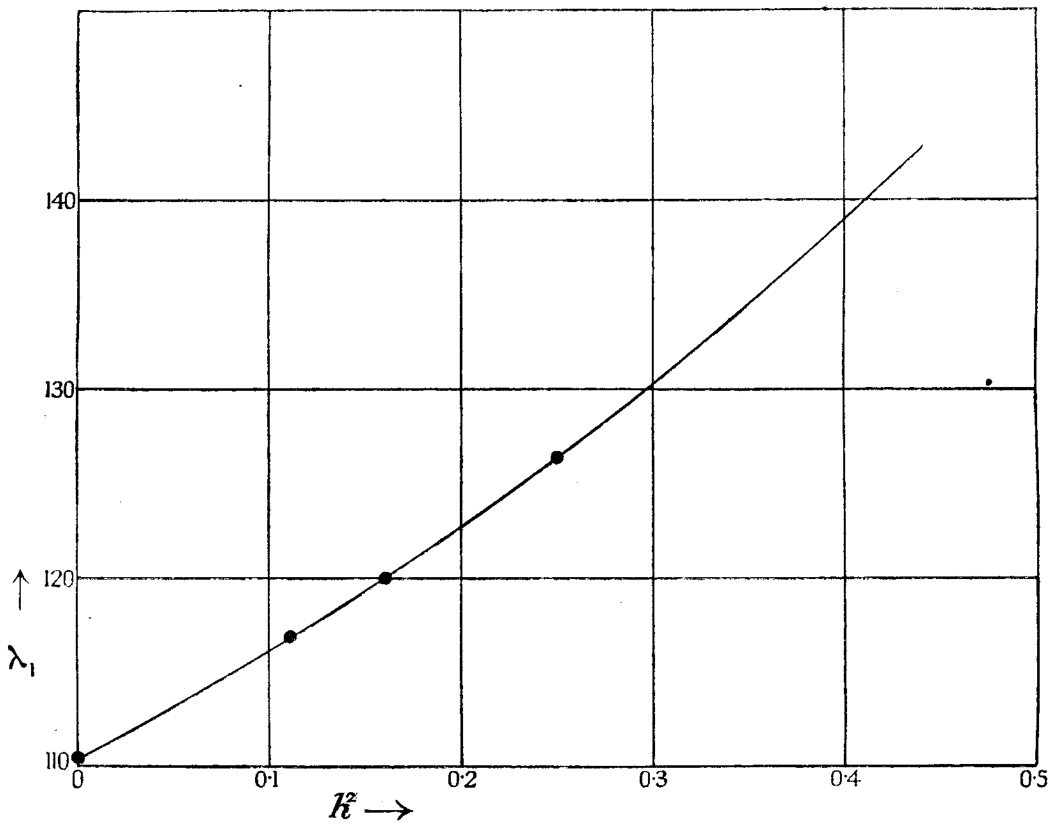

Fig. 1.

This was an h^4-extrapolation. It may be proved to be nearly equivalent to drawing a circle through three given points on a diagram having h^2 and $\phi(x, h)$ as co-ordinates. See the portion of a circle on fig. 1. The h^2-extrapolation corresponds exactly to a straight line on this diagram.

§ 4. FAILURES AND DIFFICULTIES.

§ 4.1. *Discontinuities.*

Let us see whether the person who is computing $\phi(x, h)$ by a numerical process would receive in the course of the work any warning that the unknown function $f(x)$ to which he is approximating is likely to possess a discontinuity. For if any such danger-signal should appear, it will often be possible to investigate $f(x)$ analytically in a very short range of x in which a discontinuity is suspected, even when analytic methods are not convenient for finding $f(x)$ over wide ranges of x. For purpose of illustration, however, we must choose examples in which $f(x)$ has been found exactly over a wide range.

The position of discontinuities can often be foreseen, for instance, with regard to the linear equation

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z) \cdot u = 0,$$

WHITTAKER and WATSON* prove that the solution u is analytic except where $p(z)$ or $q(z)$ is not. G. H. HARDY† classifies discontinuities into four chief varieties. If $f(x)$ tends to a limit as $x \to a$ from either side, these limits are denoted by $f(a-0)$ and $f(a+0)$. For continuity it is necessary and sufficient that $f(x)$ should be defined when $x = a$ and that $f(a-0) = f(a) = f(a+0)$.

Variety I. $f(a-0) = f(a+0)$ but $f(a)$ is either not defined or else differs from $f(a-0)$ and $f(a+0)$. For example let $f(x) = x^2$ except when $x = 1$. This variety is not likely to arise in the solution of a problem in the calculus, and in the arithmetical method all except discrete values are missing anyway, so no new difficulty is likely to arise.

Variety II. $f(a-0)$ is not equal to $f(a+0)$. This might arise in physical problems where there is a discontinuity at an interface, but if so the problem is usually quite simply arranged by adjusting two constants of integration, and need not detain us.

Varieties III and IV, the *infinity* and the *oscillatory discontinuity*, seem to present more difficulty and so will each be illustrated by an example.

Variety III. *Infinity.*—To illustrate an infinity and the convergence in its neighbourhood, a graph is shown of the successive approximations to the function defined jointly by $dy/dx = 1/x$ and $y = 0$ when $x = 1$. The solution, of course, is

$$y = f(x) = \int_1^x \frac{dx}{x} = \log_e x,$$

which goes off to $-\infty$ at $x = 0$. The corresponding problem in differences has been solved by marching from $x = 1$ by n equal steps to $x = 0$. Thus $h = 1/n$. The above integral is replaced by the sum of the reciprocals of x at the centres of its steps. Let suffixes to ϕ denote the number of steps. It is found thus that where s is an integer between 0 and n

$$\phi_n\left(\frac{s}{n}\right) = -2\left\{\frac{1}{2s+1} + \frac{1}{2s+2} + \dots + \frac{1}{2n-1}\right\}$$

and

$$\phi_n(0) = -2\left\{1 + \tfrac{1}{3} + \tfrac{1}{5} \dots + \frac{1}{2n-1}\right\}.$$

The series for $\phi_n(0)$ diverges to $-\infty$ as $n \to \infty$ thus showing the discontinuity. For various values of x the results are set out in fig. 2.

* 'Modern Analysis,' 3rd edn., § 10.21.

† "Pure Mathematics," Camb. Press, p. 178 (1925).

The numbers next the points are the values of n. The function $\log_e x$ is also plotted. We see from the graph that whereas $\phi_n(x)$ has always an infinitely great error at the point $x = 0$, yet *if by equal finite steps we approach as near to the infinity as possible, that is to say only one step away from it, the error of $\phi_n(x)$ is quite small.* This, if true in general,

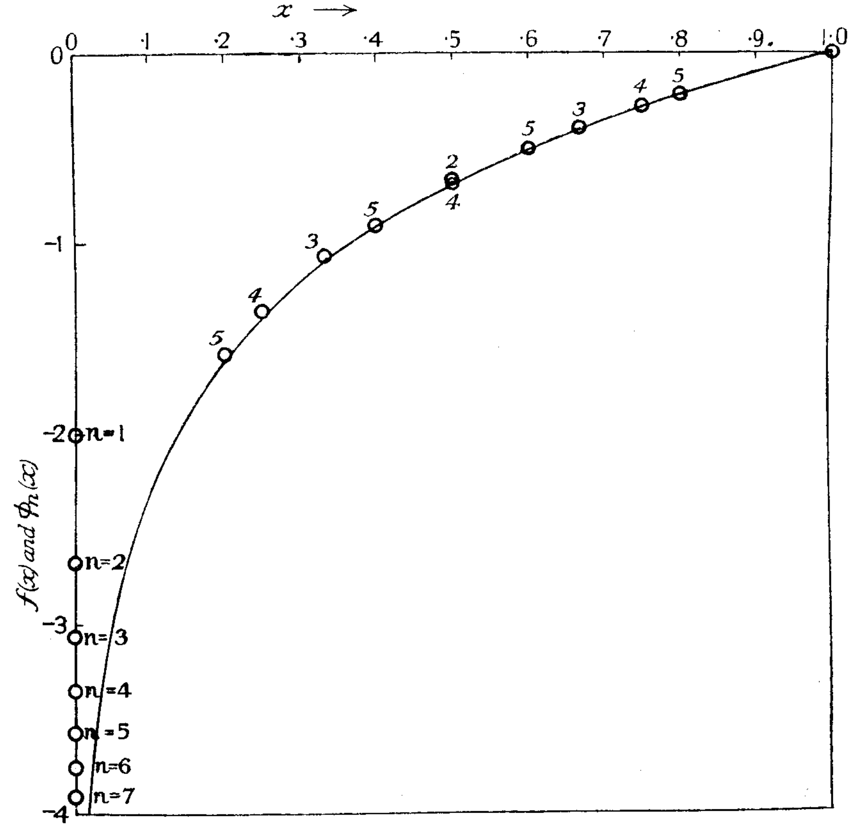

Fig. 2.

or even usually, will be very important; for it will mean that all but the last step can be done by arithmetic. In the above example the error of $\phi_n(x)$ when expressed as a fraction of $f(x)$ actually becomes less as we approach the infinity of $f(x)$, thus :—

at $x = 1/n$ where $n =$	2	3	4	5
$\dfrac{\phi_n(x) - f(x)}{f(x)}$	0·038	0·028	0·025	0·022

Suppose that the computer had not noticed that there was an infinity, but had calculated $\phi_n(0)$ for $n=1, 2, \ldots 8$ and proceeded to make h^2-extrapolations. Let $\phi_{rs}(x)$ denote the extrapolated value obtained from $\phi_r(x)$ and $\phi_s(x)$. The table shows what he would find for $x = 0$ and $x = \frac{1}{2}$.

r	s	$\phi_{rs}(\frac{1}{2})$	$\phi_{rs}(0)$
2	4	—0·69206	—3·581
4	6	—0·69299	—4·080
6	8	—0·69306	—4·413

The marked contrast between the rapid convergence of $\phi_{rs}(\frac{1}{2})$ and the behaviour of $\phi_{rs}(0)$ should make the computer suspicious about the latter.

Variety IV.—*Oscillatory discontinuity* such as that of

$$f(x) = \sin\left(\frac{2\pi}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as $x \to 0$.

In order to make this function arise as the solution of a problem in the calculus we note that

$$f'(x) = -\frac{2\pi}{x^2}\cos\left(\frac{2\pi}{x}\right), \quad \text{also} \quad f(4) = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2), (3)$$

Let (2) and (3) be the equations which it is required to solve by differences. Divide the range between $x = 0$ and $x = 4$ into n equal steps each of h, so that

$$nh = 4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Let a difference-equation to replace (2) be formed at the centre of each step. These centres are at $x = (s - \frac{1}{2})h$ where $s = 1, 2, 3, \ldots n$. In accordance with the rule laid down in § 2 the replacement results in

$$\phi_n\{(s-1)h\} = \phi_n(sh) + \frac{2\pi}{(s-\frac{1}{2})^2 h}\cos\left\{\frac{2\pi}{(s-\frac{1}{2})h}\right\}. \quad . \quad . \quad . \quad . \quad (5)$$

The process defined by equation (5) is equivalent to drawing a polygon on an (x, f) diagram starting from the point $x = 4, f(x) = 1$, such that its sides have equal projections on the x-axis, and are parallel to the tangent to the $f(x)$ curve at the value which x has at the midpoint of this projection. It is evident from a graph that the $\phi(x)$ polygon will have some resemblance to the $f(x)$ curve if there is at least one step h per quarter wave-length ; but that when steps are twice as long as this critical value, all resemblance has ceased.

The difference-problem yields a definite value of $\phi(0)$ in contrast with $f(0)$ which has any value between -1 and $+1$. The solution of the difference-problem is thus misleading at the discontinuity ; but the warning given by the presence of x^{-2} in equation (2) is probably sufficient to prevent anyone being misled.

Incidentally it appears that h^2-extrapolation is valid where there is at least one step per quarter wave-length. This is shown by the values at $x = 2$ which are

$$\phi_2(2) = 0{\cdot}3019, \quad \phi_4(2) = 0{\cdot}0726, \quad \phi_8(2) = 0{\cdot}0160, \quad f(2) = 0.$$

If we extrapolate from ϕ_2 and ϕ_4 we obtain $0{\cdot}0039$; or if we extrapolate from ϕ_4 and ϕ_8 we obtain $0{\cdot}0028$; both of which are considerable improvements on the numbers from which they are derived.

On the contrary, to extrapolate at the discontinuity would lead us further away from the truth, as the following numbers show

$$\phi_1(0) = -5\cdot28, \quad \phi_2(0) = 12\cdot868, \quad \phi_4(0) = 23\cdot809, \quad \phi_8(0) = 47\cdot186.$$

A peculiar case.—If $dy/dx = q(x) \,.\, y$. Then dy/dx must vanish where y does, unless $q(x)$ is infinite. If one of our steps ends at such a point there are likely to be difficulties. But by rearranging things so that the special point lies in the middle instead of at the end of a step these difficulties disappear. For example when the equation is

$$\frac{dy}{dx} = -\frac{2\pi}{x^2}\cot\left(\frac{2\pi}{x}\right) \cdot y,$$

there is an infinity of cot $(2\pi/x)$ at $x = 2/m$ where m is an integer. It is impossible to avoid all these.

§ 4.2. *Frills.*

What is difficult by analysis is sometimes easy by arithmetic and *vice versa*. For this reason ideas such as continuity and differentiability, which are so important in analysis, may sometimes be merely flippant if applied to the arithmetical process. For example, we have to deal with the function defined to be $\sin x$ cut off after the seventh decimal place. This function has about twenty million discontinuities between $x = 0$ and $x = \pi$, and its derivative is alternately zero and non-existent. Yet the computer finds it smooth and pleasant to deal with.

Conversely the function $\sin x + \sin(100\,x) + \sin(10000\,x)$ is everywhere continuous and differentiable to any order. The analyst finds it pleasant, but to the computer it is an intractable horror. A step h which is large enough to allow satisfactory progress in exploring the variation of $\sin x$ is far too large to reveal the detail of $\sin(10000\,x)$. Let us call these rapid oscillations, superposed on much slower variations, by the name "frills."

To discover types of differential equation of which the computer should beware we can take a primitive with a frill, say

$$y = \sin x + \mathrm{A} \sin mx \qquad (1)$$

where A and m are constants and $m > 20$ say, and then proceed to form differential equations by eliminating A or m or both between y and its derivatives $y', y'', \ldots$ with respect to x. Such differential equations include the following

$$m(\cos mx)(y - \sin x) = (\sin mx)(y' - \cos x) \qquad (2)$$

$$y'' + m^2 y = (m^2 - 1)\sin x \qquad (3)$$

$$(y - \sin x)^2(y'' + \sin x) - (y' - \cos x)^2(y - \sin x) = \mathrm{A}^2(y'' + \sin x) \qquad (4)$$

$$\sqrt{\left\{\frac{y'' + \sin x}{-y + \sin x}\right\}} \cot\left\{x\sqrt{\frac{y'' + \sin x}{-y + \sin x}}\right\} . (y - \sin x) = y' - \cos x. \qquad (5)$$

§ 5. Standards of Neglect.

§ 5.1. *Introduction.*

An error of form which is negligible in a haystack would be disastrous in a lens. Thus negligibility involves both mathematics and purpose. In this paper we discuss the mathematics, leaving the purposes to be discussed when they are known.

Suppose for the sake of argument that $\phi(x, h)$, the solution of the difference-problem, is related to h by the formula

$$\phi(x, h) = f(x) + f_2(x) \cdot h^2 + \mathrm{R}(x, h)$$

where $\mathrm{R}(x, h)$ when expanded in positive powers of h contains only powers higher than h^2. By hypothesis an h^2-extrapolation is to be made by taking two values h_1, h_2, and eliminating $f_2(x)$ so as to obtain

$$f(x) = \frac{h_2^2\{\phi(x, h_1) - \mathrm{R}(x, h_1)\} - h_1^2\{\phi(x, h_2) - \mathrm{R}(x, h_2)\}}{h_2^2 - h_1^2} \quad . \quad . \quad . \quad (1)$$

Thus it does not matter how large $f_2(x)$ may be. But $\mathrm{R}(x, h)$ must in some sense be made small; and that for both h_1 and h_2.

§ 5.2. *Choice of Standard.*

We have now to find a standard suitable for the measurement of $\mathrm{R}(x, h)$. Possible standards are:—

(i) $\phi(x, h)$. If R were negligible in comparison with ϕ all would be well. In the arithmetic ϕ is obtained simply. But in the analytical discussion ϕ often has to be expressed in complicated algebra and $f(x)$ is simpler.

(ii) $f(x)$. If R were negligible in comparison with $f(x)$ all would be well. But it is impossible to satisfy this condition at $x = c$ if $f(c) = 0$.

(iii) The difference between the greatest and least values of $f(x)$ in the range of x under investigation. This has the advantage of never vanishing except in the unimportant case $df/dx = 0$. This standard is likely to suit many purposes. But, like $f(x)$ it is not known until the problem is fully solved, and is decidedly awkward in the analysis.

(iv) All the subsequent investigations proceed by the expansion of $f(x)$ in Taylor's series and the comparison of that series with another in which $\phi(x, h)$ is expanded in the sequence of the central differences of $\phi(a, h)$, which is not the sequence of the powers of x. A simple and convenient standard, from the mathematical aspect, is the difference between the greatest and least values of any one term in Taylor's series, in the range of x under investigation. This standard will be adopted.

(v) If a different type of expansion were in use, a correspondingly different standard of error would be suitable.

§ 5.3. *Application.*

It will be proved in §§ 7, 9, 10, that these terms in differences of ϕ in the expansion of $\phi(x, h)$ commonly involve h in forms such as a factor

$$\frac{(1+\alpha_1 h^2)(1+\alpha_2 h^2)(1+\alpha_3 h^2)\ldots(1+\alpha_s h^2)}{(1-\beta_1 h^2)(1-\beta_2 h^2)\ldots(1-\beta_r h^2)} = \mathrm{J} \quad \text{say.} \quad \ldots\ldots \quad (1)$$

When $h \to 0$ then $\mathrm{J} \to 1$ and $\phi(x, h) \to f(x)$ so that the coefficient of J is the standard which has just been adopted.

Now if J be expanded in ascending powers of h^2

$$\mathrm{J} - 1 - (\beta_1 + \beta_2 \ldots + \beta_r + \alpha_1 + \alpha_2 \ldots + \alpha_s) h^2 = \xi \quad \ldots\ldots \quad (2)$$

where ξ involves $h^4, h^6, h^8 \ldots$ but not h^2 simply. The question is how small h must be in order to make ξ negligible compared with unity. The term in h^2 is of no interest. It may sometimes be not too laborious to calculate J exactly and so to settle the question. Yet an easier rough rule is desirable, provided it keeps on the safe side, making h smaller than is strictly necessary.

Since

$$(1-\beta_1 h^2)^{-1} = 1 + \beta_1 h^2 + \beta_1^2 h^4 + \ldots + \beta_1^\nu h^{2\nu} \ldots \text{ to inf.,}$$

therefore

$$\mathrm{J} = (1-\alpha_1 h^2)(1-\alpha_2 h^2)\ldots(1-\alpha_s h^2)\left\{1+\sum_1^\infty (\beta_1 h^2)^\nu\right\}\left\{1+\sum_1^\infty (\beta_2 h^2)^\nu\right\}\ldots\left\{1+\sum_1^\infty (\beta_r h^2)^\nu\right\}.$$

The coefficient of h^{2n} in this continued product consists of the sum of all possible products of $\alpha_1, \alpha_2 \ldots \alpha_s, \beta_1, \beta_2 \ldots \beta_r$ taken n at a time with the proviso that each α may occur once only in each product, but each β may occur any number of times. The varieties of products would be more numerous if we removed the restriction that any α occurs only once. Let G be the greatest of the absolute values of $\alpha_1, \alpha_2, \ldots \alpha_s, \beta_1, \beta_2 \ldots \beta_r$. . . (3)

We work the first two terms of the expansion strictly, subtract them from J, and make changes which, if anything, increase all the other terms.

Then

$$|\xi| < \sum_{n=2}^{n\to\infty} (\mathrm{G}h^2)^n \cdot {}_{r+s}\mathrm{H}_n, \quad \ldots\ldots \quad (4)$$

a positive quantity where ${}_{r+s}\mathrm{H}_n$ is, as usual, the number of n-combinations of $(r+s)$ letters when any letter may be repeated any number of times up to n.

Now

$${}_{r+s}\mathrm{H}_n = \frac{(r+s)}{1} \times \frac{(r+s+1)}{2} \times \frac{(r+s+2)}{3} \times \ldots \times \frac{(r+s+n-1)}{n}.$$

And the fractions in the second member are arranged in order of magnitude, the first being the greatest.

So that ${}_{r+s}\mathrm{H}_n < (r+s)^n$.

And

$$|\xi| < \sum_{n=2}^{n\to\infty} \{\mathrm{G}h^2(r+s)\}^n \qquad \text{a geometric series.}$$

Hence

$$|\xi| < \frac{\mathrm{G}^2h^4(r+s)^2}{1-\mathrm{G}h^2(r+s)}. \qquad (5)$$

Thus, *when h appears only as a factor of the form* J *multiplying any difference-ratio of* ϕ, *an* h^2*-extrapolation will be valid for the corresponding derivative if* $h^2\,\mathrm{G}\,(r+s)$ *is small compared with unity,* G *being defined by* (3) *and* $(r+s)$ *being the total number of factors in numerator and denominator together* . . . . (6)

§ 6. Central Difference Operators.

Before we can proceed with various necessary operations, we must be familiar with the rules for "differencing" as with those for differentiating. For instance, do the central differences of a product behave like the derivatives in Leibnitz theorem ?

The differencer δ and Sheppard's averager μ have already been defined in § 2 equations (2), (3).

It will also be convenient to define the symbol $\,|\,$ to mean that preceding operators do not operate beyond this "wall," as it will be called. Apart from preventing operators going too far, a task which it shares with the + sign, $|$ will be defined to be a mere multiplication sign.

Sheppard has shown that

$$\mu^2 = 1 + \tfrac{1}{4}\delta^2. \qquad (1)$$

As in the Infinitesimal Calculus certain derivatives are obtained from first principles while others follow from them by rules of operation such as Leibnitz' theorem; so it is here with μ and δ.

To find the *first difference of a product*. From the definition of δ

$$\delta\{\theta(x)\,.\,\chi(x)\} = \theta(x+\tfrac{1}{2}h)\,.\,\chi(x+\tfrac{1}{2}h) - \theta(x-\tfrac{1}{2}h)\,.\,\chi(x-\tfrac{1}{2}h)$$
$$= \quad \beta \quad \times \quad b \quad - \quad \alpha \quad \times \quad a,$$

where β, b, α, a are equal *respectively* to the expressions above. Now by algebra

$$\frac{b+a}{2}(\beta-\alpha) + \frac{\beta+\alpha}{2}(b-a) \equiv \beta b - \alpha a.$$

But the first member is

$$\mu\chi(x)\,|\,\delta\theta(x) + \mu\theta(x)\,|\,\delta\chi(x).$$

Hence

$$\delta(\theta\,.\,\chi) = \mu\chi\,|\,\delta\theta + \mu\theta\,|\,\delta\chi. \qquad (2)$$

Contrast the analogous formula in which $D = d/dx$

$$D(\theta \, . \, \chi) = \chi D\theta + \theta D\chi$$

To find the *mean of a product*

$$\mu\{\theta(x) \, . \, \chi(x)\} = \tfrac{1}{2}\{\beta b + \alpha a\},$$

where β, b, α, a have the same meanings as above.

Now by algebra

$$\frac{(b+a)(\beta+\alpha)}{4} + \frac{(\beta-\alpha)(b-a)}{4} \equiv \tfrac{1}{2}(\beta b + \alpha a).$$

Hence

$$\mu\{\theta \, . \, \chi\} = \mu\theta \,|\, \mu\chi + \tfrac{1}{4}\delta\theta \,|\, \delta\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Higher means or differences of a product then follow by the algebraic properties of μ and δ from (1), (2), (3). Thus it may be proved that

$$\mu\delta\{\theta \, . \, \chi\} = (\chi + \tfrac{1}{2}\delta^2\chi) \,|\, \mu\delta\theta + (\theta + \tfrac{1}{2}\delta^2\theta) \,|\, \mu\delta\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the second member of (4) occur differences $\delta^2\chi$, $\delta^2\theta$ of an order one beyond that of the highest derivatives that we should find if we were taking d/dx instead of $\mu\delta$ of the product. This extra order of difference appears, because μ brings in δ, as shown by (3).

By a second application of (2) and (1) it follows that

$$\delta^2\{\theta \, . \, \chi\} = \theta \, . \, \delta^2\chi + 2\mu\delta\theta \,|\, \mu\delta\chi + \chi \, . \, \delta^2\theta + \tfrac{1}{2}\delta^2\theta \,|\, \delta^2\chi \quad . \quad . \quad . \quad . \quad (5)$$

the term $\tfrac{1}{2}\delta^2\theta \,|\, \delta^2\chi$ makes (5) unlike the second derivative of a product.

Again

$$\delta^3\{\theta \, . \, \chi\} = \delta^3\theta \,|\, (\mu\chi + \mu\delta^2\chi) + 3\mu\delta^2\theta \,|\, \delta\chi + 3\delta\theta \,|\, \mu\delta^2\chi + (\mu\theta + \mu\delta^2\theta) \,|\, \delta^3\chi \quad (6)$$

In the second member, δ^3 is the highest difference and the differences are centered with $\mu\theta$ or $\mu\chi$.

Difference of a product of three factors.—By repeated application of (2) and (3)

$$\begin{aligned}\delta(\theta \, . \, \rho \, . \, \chi) &= \delta\theta \,|\, \mu(\chi\rho) + \mu\theta \,|\, \delta(\chi \, . \, \rho) \\ &= \delta\theta \,|\, \mu\chi \,|\, \mu\rho + \mu\theta \,|\, \delta\chi \,|\, \mu\rho + \mu\theta \,|\, \mu\chi \,|\, \delta\rho + \tfrac{1}{4}\delta\theta \,|\, \delta\chi \,|\, \delta\rho \quad . \quad . \quad . \quad (7)\end{aligned}$$

in which the factors of the separate terms are acted on once only by either μ or else δ.

Many factors.

$$\delta(\theta \, . \, \chi \, . \, \rho \, . \, \sigma \ldots \tau) = \mu\theta \,|\, \delta(\chi \, . \, \rho \, . \, \sigma \ldots \tau) + \delta\theta \,|\, \mu(\chi \, . \, \rho \, . \, \sigma \ldots \tau) \quad . \quad . \quad . \quad (8)$$

$$\mu(\theta \, . \, \chi \, . \, \rho \, . \, \sigma \ldots \tau) = \mu\theta \,|\, \mu(\chi \, . \, \rho \, . \, \sigma \ldots \tau) + \tfrac{1}{4}\delta\theta \,|\, \delta(\chi \, . \, \rho \, . \, \sigma \ldots \tau) \quad . \quad . \quad (9)$$

Thus we have split off the first factor θ leaving δ and μ of the product of the rest. Similarly χ, ρ, σ ... can be split off in turn and each will be acted on once only by either δ or by μ but not by both δ and μ. This fact will be of importance in § 9.3.

The advancing and retarding operators.—From the definitions of μ and δ

$$(\mu + \tfrac{1}{2}\delta)\,\chi(x) = \chi(x + \tfrac{1}{2}h) \qquad (10)$$

thus operating with $\mu + \frac{1}{2}\delta$ is equivalent to increasing x by $\frac{1}{2}h$.

In like manner

$$(\mu - \tfrac{1}{2}\delta)\,\chi(x) = \chi(x - \tfrac{1}{2}h). \qquad (11)$$

Operations on a reciprocal.—From the definition of δ

$$\delta\left\{\frac{1}{\chi(x)}\right\} = \frac{1}{\chi(x + \frac{1}{2}h)} - \frac{1}{\chi(x - \frac{1}{2}h)}.$$

Reducing to a common denominator and using the advancing and retarding operators

$$\delta\left\{\frac{1}{\chi(x)}\right\} = \frac{-\,\delta\chi(x)}{(\mu + \frac{1}{2}\delta)\,\chi(x) \,|\, (\mu - \frac{1}{2}\delta)\,\chi(x)} = \frac{-\,\delta\chi(x)}{\{\mu\chi(x)\}^2 - \frac{1}{4}\{\delta\chi(x)\}^2} \qquad (12)$$

In like manner the mean of a reciprocal is found to be

$$\mu\left\{\frac{1}{\chi}\right\} = \frac{\mu\chi}{(\mu\chi)^2 - \frac{1}{4}(\delta\chi)^2}, \qquad (13)$$

the denominator which is the same for μ and δ will be denoted by Z. The higher differences follow by repeated applications of these formulæ, and of those for products. Thus

$$\delta^2\left\{\frac{1}{\chi}\right\} = -\,\delta\left\{\frac{\delta\chi}{Z}\right\} = -\,\delta^2\chi \,|\, \mu\left(\frac{1}{Z}\right) - \mu\delta\chi \,|\, \delta\left(\frac{1}{Z}\right)$$

$$= \frac{-\,\delta^2\chi \,|\, \mu Z + \mu\delta\chi \,|\, \delta Z}{(\mu Z)^2 - \frac{1}{4}(\delta Z)^2}$$

Also it is found that

$$\mu Z = \chi\,(\chi + \tfrac{1}{2}\,\delta^2\chi) \text{ and } \delta Z = 2\chi \,.\, \mu\delta\chi.$$

So that

$$\delta^2\left(\frac{1}{\chi}\right) = \frac{-\,\delta^2\chi \,|\, (\chi + \frac{1}{2}\delta^2\chi) + 2\,(\mu\delta\chi)^2}{\chi\,(\chi + \frac{1}{2}\delta^2\chi)^2 - \chi\,(\mu\delta\chi)^2} \qquad (14)$$

Limits as $h \to 0$.

If $\chi(x)$ is continuous then

$$\operatorname*{Lt}_{h\to 0}\ \mu\chi(x) = \chi(x).$$

If $\chi(x)$ is differentiable then

$$\operatorname*{Lt}_{h\to 0}\ \delta\chi(x)/h = d\chi(x)/dx,$$

and $\mu\delta\chi(x)/h$ has the same limit.

If $\chi(x)$ is " analytic " throughout the range, then

$$\mathop{\mathrm{Lt}}_{h\to 0} \mu^2\chi(x) = \mathrm{Lt}\left\{\chi(x) + \frac{h^2}{4}\frac{\delta^2\chi(x)}{h^2}\right\} = \chi(x).$$

This μ^2 is the source* of h wherever it appears other than in the combination δ/h, and hence we obtain the right limits if we omit μ and put d/dx in place of every δ/h.

§ 7. The Two Series Analogous to Taylor's.

These series will be required for the solution of marching problems in § 9, jury problems in § 10, and integral equations in § 11.

To fit with the conventions which have been adopted in § 2 in order to keep the arithmetic simple, the series must bring in μ with alternate powers of δ.

When the given range equation is of even order the required series is

$$\phi(a+nh) = \phi(a) + n\mu\delta\phi(a) + \frac{n^2}{2!}\delta^2\phi(a) + \frac{n(n^2-1^2)}{3!}\mu\delta^3\phi(a)$$
$$+\frac{n^2(n^2-1^2)}{4!}\delta^4\phi(a) + \frac{n(n^2-1^2)(n^2-2^2)}{5!}\mu\delta^5\phi(a) + \ldots \quad (1)$$

It is known as the " Newton-Stirling "† formula and may be obtained by repeated application of the advancing operator, thus,

$$\phi(a+nh) = (\mu + \tfrac{1}{2}\delta)^{2n}\phi(a),$$

the expression $(\mu + \frac{1}{2}\delta)^{2n}$ being expanded by the binomial theorem, and superfluous even powers of μ then removed by the aid of $\mu^2 = 1 + \frac{1}{4}\delta^2$

The general terms, according to Whittaker and Robinson, are

$$\tfrac{1}{2}\{(n+r)_{2r} + (n+r-1)_{2r}\}\,\delta^{2r}\phi(a) + (n+r)_{2r+1}\,\mu\delta^{2r+1}\,\phi(a)$$

where $(n)_r$ denotes

$$n(n-1)(n-2)\ldots(n-r+1)/r!.$$

Let us now suppose that the integral $\phi(x, h)$ has been marched from $x = a$ to $x = a + l$ by an integral number n of steps each of h. We have then

$$nh = l, \quad (1\mathrm{A})$$

and if we put

$$h/l = \lambda = 1/n \quad (2)$$

* In § 6, but see also § 8, § 9.5.

† Whittaker and Robinson, 'The Calculus of Observations' (Blackie & Son, Ltd.), p. 43.

series (1) becomes

$$\phi(a+l) = \phi(a) + l\frac{\mu\delta\phi(a)}{h} + \frac{l^2}{2!}\frac{\delta^2\phi(a)}{h^2} + \left\{\frac{l^3}{3!}\frac{\mu\delta^3\phi(a)}{h^3} + \frac{l^4}{4!}\frac{\delta^4\phi(a)}{h^4}\right\}(1-\lambda^2)$$

$$+\left\{\frac{l^5}{5!}\frac{\mu\delta^5\phi(a)}{h^5} + \frac{l^6}{6!}\frac{\delta^6\phi(a)}{h^6}\right\}(1-\lambda^2)(1-2^2\lambda^2)\ldots$$

$$+\left\{\frac{l^{2s-1}}{(2s-1)!}\frac{\mu\delta^{2s-1}\phi(a)}{h^{2s-1}} + \frac{l^{2s}}{(2s)!}\frac{\delta^{2s}\phi(a)}{h^{2s}}\right\}(1-\lambda^2)(1-2^2\lambda^2)(1-3^2\lambda^2)\ldots\{1-(s-1)^2\lambda^2\}$$

$$+\ldots \text{ to an end specified below in (7A)}, \qquad (3)$$

where s is a positive integer.

In comparison with TAYLOR's series this contains difference-ratios in place of derivatives, also the factors in λ. Unlike TAYLOR's series, (3) is not simply arranged in powers of l, because λ involves l.

When the given range-equation is of odd order, it has been shown in § 2 statement (5) that the differences must be "centered with $\mu\phi$." That is to say the boundary $x = a$ falls midway between two values of x at which ϕ is tabulated, and the tabular values are

$$\phi\{a + (n+\tfrac{1}{2})h\} = (\mu + \tfrac{1}{2}\delta)^{2n+1}\phi(a), \qquad (4)$$

where n is an integer.

The required series is called the "NEWTON-BESSEL" formula.* It may be obtained by operating on series (1) with $\mu + \frac{1}{2}\delta$ and afterwards re-arranging the terms according to powers of δ

$$\phi\{a + (n+\tfrac{1}{2})h\} = \mu\phi(a) + (n+\tfrac{1}{2})\delta\phi(a) + \frac{n(n+1)}{2}\mu\delta^2\phi(a)$$

$$+\frac{n(n+\frac{1}{2})(n+1)\delta^3\phi(a)}{3!}$$

$$+\frac{(n-1)n(n+1)(n+2)}{4!}\mu\delta^4\phi(a)$$

$$+\frac{(n-1)n(n+\frac{1}{2})(n+1)(n+2)}{5!}\delta^5\phi(a)$$

$$+\ldots \qquad (5)$$

We observe that the coefficient of δ^r contains r factors linear in n; that these factors are arranged symmetrically about $(n+\frac{1}{2})$ as centre; but that $(n+\frac{1}{2})$ appears only with the odd differences.

Now if the integral has to be marched from $x = a$ to $x = a + k$ we must put

$$(n+\tfrac{1}{2})h = k$$

in order to have a tabular value at $x = a + k$.

* WHITTAKER and ROBINSON, 'The Calculus of Observations' (Blackie & Son, Ltd.), pp. 39, 42, 47.

Let

$$h/k = \kappa = 1/(n + \tfrac{1}{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and eliminate n from the foregoing expansion, then it is found that

$$\begin{aligned}\phi(a+k) &= \mu\phi(a) + k\frac{\delta\phi(a)}{h} \\ &+ \left\{\frac{k^2}{2!}\frac{\mu\,\delta^2\phi(a)}{h^2} + \frac{k^3}{3!}\frac{\delta^3\phi(a)}{h^3}\right\}\left(1 - \frac{\kappa^2}{2^2}\right) \\ &+ \left\{\frac{k^4}{4!}\frac{\mu\,\delta^4\phi(a)}{h^4} + \frac{k^5}{5!}\frac{\delta^5\phi(a)}{h^5}\right\}\left(1 - \frac{\kappa^2}{2^2}\right)\left(1 - \frac{3^2\kappa^2}{2^2}\right) \\ &+ \dots \\ &+ \left\{\frac{k^{2s}}{(2s)!}\frac{\mu\,\delta^{2s}\phi(a)}{h^{2s}} + \frac{k^{2s+1}\,\delta^{2s+1}\phi(a)}{h^{2s}}\right\}\left(1 - \frac{\kappa^2}{2^2}\right)\left(1 - \frac{3^2\kappa^2}{2^2}\right)\left(1 - \frac{5^2\kappa^2}{2^2}\right)\dots \\ &\qquad \dots\left\{1 - \frac{(2s-1)^2\kappa^2}{2^2}\right\} \\ &+ \dots \text{ to an end specified below in (7A)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)\end{aligned}$$

where s is a positive integer. This series is not simply arranged in powers of k, because κ also involves k.

* *Extended definition of* ϕ.—When the range $a < x < a + l$ is divided into n equal steps for the practical arithmetical process then the NEWTON-STIRLING series, because it involves differences centered at $x = a$, must involve discrete values of ϕ arranged symmetrically on the opposite side† of $x = a$. That is $2n + 1$ discrete values of ϕ altogether. These suffice to define differences up to order $2n$.

So far $\phi(x, h)$ is undefined, except at these $2n + 1$ equally spaced values of x. An h^2-extrapolation is so far possible only where there is a coincidence, the same value of x occurring for two values of h. This has hitherto restricted the usefulness of the h^2-extrapolation. However, the NEWTON-STIRLING series (3) defines a family of continuous functions of x, each function passing through the $2n + 1$ discrete values of ϕ, which are given by the practical arithmetical process for solving the difference problem. The members of the family vary from one another only in consequence of variations of differences of $\phi(a)$ of order higher than $2n$. So far these higher differences are undefined. They appear to be useless in practice and therefore we establish:—

A *convention* that at $x = a$ differences of order higher than $2l/h$ are each zero in series (3). Similarly those higher than $2k/h$ are zero in series (7). . . . . (7A)

These series are then polynomials of degree $2n$ in l and $(2n + 1)$ in k. And it is the object of the ordinary rules of interpolation‡ to pass a polynomial through the given

* *Revised March*, 1927, here and in its implications in § 9, § 10.

† Otherwise for economy $x = a$ might be at the midpoint of the practical range.

‡ WHITTAKER and ROBINSON, 'Calculus of Observations' (Blackie).

points. Thus it will be an easy operation to find a small piece of a polynomial bridging across a short gap for one set n_1, so as to provide a coincidence with a point of another set n_2. The only uncertainty is that ordinary rules of interpolation may not provide a polynomial of a degree as high as $2n$. However, for a short bridge that is not likely to be of importance.

The aim therefore of the following discussion will be to investigate the h^2-extrapolation from $\phi(x, h_1)$ and $\phi(x, h_2)$ both considered as *continuous* functions of x defined by series (3) and (7). However, let $\phi(x, h)$ remain a discontinuous function of h, defined only for discrete values of h, for the present; although ultimately we shall obtain for $\phi(x, h)$ an explicit function of h which defines ϕ for all values of h.

We may regard these series, analogous to TAYLOR'S, as finite, for the difference-ratios conventionally vanish when $s \geq n+1$. Then the number of terms in the series increases steadily as h diminishes and tends to infinity as $h \to 0$. For each particular value of h we have a particular formula for the aberration, different from the formulæ for all other values of h; and comparison is difficult. It is often convenient, and equally correct, to regard the series as consisting of infinitely many terms, each of which has a value for all permissible values of h. Excluding the arithmetically impracticable case of $h = 0$, we can then always find a term such that all beyond it are zero, so that the series are convergent.

The behaviour of any particular term as a function of h cannot be studied fully until we know the behaviour of the alternating difference of $\phi(a)$ which it contains as a factor. But in the meantime let us study the factors in λ or κ.

The coefficients in (3) *involving* λ.—For reference denote these by L defined thus

$$\left.\begin{aligned} L_{2s} &= (1-\lambda^2)(1-2^2\lambda^2)(1-3^2\lambda^2)\ldots\{1-(s-1)^2\lambda^2\} \\ L_{2s-1} &= L_{2s} \end{aligned}\right\} \quad \ldots \quad (8)$$

s being a positive integer. When s is fixed L_{2s} is a polynomial in λ and *odd powers of* λ *are absent.* If we were to make the substitution $\lambda = h/l$ in accordance with (2) above, then *odd powers of h would be absent,* for l is fixed.

The result of § 5 shows that, as far as L_{2s} alone is concerned, an h^2-extrapolation would be valid if $(s-1)^2(s-1)\lambda^2$ were small compared with unity. This condition is unnecessarily strict, as may be seen from the example $s = 5$. The condition then requires that λ should be small in comparison with $1/8$. Now

$$L_{2s} = L_{10} = 1 - 39\lambda^2 + 399\lambda^4 - 1261\lambda^6 + 900\lambda^8,$$

the largest of the set of discrete values of λ for which L_{10} does not vanish is $\lambda = 1/5$ and the terms of L_{10} are then respectively

$$1 - \frac{39}{25} + \frac{399}{625} - \frac{1261}{15625} + \frac{900}{390625}.$$

Even here the λ^2 term predominates over the higher powers.

The fact that L_{2s} vanishes when $\lambda = \pm 1/(s-1)$ and when $\lambda = \pm 1/(s-2)$ shows that it is not possible to choose λ, independently of s, so that L_{2s} may remain approximately of the form $1 - X\lambda^2$, where X is independent of λ, as $s \to \infty$. Consequently, *as far as the polynomials* L_{2s} *are concerned, it is not possible to fix h so as to make an* h^2*-extrapolation valid for every s. But for* L_{2s} *it will suffice to make n, the number of steps in the range l large relatively to* $(s-1)^{3/2}$. . . . .(9)

This condition is the same for all range and boundary equations, of even order. The behaviour of $\delta^{2s}\phi(a)/h^{2s}$ and $\mu\delta^{2s+1}\phi(a)/h^{2s+1}$ *must also be considered; it is different in different problems.* If λ be fixed while the integer s increases, then L_{2s} vanishes as soon as $(s-1)^2 \lambda^2 = 1$ and remains zero for all greater values of s. Let s_m be the greatest s for which L_{2s} does not vanish. Then

$$s_m{}^2\lambda^2 = 1 \qquad \text{so} \qquad s_m = n.$$

But differences of order higher than $2n$ are conventionally zero. Also L_{2n} is a coefficient of $\delta^{2n}\phi(a)$ in the expansion (3).

Thus the L_{2s} *that are not even roughly of the form* $1 + Xh^2$ *because they are zero when* $1 + Xh^2$ *is merely small, are just those* L_{2s} *that are multiplied by differences* $\delta^{2s}\phi(a)$ *which are conventionally zero.* . . . .(10)

Usually $f(x)$ will be represented by a series without end, and whether the termination of the series for $\phi(x, h)$ is of importance, is a further question.

The coefficients in (7) *involving* κ.—The discussion of series (7) is so like that of series (3) that it need not be recorded in detail. In order that the terms in $\mu\delta^{2s}\phi(a)/h^2$ and $\delta^{2s+1}\phi(a)/h^{2s+1}$ should be approximately of the form $1 + X\kappa^2$ where X is independent of κ it will suffice in accordance with § 5, that $s^3\kappa^2$ should be small in comparison with unity.

§ 8. The Permissible Step h for Functions of x but not of h.

§ 8.1. *General.*

In the approximate solution of differential equations we meet not merely differences taken with step h, but also differences of a function of h, namely, $\phi(x, h)$. Leaving aside the latter question, § 8 treats of the differences of the given functions of x, independent of h, which may occur as coefficients in the range-equation.

The necessary series are given by SHEPPARD.* When δ, μ are treated as operators along with $D \equiv d/dx$ the relations may be written in brief

$$\frac{\delta^n}{h^n} p(x) = D^n \left\{\frac{2 \sinh \frac{1}{2}hD}{hD}\right\}^n p(x). \qquad (1)$$

$$\frac{\mu\delta^n}{h^n} p(x) = D^n \left\{\frac{2 \sinh \frac{1}{2}hD}{hD}\right\}^n \cosh \tfrac{1}{2}hD \,.\, p(x). \qquad (2)$$

* W. F. SHEPPARD, 'Lond. Math. Soc. Proc.,' vol. 31, p. 464 (1899).

The first members of these equations are difference-ratios. The first factor of the second member if acting alone would produce the corresponding derivative. What is important for present purposes is that the other factor of the second member can in all cases be expanded in a series beginning with unity and proceeding by even powers of hD. For example

$$\frac{\mu\delta}{h} p(x) = \mathrm{D}\left\{1 + \frac{1}{6}h^2\mathrm{D}^2 + \frac{1}{120}h^4\mathrm{D}^4 + \frac{1}{5040}h^6\mathrm{D}^6 \ldots\right\} p(x). \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\delta^2}{h^2} p(x) = \mathrm{D}^2\left\{1 + \frac{1}{12}h^2\mathrm{D}^2 + \frac{1}{360}h^4\mathrm{D}^4 + \frac{1}{20160}h^6\mathrm{D}^6 \ldots\right\} p(x). \quad . \quad . \quad . \quad (4)$$

We shall have to apply the relations (1), (2) to coefficients $p(x)$ occurring in the given range-equation. The convergency of the resulting series cannot be investigated until this function is defined. Three simple types of $p(x)$ will, however, be discussed now, so that they may be ready to serve as comparison-series (series majorantes). What we want to know is how small h must be in order that the term in $h^2\mathrm{D}^2$ shall be the last that need be taken into account. The coefficient of this term can be found thus

$$\frac{2 \sinh \frac{1}{2}h\mathrm{D}}{h\mathrm{D}} = \frac{2}{h\mathrm{D}}\left\{\frac{h\mathrm{D}}{2} + \frac{1}{3!}\left(\frac{h\mathrm{D}}{2}\right)^3 + \ldots\right\} = 1 + \frac{h^2\mathrm{D}^2}{24} + \ldots.$$

So by raising both sides to the power n it is seen that the series for (1) begins

$$\frac{\delta^n}{h^n} p(x) = \mathrm{D}^n\left\{1 + \frac{n}{24}h^2\mathrm{D}^2 \ldots\right\} p(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

while because $\cosh \frac{1}{2}h\mathrm{D} = 1 + \frac{3}{24}h^2\mathrm{D}^2 + \ldots$, the series for (2) begins

$$\frac{\mu\delta^n}{h^n} p(x) = \mathrm{D}^n\left\{1 + \frac{(n+3)}{24}h^2\mathrm{D}^2 \ldots\right\} p(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

§ 8.2. *Exponential Type.*

A simple type that presents itself is that for which

$$h^2\mathrm{D}^2 p(x) = p(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for then the separate terms of series such as § 8.1 (3), (4), (5), (6) have $p(x)$ as a common factor and (5) and (6) may be written

$$\frac{\delta^n}{h^n} p(x) = \mathrm{D}^n p(x) \,.\left\{1 + \frac{n}{24} \ldots\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{\mu\delta^n}{h^n} p(x) = \mathrm{D}^n p(x) \,.\left\{1 + \frac{n+3}{24} \ldots\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

At the same time the sum to infinity of the series in the bracket is given by putting $hD = 1$ in § 8.1 (1), (2). The question then is: how small must h be in order that

$$(2 \sinh \tfrac{1}{2})^n = 1 + \frac{n}{24} + \dots \qquad (4)$$

and

$$(2 \sinh \tfrac{1}{2})^n \cosh \tfrac{1}{2} = 1 + \frac{n+3}{24} + \dots, \qquad (5)$$

with satisfactory approximation at the last term stated? The question involves the purpose of the calculation, and so cannot be decided in general; but some illustrations will be given.

$$2 \sinh \tfrac{1}{2} = 1 \cdot 0422_0, \qquad \cosh \tfrac{1}{2} = 1 \cdot 12763$$

n	$(1 \cdot 04220)^n$.	$1 + \frac{n}{24}$.	$(1 \cdot 04220)^n \times 1 \cdot 12763$.	$1 + \frac{n+3}{24}$.
1	1·0422	1·0417	1·1752	1·1667
10	1·5119	1·4166	1·7048	1·5417

The table shows that even for tenth differences the terms $n/24$ and $(n+3)/24$ form the greater parts of the excess above unity of the sum to infinity.

The convergence is faster if $h^2D^2p(x) < p(x)$. That is to say if $p(x) = e^{ax}$ the convergence will be faster than that indicated in the table provided that

$$|ha| < 1 \qquad (6)$$

But it is not possible to fix h so that the h^2-extrapolation remains valid for obtaining $D^n e^{ax}$ *from* $\delta^n e^{ax}/h^n$ *as* $n \to \infty$ . . . . (7)

For on comparing § 8.1 (5) with § 8.1 (1), in view of the fact that $De^{ax} = ae^{ax}$, the question is whether

$$\left\{\frac{2 \sinh (ha/2)}{ha}\right\}^n = 1 + \frac{nh^2a^2}{24} + \dots$$

with sufficient approximation at the last term shown? The first member is an exponential function of n, the second member is a linear function of n, and they cannot remain approximately equal as $n \to \infty$.

§ 8.3. *Cosine Type.*

If $h^2D^2p(x) = -p(x)$, we put $hD = \sqrt{-1}$ in § 8.1 (1), (2), (5), (6), thus changing hyperbolic to circular functions; so that in place of § 8.2 (4), (5), we have

$$(2 \sin \tfrac{1}{2})^n = 1 - \frac{n}{24} + \dots, \qquad (1)$$

$$(2 \sin \tfrac{1}{2})^n \cos \tfrac{1}{2} = 1 - \frac{n+3}{24} + \dots \qquad (2)$$

Now

$$2 \sin \tfrac{1}{2} = 0 \cdot 95885, \qquad \cos \tfrac{1}{2} = 0 \cdot 87758.$$

n	$(0\cdot95885)^n$.	$1 - \dfrac{n}{24}$.	$(0\cdot95885) \times 0\cdot87758$.	$1 - \dfrac{n+3}{24}$.
1	0·959	0·952	0·842	0·833
10	0·657	0·583	0·576	0·458

That is to say if $p(x) = A \cos(bx + \varepsilon)$ where A, b, ε are independent of x, then the convergence of SHEPPARD's series will be faster than that indicated in the table provided that

$$|hb| < 1. \qquad (3)$$

But it may be proved, as in § 8.2, that it is not possible to fix h so that $h^{-n}\delta^n \cos(bx + \varepsilon)$ should remain of the form $D^n \cos(bx + \varepsilon) + h^2 X$, where X is independent of h, as $n \to \infty$. . . . . (4)

§ 8.4. [*Revised March*, 1927.]

$$\textit{Type } p(x) = e^{ax}(c \cos bx + s \sin bx), \qquad (1)$$

where a, b, c, s are independent of x. It is not in general possible with this type of $p(x)$ to make $h^2D^2p(x)$ small in comparison with $p(x)$ by any choice of h, for $p(x)$ vanishes where $D^2p(x)$ does not. We must have some other standard of comparison. Let us choose as the standard

$$e^{ax} \, | \sqrt{(c^2 + s^2)} |$$

and let this be called the " amplitude " of $p(x)$. . . . . (2)

Now

$$h^{2n}D^{2n}p(x) = e^{ax}(hD + ha)^{2n} p(x) \qquad (3)$$

a proof of which is given in FORSYTH's ' Treatise on Differential Equations ' (1903), p. 56.

Also

$$hD(c \cos bx + s \sin bx) = hb(-c \sin bx + s \cos bx)$$

so that the operation hD acting on $(c \cos bx + s \sin bx)$ changes the amplitude in the ratio $|hb|$.

Therefore $h^{2n}D^{2n}$ acting on $e^{ax}(c \cos bx + s \sin bx)$ will decrease its amplitude if

$$h^{2n} \{ |a| + |b| \}^{2n} < 1. \qquad (4)$$

Now suppose that we take the series

$$p(x) + \frac{n}{24} h^2 D^2 p(x) + \ldots$$

$$p(x) + \frac{n+3}{24} h^2 D^2 p(x) + \ldots$$

the higher terms being defined as in § 8.1, and form other two series in which $p(x)$, $D^2 p(x)$, $D^4 p(x) \ldots$ are replaced by their amplitudes. Then these latter series will converge more rapidly than that tabulated in § 8.2 provided that (4) is satisfied. Now the actual value of any quantity of the form $e^{ax}(c \cos bx + s \sin bx)$ cannot exceed its amplitude. Therefore the terms which we have neglected will be truly negligible to the extent shown in the table of § 8.2 relative to the amplitude of the first term. The actual value of the first term will be zero for some values of x.

§ 9. Marching Problems.

§ 9.1. *Introduction.*

The range and boundary equations taken together usually provide derivatives of $f(x)$ of any order at one end of the range, so that the expansion of $f(x)$ in a Taylor series can be formed.

We next arrange analogous processes in order to solve the corresponding difference-problem. These give $\phi(x, h)$ as a series involving x and h together with differences of the coefficients in the range equation. The differences of these given functions are expressed as infinite series involving powers of h and derivatives of the same functions by means of Sheppard's formulæ of § 8.

It is found that only even powers of h occur. As h is then explicit, it is possible to discuss how small it must be in order to make the h^2-extrapolation valid.

Formulæ applicable to an unrestricted range-equation would be cumbrous. It is more convenient to show that the process as described above in words, is of very general application, and then to illustrate it by formulæ in special cases.

§ 9.2. *The expansion of $f(x)$ in Taylor's series.*

The derivatives of $f(x)$ at $x = a$ to any order are usually determinate.

For, denoting the x-derivative of order n by $f^{(n)}(x)$ let the given range equation be

$$0 = \psi\{f^{(n)}(x),\ f^{(n-1)}(x),\ f^{(n-2)}(x),\ \ldots\ f^{(1)}(x),\ f(x), x\}. \quad \ldots \quad (1)$$

where ψ denotes some specified function.

To make a marching problem it is necessary that for one point of the range, say $x = a$ we should have given

$$f^{(n-1)}(a),\ f^{(n-2)}(a),\ \ldots\ f^{(1)}(a),\ f(a). \quad \ldots \quad (2)$$

Together with (1) these determine $f^{n}(a)$.

On differentiating (1) once, whatever be the form of ψ, no derivative of f of order higher than $f^{(n+1)}(x)$ can appear. So $f^{(n+1)}(a)$ is usually determined. And so on to any order.* Exceptions occur, see below for an example.

There are subtle questions† connected with the expansion of $f(x)$ in the form

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \dots$$
$$\dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta x), \quad . \quad . \quad (3)$$

where θ is some suitably chosen number between 0 and 1.

(i) Does the series, omitting the θx term, converge ?

(ii) If so, does the θx term tend to zero as $n \longrightarrow \infty$?

In some cases it is possible, without solving the differential equation, to be sure that its integral is analytic and therefore‡ that both questions (i) and (ii) have affirmative answers. For example if the range-equation be

$$f''(x) + p(x) \,.\, f'(x) + q(x) \,.\, f(x) = 0$$

it is known that $f(x)$ is analytic for values of x at which both $p(x)$ and $q(x)$ are analytic. And similarly for a linear differential equation of any order, the solution is analytic except at the singularities of coefficients of the equation§ . . . . (4)

In this connection let us consider the function

$$f(x) = e^{-1/x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the TAYLOR expansion of which, starting from $x = 0$ was shown by CAUCHY‖ to consist only of the θx term, all the other terms being zero, so that the answers to the questions are (i) Yes, (ii) No. When x is real, e^{-1/x^2} is zero at $x = 0$ and increases continuously to unity as $x \longrightarrow \infty$. The singularity is revealed if we put $x = y\sqrt{-1}$ and then let $y \longrightarrow 0$; for it follows that $f(0) \longrightarrow \infty$ instead of to zero. It is interesting to see what would happen if we try to make this function arise as the solution of a marching problem, starting from $x = 0$. Differentiating (5)

$$f'(x) = \frac{2}{x^3} f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

a linear equation, but with a coefficient, $2x^{-3}$, having a singularity at $x = 0$; so that we should be warned by (4). To define the marching problem we take $f(0) = 0$ . . . (7)

If, ignoring the warning, we proceed in the usual way, putting $x = 0$ in (6) and substituting the value of $f(0)$ from (7) we obtain $f'(0)$, but it is in the indeterminate form 0/0. As nothing further could be done, no error would be committed.

* For restrictions, see DE LA VALLÉE POUSSIN, 'Cours d'Analyse' (1926), Ch. IV.

† GOURSAT, 'Cours d'Analyse' (1924), art. 170 ; VOSS, 'Encyk. Math. Wiss. II.,' A. 2, art. 14.

‡ WHITTAKER and WATSON, 'Modern Analysis' (1920), § 5.4.

§ WHITTAKER and WATSON, 'Modern Analysis' (1920), § 10.2, § 10.21.

‖ GOURSAT, 'Cours d'Analyse' (1924), art. 170.

For non-linear equations the question (i) can be discussed in the particular cases after the general term of the series has been found. But then there is no obvious way of answering question (ii); for although we have a way of finding any derivative at $x = a$, this tells us nothing about the behaviour of any derivative in the range of x, except in so far as we assume that the derivative can be expanded in TAYLOR's series; and that brings up question (ii) as regards the expansion of the derivative.

The "Calcul des limites" of CAUCHY* and the method of LIPSCHITZ† are both in a rather similar difficulty; for both assume that a certain upper bound M is known; but for non-linear equations M depends on $f(x)$, and $f(x)$ is unknown.

In § 9.4.3 a special device is employed.

To explore such questions would distract attention from the main purpose of this memoir. It seems better to admit that, for non-linear equations, casualties may occur, and to go on.

The following question (iii), relating to series in general, includes some of the finesse of questions (i) and (ii), but, unlike them,‡ question (iii) is also adapted to the practical needs of the computer.

(iii) Are n terms of the series enough without any remainder term?

For e^{-1/x^2} in relation to its TAYLOR series, the answer is no. If for another function the answer be "yes, when $n = 10^6$," then the series is valid, but probably useless. If the answer be "yes, when $n = 6$" then the computer can probably deal with it. In § 9.4.1 and § 10 the question (iii) will be answered in some special cases.

§ 9.3. *Finding the central differences of $\phi(x, h)$ at $x = a$ from the range- and boundary equations.*

As we here regard h as fixed, $\phi(x, h)$ may be contracted to $\phi(x)$.

For the difference equation analogous to § 9.2(1) we must choose a tabulation that avoids μ in the highest difference, for the reasons explained in § 2. Thus if n be even we must use

$$0 = \psi\left\{\frac{\delta^{2s}\phi(x)}{h^{2s}}, \quad \frac{\mu\delta^{2s-1}\phi(x)}{h^{2s-1}}, \quad \frac{\delta^{2s-2}\phi(x)}{h^{2s-2}}, \ldots \frac{\delta^{2}\phi(x)}{h^{2}}, \quad \frac{\mu\delta\phi(x)}{h}, \quad \phi(x), \quad x\right\}. \quad (1\text{A})$$

in which μ occurs with differences of odd order, s being an integer. But if n be odd we must use

$$0 = \psi\left\{\frac{\delta^{2s-1}\phi(x)}{h^{2s-1}}, \quad \frac{\mu\delta^{2s-2}}{h^{2s-2}}, \ldots \frac{\mu\delta^{2}\phi(x)}{h^{2}}, \quad \frac{\delta\phi(x)}{h}, \quad \mu\phi(x), \quad x\right\}, \quad . \quad . \quad (1\text{B})$$

in which μ occurs with the even orders.

* GOURSAT, 'Cours d'Analyse' (1924), art. 383.

† GOURSAT, 'Cours d'Analyse' (1924), art. 391.

‡ See H. JEFFREYS, F.R.S., 'Phil. Mag.,' July, 1926.

To make a marching problem, we must then have given at one end of the range all differences belonging to the same alternating set of all lower orders down to and including zero, namely, either for use with (1A)

$$\frac{\mu\delta^{2s-1}\phi(a)}{h^{2s-1}} = f^{(2s-1)}(a), \ldots \frac{\mu\delta\phi(a)}{h} = f^{(1)}(a), \quad \phi(a) = f(a), \quad . \quad . \quad . \quad (2\text{A})$$

or else for use with (1B)

$$\frac{\mu\delta^{2s-2}\phi(a)}{h^{2s-2}} = f^{(2s-2)}(a), \ldots \frac{\delta\phi(a)}{h} = f^{(1)}(a), \quad \mu\phi(a) = f(a). \quad . \quad . \quad . \quad (2\text{B})$$

The boundary equations (2A), in which the derivatives are given numbers, when inserted in (1A) determine the numerical value of $\delta^{2s}\phi(a)$. In like manner the conditions (2B) when inserted in (1B) determine $\delta^{2s-1}\phi(a)$. But let us leave (1B) and (2B) aside for the present.

To find the next higher difference we now seek a process analogous to differentiating (1A) or (1B) and substituting the values of the derivatives of lower orders. In order to simplify the discussion let ψ be restricted to be a *rational function* of the "alternating" differences of ϕ, the coefficients of the powers and products of these differences being any sort of analytic functions of x, provided that they are known . . . . (3)

The possibility that these coefficients may be transcendental functions of x will be taken into consideration later by the aid of SHEPPARD's series as in § 8.

Say then for (1A)

$$\psi = \mathrm{T}/\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\mathrm{T} = \sum_m p_m(x)\left(\frac{\delta^{2s}\phi}{h^{2s}}\right)^{a_m}\left(\frac{\mu\,\delta^{2s-1}\phi}{h^{2s-1}}\right)^{b_m}\cdots\left(\frac{\mu\,\delta\phi}{h}\right)^{y_m}\phi^{z_m}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and V is a similar sum of products with different $p, a, b \ldots z$, say $p', a', b' \ldots z'$.

By § 6 (2),

$$\delta\psi = \mu\mathrm{T} \,|\, \delta\left(\frac{1}{\mathrm{V}}\right) + \delta\mathrm{T} \,|\, \mu\left(\frac{1}{\mathrm{V}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

And by § 6 (12), (13)

$$\delta\psi = \frac{-\mu\mathrm{T}\,|\,\delta\mathrm{V} + \delta\mathrm{T}\,|\,\mu\mathrm{V}}{(\mu\mathrm{V})^2 - \frac{1}{4}(\delta\mathrm{V})^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The rules for differencing and averaging products can then be applied repeatedly to the separate terms of T and V until, in the expansion of $\delta\psi$, the operators δ and μ no longer act on any *product* of functions of x, but act directly on $\phi(x)$ and on the coefficients $p_m(x)$. $p'_m(x)$. Similar remarks apply to $\mu\psi$. Then $\mu^2 = 1 + \frac{1}{4}\delta^2$ may be used to remove all powers of μ above the first. The effect of μ and δ when acting on ψ is thus to produce another rational function of differences of ϕ. The argument can be extended from order to order so that $\delta^r\psi$ and $\mu\,\delta^{r-1}\psi$ are also rational functions of the differences of $\phi(x)$, where r is any integer.

Consider equations (1A) and (2A) which are centered with ϕ. The next higher difference that is so centered is $\mu\,\delta^{2s+1}\phi$. To produce this we first act on (1A) with δ. In view of the previous discussion, especially (4), (5), (7) and the part in § 6 about "many factors" the operation δ is seen to involve acting separately with both μ and δ on each of the differences already present in the second member of (1A), but not with both μ and δ on the same difference. Thus because ψ is a rational function of

$$\phi, \quad \mu\,\delta\phi, \quad \delta^2\phi, \quad \ldots \quad \mu\,\delta^{2s-1}\phi, \quad \delta^{2s}\phi \quad \ldots \ldots \tag{8}$$

Therefore $d\psi$ is some other rational function of

$$\left.\begin{array}{l} \mu\phi, \quad \mu^2\,\delta\phi, \quad \mu\,\delta^2\phi, \quad \ldots \quad \mu^2\,\delta^{2s-1}\phi, \quad \mu\,\delta^{2s}\phi \\ \text{and of} \qquad \delta\phi, \quad \mu\,\delta^2\phi, \quad \ldots \quad , \quad \mu\,\delta^{2s}\phi, \quad \delta^{2s+1}\phi \end{array}\right\} \ldots \tag{9}$$

Putting $\mu^2 = 1 + \frac{1}{4}\delta^2$ in (9) we see that $\delta\psi$ is some third rational function of

$$\mu\phi, \quad \delta\phi, \quad \mu\,\delta^2\phi, \quad \ldots \quad \delta^{2s-1}\phi, \quad \mu\,\delta^{2s}\phi, \quad \delta^{2s+1}\phi, \ldots \ldots \tag{10}$$

which are all centered with $\mu\phi$, unlike the given boundary values. To produce $\mu\delta\psi$ we next act on $\delta\psi$ with μ. The previous discussion shows that this involves acting on each of the differences in the list (10) with μ and δ separately. That is to say $\mu\delta\psi$ is some fourth rational function of

$$\left.\begin{array}{l} \mu^2\phi, \quad \mu\,\delta\phi, \quad \mu^2\,\delta^2\phi, \quad \ldots \quad \mu\,\delta^{2s-1}, \quad \mu^2\,\delta^{2s}\phi, \quad \mu\,\delta^{2s+1}\phi \\ \text{and of} \qquad \mu\,\delta\phi, \quad \delta^2\phi, \quad \ldots \quad \delta^{2s}\phi, \quad \mu\,\delta^{2s+1}\phi, \quad \delta^{2s+2}\phi \end{array}\right\}, \ldots \tag{11}$$

and because $\mu^2 = 1 + \frac{1}{4}\delta^2$ therefore $\mu\delta\psi$ is some fifth rational function of

$$\phi, \quad \mu\delta\phi, \quad \delta^2\phi, \quad \delta^{2s}\phi, \quad \mu\,\delta^{2s+1}\phi, \quad \delta^{2s+2}\phi, \quad \ldots \tag{12}$$

which are all centered with ϕ, like the given boundary values.

Thus either of the operations μ and δ, acting alone on a rational function of differences, changes the centering, and therefore $\mu\delta$ leaves the centering unchanged; so does δ^2. The highest possible difference in $\mu\delta\psi$ is seen to be *two* orders beyond the highest difference in ψ. One of these orders is brought in by δ and the other by μ via

$$\mu\,(\theta\,.\,\chi) = \mu\theta \,|\, \mu\chi + \tfrac{1}{4}\delta\theta \,|\, \delta\chi.$$

There is nothing like this in the analogous process with derivatives. Thus $\mu\,\delta\psi = 0$ *is one equation connecting given and found boundary values with the two unknowns* $\delta^{2s+2}\phi\,(a)$ and $\mu\,\delta^{2s+1}\phi(a)$. To find these separately we need another equation of no higher order. This is obtained by returning to $\delta\psi$ and operating on it with δ instead of μ obtaining $\delta^2\psi = 0$. The differences produced are the same whether δ or μ

be the operator, namely, those set out in the list (12) but $\mu\delta\psi$ and $\delta^2\psi$ will in general be different rational functions of these differences, so that the two equations necessary for the determination of $\delta^{2s+2}\phi(a)$ and $\mu\,\delta^{2s+1}\phi(a)$ are provided.

Let this process be continued. Starting with $\mu\delta\psi = 0$ and $\delta^2\psi = 0$, each of which contains no differences except those in the list (12), operate on each of them with $\delta\delta$ thus obtaining the two equations

$$\mu\,\delta^3\psi = 0, \qquad (13)$$

$$\delta^4\psi = 0. \qquad (14)$$

The first operation δ changes the list (12) to

$$\left.\begin{array}{l} \delta\phi,\quad \mu\,\delta^2\phi \quad \ldots \quad \delta^{2s+1}\phi,\quad \mu\,\delta^{2s+2}\phi,\quad \delta^{2s+3}\phi \\ \mu\,\phi,\quad \mu^2\,\delta\phi \quad \ldots\ldots \quad \mu^2\,\delta^{2s+1}\phi,\quad \mu\delta^{2s+2}\phi,\quad \delta^{2s+3}\phi \end{array}\right\} \qquad (15)$$

together with

And the second operation δ changes the list (15) to

$$\phi,\quad \mu\,\delta\phi \quad \ldots\ldots \quad \mu\,\delta^{2s+3}\phi,\quad \delta^{2s+4}\phi, \quad \ldots \qquad (16)$$

the last two of which can, therefore, be determined at $x = a$.

Proceeding in this manner the required differences can be found in pairs to any order at $x = a$, *as functions of differences of the coefficients* $p_m(x)$, $p'_m(x)$ *which are given functions of* x.

The discussion of equations (1B) and (2B) would be very similar to that of (1A) and (2A) and need not be recorded.

In the range and boundary equations every power of δ is divided by the same power of h. If in operations upon these we could always use h in the combination δ/h, then the result would involve h only in those forms which tend to derivatives as $h \to 0$. This can be done with one important exception arising from $\mu^2 = 1 + \frac{1}{4}h^2(\delta^2/h^2)$. This brings in even powers of h. *Except in forms which tend to derivatives as* $h \to 0$, *odd powers of* h *are absent from the expressions for the differences of* ϕ *as functions of the coefficients of the range and boundary equations.*

Now if we compare these operations with $\mu\,\delta$ and δ on the difference equation with the analogous operations with D on the differential equation in the light of the remarks at the end of § 6 it is seen that $\delta^r\phi(a, h)/h^r$ and $\mu\,\delta^r\phi(a, h)/h^r$ both tend to the same limit $d^rf(x)/dx^r$ as $h \to 0$.

§ 9.4. *Special Cases.*

§ 9.4.1. *Exponential type.*—Let

$$0 = \mathrm{D}f(x) + pf(x) \qquad (1)$$

where p is independent of x.

Also

$$f(a) = 1. \qquad (2)$$

We replace these by

$$0 = \delta\phi(x, h)/h + p \,.\, \mu\phi(x, h). \quad (3)$$

also

$$\mu\phi(a, h) = 1. \quad (4)$$

From (3) and (4)

$$0 = \delta\phi(a, h)/h + p \quad (5)$$

We operate on (3) with $\mu\delta/h$ obtaining

$$0 = \tfrac{1}{4}p \,.\, h^2\delta^3\phi/h^3 + \mu\delta^2\phi/h^2 + p\delta\phi/h, \quad (6)$$

and on (3) with δ^2/h^2 obtaining

$$0 = \delta^3\phi/h^3 + p\mu\delta^2\phi/h^2. \quad (7)$$

Specialising (6) and (7) for $x = a$ and using (5)

$$0 = \tfrac{1}{4}ph^2\delta^3\phi(a)/h^3 + \mu\delta^2\phi(a)/h^2 - p^2. \quad (8)$$

$$0 = \delta^3\phi(a)/h^3 + p\mu\delta^2\phi(a)/h^2. \quad (9)$$

Solving for the two unknowns $\mu\delta^2\phi$ and $\delta^3\phi$

$$\mu\delta^2\phi(a)/h^2 = p^2/(1 - p^2h^2/4). \quad (10)$$

and

$$\delta^3\phi(a)/h^3 = -p^3/(1 - p^2h^2/4). \quad (11)$$

In general when s is a positive integer it may be shown by continuing the process described in § 9.3 that

$$\mu\delta^{2s}\phi(a)/h^{2s} = p^{2s}/(1 - \tfrac{1}{4}p^2h^2)^s. \quad (12)$$

and

$$\delta^{2s+1}\phi(a)/h^{2s+1} = -p^{2s+1}/(1 - \tfrac{1}{4}p^2h^2)^s. \quad (13)$$

Contrast

$$D^{2s}f(a) = p^{2s} \quad \text{and} \quad D^{2s+1}f(a) = -p^{2s+1}. \quad (14), (15)$$

These differences of $\phi(a)$ are next inserted in the NEWTON-BESSEL series § 7 (7) with the result that

$$\phi(a+k, h) = 1 - kp + \left\{\frac{k^2p^2}{2!} - \frac{k^3p^3}{3!}\right\}\frac{1-\kappa^2/2^2}{1-p^2h^2/4} \ldots$$

$$\ldots + \left\{\frac{(kp)^{2s}}{(2s)!} - \frac{(kp)^{2s+1}}{(2s+1)!}\right\}\frac{(1-\kappa^2/4)(1-3^2\kappa^2/4)(1-5^2\kappa^2/4)\ldots\{1-(2s-1)^2\kappa^2/4\}}{(1-p^2h^2/4)^s}$$

$$\ldots \text{ to } s = n \quad (16)$$

where, as before

$$\kappa = h/k = 1/(n + \tfrac{1}{2}). \quad (17)$$

Here at last h is fully explicit. If we apply the test of § 5 to decide how small h must be in order that the h^2-extrapolation may be valid up to at least the $(2s + 1)$th power of p we find that k^2G2s must be small compared with unity, where G is the greater of $\{(2s - 1)/(2k)\}^2$ and $p^2/4$.

For example if $k=1$, $p=1$, $2s=10$, then $G=20\cdot25$, and it will suffice if h is small in comparison with $0\cdot07$.

Actually h is restricted by (17) in which n must be a positive integer. If $n=14$, $h=1/14\cdot5=0\cdot069$. Also $\kappa=1/14\cdot5$. The first pair of neglected terms, for which the h^2-extrapolation begins to fail so that our hope depends on their smallness, are those for $s=6$, which for this value of κ are

$$\left\{\frac{1}{12!}-\frac{1}{13!}\right\}\frac{\left(1-\frac{1}{29^2}\right)\left(1-\frac{3^2}{29^2}\right)\left(1-\frac{5^2}{29^2}\right)\left(1-\frac{7^2}{29^2}\right)\left(1-\frac{9^2}{29^2}\right)\left(1-\frac{11^2}{29^2}\right)}{\left(1-\frac{1}{29^2}\right)^6}.$$

The higher terms become steadily less until they vanish, when $(2s+1)\kappa>2$; also they alternate in sign. Thus the sum to infinity of the neglected tail lies between 0 and 1/12 !

There may be quicker ways of discussing this special problem (see § 3.2), but they would not illustrate the general process of § 9.3 so well.

§ 9.4.2. *Linear equation with variable coefficient.*—Given

$$D^2 f(x)+p(x)\,.\,f(x)=0 \quad \ldots\ldots \quad (1)$$

where $p(x)$ is an analytic function.

Also

$$Df(a)=1 \quad \text{and} \quad f(a)=0. \quad \ldots\ldots \quad (2), (3)$$

As usual we replace these by

$$\delta^2\phi(x)/h^2+p(x)\,.\,\phi(x)=0 \quad \ldots\ldots \quad (4)$$

also

$$\mu\delta\phi(a)/h=1 \quad \text{and} \quad \phi(a)=0. \quad \ldots\ldots \quad (5), (6)$$

From (4) and (6) it follows that

$$\delta^2\phi(a)=0. \quad \ldots\ldots \quad (7)$$

To find the higher alternating differences of $\phi(a)$ we begin by taking $\mu\delta/h$ of (4) using the rule for $\mu\delta$ of a product as given in § 6 and writing p, ϕ for $p(x)$, $\phi(x, h)$.

$$\mu\delta^3\phi\,.\,h^{-3}+\mu\delta p\,.\,h^{-1}\,\|\,\{\phi+\tfrac{1}{2}\delta^2\phi\}+\mu\delta\phi\,.\,h^{-1}\,\|\,\{p+\tfrac{1}{2}\delta^2 p\}=0. \quad (8)$$

In this on putting $x=a$ and using (5), (6), (7) there follows

$$\mu\delta^3\phi(a)\,.\,h^{-3}+p(a)+\tfrac{1}{2}h^2\{\delta^2 p(a)/h^2\}=0. \quad \ldots\ldots \quad (9)$$

Again, taking δ^2/h^2 of (4) using the rule for δ^2 of a product given in § 6,

$$\delta^4\phi\,.\,h^{-4}+p\,.\,\delta^2\phi\,.\,h^{-2}+2\mu\delta p h^{-1}\,\|\,\mu\delta\phi\,.\,h^{-1}+\phi\delta^2 p\,.\,h^{-2}+\tfrac{1}{2}\delta^2\phi\,\|\,\delta^2 p h^{-2}=0. \quad (10)$$

Now let $x=a$ and use (5), (6), (7) obtaining

$$\delta^4\phi(a)\,.\,h^{-4}+2\mu\delta p(a)\,.\,h^{-1}=0. \quad \ldots\ldots \quad (11)$$

The alternating differences of $\phi(a)$ are here being determined singly instead of in the usual pairs, because the derivative next below the highest is missing from the range equation. The process can be continued indefinitely, giving the differences of ϕ as functions of differences of p both at $x = a$. The expressions become lengthy unless $p(x)$ is some simple function. The fourth order will suffice for illustration.

Next the differences of $\phi(a)$ are inserted in the NEWTON-STIRLING series § 7 (3) with the result that

$$\phi(a+l) = 0 + l + 0 - \frac{l^3}{3!}\left\{p(a) + \frac{h^2}{2}\frac{\delta^2 p(a)}{h^2}\right\}\left(1 - \frac{h^2}{l^2}\right) - \frac{l^4}{4!}\frac{2\,\mu\delta p(a)}{h}\left(1 - \frac{h^2}{l^2}\right). \quad (12)$$

Lastly, to make h fully explicit $\mu\delta p(a)$, $\delta^2 p(a)$ must be expressed in terms of derivatives of p, by SHEPPARD'S series of § 8 giving

$$\phi(a+l) = l - \frac{l^3}{3!}\left\{p(a) + \frac{h^2}{2}D^2\left(1 + \tfrac{1}{12}h^2D^2 + \tfrac{1}{360}h^4D^4 \ldots\right)p(a)\right\}\left(1 - \frac{h^2}{l^2}\right)$$

$$- \frac{l^4}{4!}\left\{D\left(1 + \tfrac{1}{6}h^2D^2 + \tfrac{1}{120}h^4D^4 \ldots\right)p(a)\right\}\left(1 - \frac{h^2}{l^2}\right) + \ldots \quad (13)$$

Only even powers of h occur. For the terms shown the h^2-extrapolation will be valid if h is not only much less than l but if also $h^2D^2p(a)$ is much less than $p(a)$ as may be seen by using the series of §8 for comparison; and possibly in other circumstances less stringent.

§ 9.4.3. *Cube of dependent variable.*—Suppose that we are given

$$D^2 f = -f^3 \quad \ldots \quad (1)$$

together with

$$f(0) = 1 \text{ and } df/dx = 0 \text{ at } x = 0. \quad \ldots \quad (2), (3)$$

The solution may be shown* to be $f(x) = \mathrm{cn}x$ when k, the modulus of this JACOBIAN elliptic function, is $1/\sqrt{2}$ . . . . . (4)

CAYLEY† gives the first few terms of the expansion

$$\mathrm{cn}x = 1 - \frac{x^2}{2!} + \frac{3x^4}{4!} - \frac{27x^6}{6!} + \frac{441x^8}{8!} + R(x). \quad \ldots \quad (5)$$

The remainder $R(x)$ can always be found, because $\mathrm{cn}x$ is known from the tables of the first elliptic integral. For example $R(1) = -0{\cdot}002$.

To see how the corresponding series for $\phi(x, h)$ begins, we replace (1), (2), (3) by

$$h^{-2}\delta^2\phi = -\phi^3\,;\ \phi(0) = 1\,;\ h^{-1}\mu\delta\phi(0) = 0. \quad \ldots \quad (6), (7), (8)$$

These give immediately

$$h^{-2}\delta^2\phi(0) = -1 \quad \ldots \quad (9)$$

* CAYLEY, 'An elementary treatise on Elliptic functions' (1895), art. 19.

† *Loc. cit.* p. 57, correcting the first term from u to 1.

Knowing from the symmetry of (6), (7), (8) that the odd terms of the series all vanish, we omit the operation $\mu\delta$ and act on (6) with $h^{-2}\delta^2$ according to § 6 (5), obtaining

$$h^{-4}\delta^4\phi = -h^{-2}\delta^2\phi^3 = -h^{-2}\{\phi\,.\,\delta^2\phi^2 + 2\mu\delta\phi \mid \mu\delta\phi^2 + \phi^2\delta^2\phi + \tfrac{1}{2}\delta^2\phi \mid \delta^2\phi^2\}.$$

A second application of § 6 (5), (4), gives

$$h^{-4}\delta^4\phi = -h^{-2}[\{2\phi\,.\,\delta^2\phi + 6(\mu\delta\phi)^2 + \tfrac{1}{2}(\delta^2\phi)^2\}\{\phi + \tfrac{1}{2}\delta^2\phi\} + \phi^2\delta^2\phi]. \qquad (10)$$

Now specialising for $x = 0$ and substituting from (7), (8), (9) we have

$$h^{-4}\delta^4\phi(0) = 3 - \tfrac{3}{2}h^2 + \frac{h^4}{4}. \quad \ldots\ldots\ldots \quad (11)$$

Inserting these difference-ratios in the series § 7, (3), it is found to begin thus

$$\phi(x, h) = 1 - \frac{x^2}{2!} + \frac{3 - \frac{3}{2}h^2 + \frac{1}{4}h^4}{4!}x^4\left(1 - \frac{h^2}{x^2}\right). \quad \ldots\ldots \quad (12)$$

§ 9.4.4. $f(x)$ *single-valued, but* $\phi(x)$ *branching.*—Let the problem be

$$df/dx = f^2 \text{ and } f(0) = 1 \quad \ldots\ldots\ldots \quad (1), (2)$$

The solution is

$$f(x) = 1/(1-x) \quad \ldots\ldots\ldots \quad (3)$$

a single valued function except at $x = 1$.

The analogous problem in differences is

$$h^{-1}\delta\phi = (\mu\phi)^2 \quad \text{and} \quad \mu\phi(0) = 1 \quad \ldots\ldots\ldots \quad (4), (5)$$

In accordance with § 9.3 the differences $\mu\delta^2\phi(0)$ and $\delta^3\phi(0)$ have to be determined from a pair of equations each of which involves both unknowns. These equations are quadratic, so that the differences are multivalued.

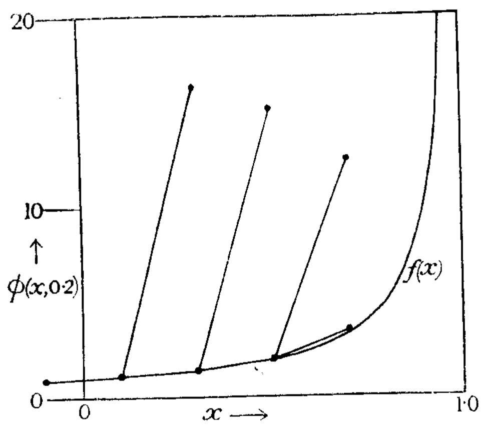

Fig. 2A.

If we step out the values of ϕ by arithmetic, they are determined by quadratic equations, which have two real roots or none. So the graph of $\phi(x)$ either bifurcates at each step or comes to an end, thus resembling the branches of some trees. In fig. 2A the seven large dots show all the discrete real values of $\phi(x, 0\cdot2)$ when x is positive. The lines joining the dots are merely to show the sequence, and do not represent ϕ as the polynomials of § 7 would do. The $f(x)$ curve lies close to one branch. If h is reduced to $0\cdot1$ one branch continues to fit $f(x)$ well; another is more steeply divergent.

§ 9.5. *Conclusion of the Marching Problem.*

It has been shown that when $\phi(x, h)$ is expanded in a series arranged according to the alternating differences of $\phi(a, h)$, then the separate terms of the series are functions of a number of expressions of the form $(1 + Xh^2 + \ldots)$ where X is independent of h, and $+ \ldots$ indicates possible terms in higher even powers of h. These expressions arise in three ways :—

(i) There are some factors like $(1 + Xh^2 \ldots)$ which are the same for all range and boundary equations. They have been discussed in § 7. They bring in h in the form $h^2/(x-a)^2$.

(ii) Other expressions like $(1 + Xh^2 + \ldots)$ are brought in by the differences of $\phi(a, h)$. They depend upon the particular form of the range and boundary equations. They do not involve x. They may be divided into two sub-classes :—

(*a*) Some appear even if the coefficients of the derivatives in the given range equations are independent of x, as in § 9.4.1, § 9.4.3.

(*b*) Others are produced by the variability of the coefficients in the range equation, as in § 9.4.2.

The production of factors $(1 + Xh^2 + \ldots)$ may be traced back for classes (i) and (ii) (*a*) to the formula $\mu^2 = 1 + \frac{1}{4}h^2(\delta^2/h^2)$. But in the class (ii) (*b*) the source of h^2 is different, and depends on the formulæ $\mu = \cosh \frac{1}{2}hD$, $\delta = 2 \sinh \frac{1}{2}hD$ as in § 8.

How to choose h so that a conglomerate of expressions like $(1 + Xh^2)$ may make the h^2-extrapolation valid, has been discussed in § 5. It is shown in § 7 that, for any fixed h, the h^2-extrapolation cannot in general remain valid for a derivative d^sf/dx^s as $s \to \infty$. This peculiarity may not hinder the purpose of the calculation.

§ 10. A Simple Jury Problem.

When there is only one independent variable, jury problems may be evolved from marching problems. The peculiarity of a jury problem is that not all of the derivatives of $f(x)$ below the order of the highest in the range equation, are given at either end of the range. However we put in symbols for those that are unknown at one end $x = a$ and write out the expansions of $f(x)$ and of $\phi(x, h)$ as if it were a marching problem starting from $x = a$. The unknowns are then determined by the given conditions at the other end of the range.

To illustrate this process in a very simple case let it be given that

$$\frac{d^2f(x)}{dx^2} + f(x) = 0, \qquad f(a) = 0, \qquad f(b) = 1 \ldots \quad (1), (2), (3)$$

We replace these by

$$\frac{\delta^2\phi}{h^2} + \phi = 0, \qquad \phi(a) = 0, \qquad \phi(b) = 1. \ldots \quad (4), (5), (6)$$

Let

$$\mu\delta\phi(a)/h = \beta, \text{ an unknown.} \ldots \quad (7)$$

Then by the process which has been described in § 9.3 and illustrated in § 9.4 it can be shown that all the even differences of ϕ all vanish at $x = a$ and that for the odd differences

$$\frac{\mu\delta^{2s+1}\phi(a, h)}{h^{2s+1}} = (-)^s \beta, \quad \ldots \quad (8)$$

where s is a positive integer.

To fit with the arithmetical practice we suppose that the range $(b - a)$ is divided into n equal steps thus fixing h to be equal to $(b - a)/n$. The NEWTON-STIRLING series § 7 (3) then shows that,

$$\phi(b) = \beta\left[(b-a) - \frac{(b-a)^3}{3\,!}\left(1 - \frac{1}{n^2}\right) + \frac{(b-a)^5}{5\,!}\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{2^2}{n^2}\right) - \ldots\right], \quad (9)$$

ending at the term of degree $2n - 1$, and in view of (6) this determines β and thereby reduces the jury problem to a marching problem.

The next question is to find how $\phi(x, h)$ depends on h at some intervening point. In view of the extended definition of ϕ given in § 7 it does not matter whether this intervening point can be reached by an integral number of steps from $x = a$. Let the point be $x = a + l$, and let r be defined by

$$l/(b - a) = r/n. \quad \ldots \quad (10)$$

Then by a second application of the NEWTON-STIRLING series

$$\phi(a + l) = \beta\left[l - \frac{l^3}{3\,!}\left(1 - \frac{1}{r^2}\right) + \frac{l^5}{5\,!}\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{2^2}{r^2}\right)\ldots\right]. \quad \ldots \quad (11)$$

So that on eliminating β between (9) and (11)

$$\phi(a + l) = \frac{l - \frac{l^3}{3\,!}\left(1 - \frac{1}{r^2}\right) + \frac{l^5}{5!}\left(1 - \frac{1}{r^2}\right)\left(1 - \frac{2^2}{r^2}\right) - \ldots}{(b-a) - \frac{(b-a)^3}{3\,!}\left(1 - \frac{1}{n^2}\right) + \frac{(b-a)^5}{5!}\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{2^2}{n^2}\right) - \ldots} \quad \ldots \quad (12)$$

both ending at terms in $(2n - 1)\,!$

The general terms of the series may be seen by reference to (8) and § 7 (3).

The value of h which is small enough to make the h^2-extrapolation valid, may be decided in the following way. An algebraic argument is given concurrently with a numerical illustration in the margin. As $(b - a)$ is given, we may choose a positive integer s so that not only is $(b - a)^{2s-1}/(2s - 1)\,!$ negligibly small but also

$$(2s - 1) > |b - a| \quad \ldots \quad (13)$$

The tail of the series in the denominator of (12) from and including the term

$$\frac{(b-a)^{2s-1}}{(2s-1)\,!}\left\{1 - \frac{h^2}{(b-a)^2}\right\}\left\{1 - \frac{2^2h^2}{(b-a)^2}\right\}\ldots\left\{1 - \frac{(s-1)^2\,h^2}{(b-a)^2}\right\}$$

$b - a = 2$
$s = 7$
$\frac{(b-a)^{2s-1}}{(2s-1)\,!} = 1 \cdot 3 \times 10^{-6}$

is then a series with alternating signs and terms steadily decreasing in modulus, because the extra factors in h^2, that come on in the higher terms, are all less than unity and vanish altogether after a certain term. So the modulus of the sum to infinity of the tail is less than the modulus of first term of the tail, which in turn is less than $|(b-a)^{2s-1}/(2s-1)\,!\,|$, which is negligible. Similarly the tail of the series in the numerator of (12) from and including the term in l^{2s-1} must be negligible because $|l|<|b-a|$.

Next we can choose h so that all the continued products in n and r, which are not in negligible terms, are of the form $1+h^2\times$ (number independent of h). For this it suffices, according to the rule of § 5 that $h^2 G(s-1)$ should be small compared with unity, where G is the greater of $(s-1)^2/(b-a)^2$ and $(s-1)^2/l^2$; that is to say h^2 should be small compared with

$$l^2/(s-1)^3. \qquad (14)$$

When h is thus restricted we may rearrange the numerator and denominator of (12) according to powers of h, neglecting powers beyond h^2. It is thus found that

$$\phi(a+l,h)=\frac{\sin l+h^2F(l)}{\sin(b-a)+h^2F(b-a)} \qquad (15)$$

in which

$$F(l)=\frac{l}{3\,!}-\frac{l^3}{5\,!}(1^2+2^2)+\frac{l^5}{7\,!}(1^2+2^2+3^2)\ldots$$

$$+(-)^{t+1}\frac{l^{2t-1}}{(2t+1)\,!}(1^2+2^2+3^2+4^2\ldots t^2)$$

$$+\ldots \text{ to the term in } (2n-1)\,! \qquad (16)$$

Now by a reapplication of the rule of § 5 the second member of (15) will be of the required form if $h^2 G_1\, 2\leqslant 1$ where G_1 is the greater of $|F(l)/\sin l|$ and $|F(b-a)/\sin(b-a)|$. (17) We have thus in (14) and (17) three separate restrictions on h, and it suffices if the strictest be observed.

If $\sin(b-a)=0$ then $G_1\to\infty$, and no value of h will make the h^2-extrapolation valid. There is a corresponding peculiarity in the solution of the problem in the infinitesimal calculus. To satisfy (1) and (2) $f(x)=c\sin(x-a)$ where c is an arbitrary constant. To satisfy (3) also $1=c\sin(b-a)$ which determines c unless $\sin(b-a)=0$.

$l=1$

$h^2\leqslant 1/6^3$

$h\leqslant 1/15$

$n\geqslant 15$

$F(1)=0{\cdot}130$

$F(2)=0{\cdot}079$

$\frac{F(1)}{\sin 1}=0{\cdot}155$

$\frac{F(2)}{\sin 2}=0{\cdot}190$

therefore

$h^2\leqslant\frac{1}{0{\cdot}38}$

$h\leqslant 1{\cdot}6$

much less strict than (14) in this case.

Thus the case in which no suitable value of h can be found is that in which the problem is indeterminate. Exceptions of this type are usual in jury problems. Physically they are connected with the disturbing effects of free periods of oscillation.

§ 11. Volterra's Integral Equation of the First Kind.

§ 11.1. *Introduction.*

The given equation is

$$0 = p(x) + \int_0^x \kappa(x, y) \cdot f(y)\, dy \quad \ldots \ldots \ldots \quad (1)$$

in which $p(x)$ and $\kappa(x, y)$ are given functions, and the problem is to find $f(y)$. The h^2-extrapolation has been used in this connection previously,* but justification was lacking.

The problem will now be approached by the aid of μ and δ and the Newton-Bessel† series. That is to say we shall compare two analogous processes; in the one $f(x)$ will be expanded in derivatives of $p(x)$ and of $\kappa(x, y)$ with powers of x in the coefficients; in the other derivatives of $f(x)$ will be replaced by difference-ratios and means of $\phi(x, h)$.

§ 11.2. *The Expansion in Derivatives.*‡

In § 11.1 (1) let $x \longrightarrow 0$; it follows that

$$p(0) = 0 \quad \ldots \ldots \ldots \ldots \quad (1)$$

$p(x)$ must be given thus, otherwise § 11.1 (1) is self-contradictory. We shall assume that $p(x)$ and $\kappa(x, y)$ possess derivatives to any order at all points of the range with respect to all the variables that they contain . . . . (2)

The process is to take x-derivatives of equation § 11.1 (1) of orders 1, 2, 3, 4, 5, . . ., and then to let $x = 0$, so as to get rid of all the integrals. This leaves us with a set of linear equations connecting $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ ... with known derivatives of $p(x)$ and of $\kappa(x, y)$.

Taking $\partial/\partial x$ of § 11.1 (1)

$$0 = \frac{\partial p(x)}{\partial x} + f(x) \cdot \kappa(x, x) + \int_0^x f(y) \cdot \frac{\partial \kappa(x, y)}{\partial x}\, dy \quad \ldots \ldots \quad (3)$$

Let $x = 0$ then

$$0 = \frac{\partial\, p(0)}{\partial x} + f(0) \cdot \kappa(0, 0) \quad \ldots \ldots \ldots \quad (4)$$

which gives $f(0)$ in terms of knowns.

Taking $\partial/\partial x$ of (3)

$$0 = \frac{\partial^2 p(x)}{\partial x^2} + \frac{\partial f(x)}{\partial x}\kappa(x, x) + f(x)\frac{\partial \kappa(x, x)}{\partial x} + f(x)\left[\frac{\partial \kappa(x, y)}{\partial x}\right]_{x=y} + \int_0^x f(y) \cdot \frac{\partial^2 \kappa(x, y)}{\partial x^2}\, dy. \quad (5)$$

* L. F. Richardson, 'Phil. Trans.,' A, vol. 223, p. 363 (1923).

† By mistake the nucleus $\kappa(x, y)$ has been printed as kappa instead of capital K. This kappa has of course no connection with the same letter in the Newton-Bessel series.

‡ E. T. Whittaker, 'Roy. Soc. Proc.,' A, vol. 94, p. 367 (1918), has discussed a less general type of nucleus, namely, $\kappa(x - y)$.

In the last we have to distinguish whether x is put equal to y before or after the differentiation. A notation like $\partial\kappa(0,0)/\partial x$ is ambiguous and causes mistakes. Let $x=0$ then

$$0=\frac{\partial^2 p(0)}{\partial x^2}+\frac{\partial f(0)}{\partial x}\cdot\kappa(0,0)+f(0)\left\{\left(\frac{\partial\kappa(x,x)}{\partial x}\right)_{x=0}+\left(\frac{\partial\kappa(x,y)}{\partial x}\right)_{x=y=0}\right\} \qquad (6)$$

which with (4) determines $\partial f(0)/\partial x$.

This process can evidently be continued without end. The results may be written compactly if we have an operator to mean "put $x=y$." Try $\|$ for this purpose, so that if $\mathrm{F}(x, y)$ is any function of x and y

$$\|\,\mathrm{F}(x,y)=\mathrm{F}(x,x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The symbol $\|$ may be pronounced "equalised." Then if D denote $\partial/\partial x$

$$\mathrm{D}\,\| \text{ is not in general equivalent to } \|\,\mathrm{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Provided that f means $f(x)$ except in the integrand, (3) may now be written

$$0=\mathrm{D}p+f\,\rule{1pt}{1em}\,\|\,\kappa+\int_0^x f(y)\,.\,\mathrm{D}\kappa\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\mathrm{A})$$

in which $\rule{1pt}{1em}$ is the "wall" of § 6.

Similarly (5) may be written

$$0=\mathrm{D}^2p+\mathrm{D}f\,\rule{1pt}{1em}\,\|\,\kappa+f\,\rule{1pt}{1em}\,\{\mathrm{D}\,\|\,\kappa+\|\,\mathrm{D}\kappa\}+\int_0^x f(y)\,\mathrm{D}^2\kappa\,dy. \quad . \quad . \quad . \quad (5\mathrm{A})$$

Now operating on (5A) with D we obtain

$$0=\mathrm{D}^3p+\mathrm{D}^2f\,\rule{1pt}{1em}\,\|\,\kappa+\mathrm{D}f\,\rule{1pt}{1em}\,\{2\mathrm{D}\,\|\,\kappa+\|\,\mathrm{D}\kappa\}+f\,\rule{1pt}{1em}\,\{\mathrm{D}^2\,\|\,\kappa+\mathrm{D}\,\|\,\mathrm{D}\kappa+\|\,\mathrm{D}^2\kappa\}$$
$$+\int_0^x f(y)\,\mathrm{D}^3\,\kappa dy. \quad (9)$$

Operating on (9) with D

$$0=\mathrm{D}^4p+\mathrm{D}^3f\,\rule{1pt}{1em}\,\|\,\kappa+\mathrm{D}^2f\,\rule{1pt}{1em}\,\{3\mathrm{D}\,\|\,\kappa+\|\,\mathrm{D}\kappa\}+\mathrm{D}f\,\rule{1pt}{1em}\,\{3\mathrm{D}^2\,\|\,\kappa+2\mathrm{D}\,\|\,\mathrm{D}\kappa+\|\,\mathrm{D}^2\kappa\}$$
$$+f\,\rule{1pt}{1em}\,\{\mathrm{D}^3\,\|\,\kappa+\mathrm{D}^2\,\|\,\mathrm{D}\kappa+\mathrm{D}\,\|\,\mathrm{D}^2\kappa+\|\,\mathrm{D}^3\kappa\}+\int_0^x f(y)\,.\,\mathrm{D}^4\kappa\,dy, \quad (10)$$

an equation which would be very cumbrous if written without the aid of $\|$. And so on. When $x=0$ the integrals all vanish, and what is left is a set of linear equations to determine f, $\mathrm{D}f$, $\mathrm{D}^2f \ldots$ at $x=0$. They are so arranged that $f(0)$ is determined by the first equation; $\mathrm{D}f(0)$ by the second equation and $f(0)$; $\mathrm{D}^2f(0)$ by the third equation together with $f(0)$ and $\mathrm{D}f(0)$; and in general $\mathrm{D}^n f(0)$ by the nth equation together with the results already extracted from the previous $n-1$ equations.

It will be assumed that f is such that it can be expanded by TAYLOR's theorem . (11)

For example, if $\kappa(x,y)=1+e^{-xy^2}$, an unsymmetrical kernel, it may be shown in this way that the series for $f(x)$ begins thus

$$f(x)=-\tfrac{1}{2}\left\{\mathrm{D}p(0)+x\mathrm{D}^2p(0)+\frac{x^2}{2!}\mathrm{D}^3p(0)+\frac{x^3}{3!}(\mathrm{D}^4p(0)+4\mathrm{D}p(0))+\ldots\right\}. \quad (12)$$

§ 11.3. *The Analogous Process in Differences.*

Along a vertical line on which $\kappa(x, y) \,.\, f(y)$ is tabulated we seek to define a sum $h\Sigma(x)$ which shall be a simple but good approximation, vanishing at $x = 0$, to

$$\int_0^x \kappa(x, y) \,.\, f(y) \,.\, dy.$$

For brevity let $\kappa(x, y) \,.\, f(y)$ be denoted by

$$\theta(x, y) \ldots \qquad (1)$$

Let it be granted that θ is tabulated at equally spaced values of y and that the only alternative is that the ends of the range $0 \leqslant y \leqslant x$ may either coincide with points where θ is tabulated or fall midway between them. In either case let $\Sigma(x)$ be defined, when x is positive, to be the sum of the tabular values of θ lying within the range, plus half of the tabular values, if any, of θ lying at the termini of the range. When x is negative $\Sigma(x)$ is defined as above but with the opposite sign . . . . (2)

The integral equation is replaced by

$$0 = p(x) + h\Sigma(x). \qquad (3)$$

The Tabulation.—The choice of tabulation is important. In the diagrams let the sloping line be $x = y$ and let the dots indicate the points at which $\kappa(x, y)$ and $f(y)$ are both tabulated. The x-axis is horizontal.

The co-ordinate lines are $x = \pm nh$, $y = \pm nh$, where n is an integer.

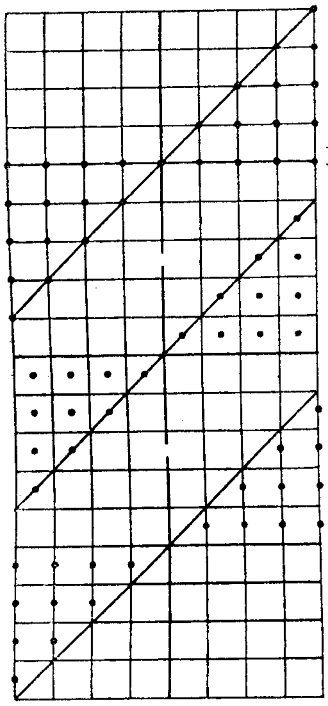

Fig. 3.

Fig. 4.

Fig. 5.

At first sight fig. 3 looks harmless. $\Sigma(h)$ involves $f(h)$ and $f(0)$ by the rule (2). $\Sigma(-h)$ involves $f(-h)$ and $f(0)$. At the origin we can put $x = 0$ in (3), but as both terms of the equation then vanish we get no information. Next, by operating with $\mu_x\delta_x$ and δ_x^2 in which suffix x means that y is constant we can form two equations

$$0 = \mu_x\delta_x p(0) + \mu_x\delta_x\Sigma(0),$$

$$0 = \delta_x^2 p(0) + \delta_x^2\Sigma(0),$$

but these involve three unknowns $f(-h)$, $f(0)$, $f(h)$ and so are insoluble. The next pair of alternating differences $\mu_x\delta_x^3$ and d_x^4 would bring in two more unknowns $f(-2h)$ and $f(2h)$; and so on for higher pairs. Thus the tabulation of fig. 3 produces an indeterminate problem and must be rejected. We have already met unworkable tabulations in § 2.

The above difficulty is avoided by the tabulation of fig. 4, for with this, if we act on equation (3) separately with μ_x and δ_x, we obtain two equations sufficient to determine $f(\tfrac{1}{2}h)$

and $f(-\frac{1}{2}h)$ which are the only unknowns. Next $\mu\delta_x^2$ and δ_x^3 of (3) determine $f(\frac{3}{2}h)$ and $f(-\frac{3}{2}h)$, and so on, in pairs without end. But $\Sigma(\frac{1}{2}h)$ then depends on κ at one end only of the range of integration. This is an ill-balanced arrangement, likely to produce error. Indeed, an investigation which would occupy two pages, shows that the limit of $h^{-1}\delta\phi$ as $h \to 0$ is at the origin $\{\mathrm{D}p \,|\, \mathrm{D} \,\|\, \kappa - \frac{1}{2}\mathrm{D}^2 p \,|\, \|\, \kappa\}(\|\,\kappa)^{-2}$. This does not agree with Df, which according to § 11.2 (3A, 5A), is at the origin

$$\{\mathrm{D}p \,|\, (\mathrm{D} \,\|\, \kappa + \|\, \mathrm{D}\kappa) - \mathrm{D}^2 p \,|\, \|\, \kappa\}(\|\,\kappa)^{-2}.$$

So if we were to use the tabulation of fig. 4, then the limit of ϕ as $h \to 0$ would not be a solution of the given integral equation.

A tabulation which avoids both the indeterminancy of fig. 3 and the wrong limit of fig. 4 is shown in fig. 5. It was used by the present writer in connection with "Spheres shot upwards"* and will now be examined more critically.

The specialising operator.—In § 11.2 (8) we have begun to use an algebra which is in part non-commutative. It is desirable for consistency that *the symbols for all operations should be written down in the sequence in which the operations are to be performed; or else in the reverse sequence*, which happens to be customary; but not in a mixture of the two sequences. Now such a mixture occurs if D$f(a)$ be used to denote the value of D$f(x)$ when $x = a$. It should be (a) Df, or to make it more distinct we can use a special bracket, writing $[a]\,\mathrm{D}f$. Here $[a]$ means "put $x = a$." . . . . (4)
It may be called a "localising" or "specialising" operator.

Similarly when we have two independent variables $[a, b]\,\kappa$ will mean the value of $\kappa(x, y)$ when $x = a$, $y = b$ . . . . (5)

With the aid of this new operator it is possible to avoid ambiguity.

On the same scheme we might use $[x, x]$ to mean "put $y = x$" But $\|$ is easier to write, to read and to print and will be used instead.

The definitions of μ and δ now appear thus, where θ is any function of x

$$[x]\,\mu\theta = \tfrac{1}{2}[x + \tfrac{1}{2}h]\,\theta + \tfrac{1}{2}[x - \tfrac{1}{2}h]\,\theta \quad (6)$$

$$[x]\,\delta\theta = [x + \tfrac{1}{2}h]\,\theta - [x - \tfrac{1}{2}h]\,\theta \quad (7)$$

Halving the second of the equations and adding, we obtain the advancing operator $(\mu + \frac{1}{2}\delta)$ in the relation

$$[x](\mu + \tfrac{1}{2}\delta)\,\theta = [x + \tfrac{1}{2}h]\,\theta \quad (8)$$

And by subtraction the retarding operator $\mu - \frac{1}{2}\delta$ thus

$$[x](\mu - \tfrac{1}{2}\delta)\theta = [x - \tfrac{1}{2}h]\,\theta \quad (9)$$

* 'Phil. Trans.,' A, vol. 223, p. 361 (1923).

Formation of the sum $\Sigma(x)$.—Let the numerical values of $\theta(x, y)$ be A, B, C, ... a, b, c ... arranged as in fig. 6.

Then in accordance with (2) we replace $\int_0^x \theta(x, y)\,dy$ by $h\Sigma$ where, for example, at $x = 3h$ we have $\Sigma = \mathrm{D} + \mathrm{E} + \mathrm{F}$ simply. We are concerned only with means and differences in the x-direction so that μ and δ may be used to denote these, without needing x as a suffix. Let us form in turn $\mu\delta$, δ^2, $\mu\delta^3$, δ^4 ... of Σ at $x = 0$.

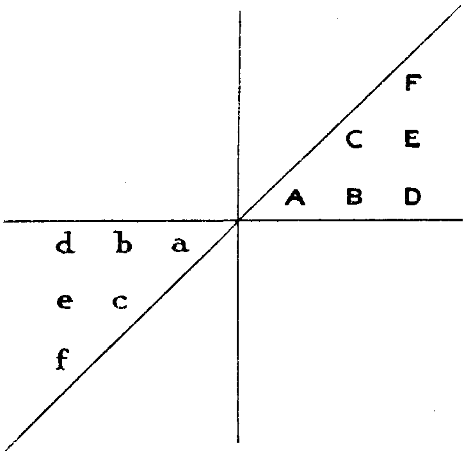

Fig. 6.

The first pair $\mu\delta$ *and* δ^2.—$[0]\,\mu\delta\Sigma = \frac{1}{2}(\mathrm{A} + a)$ the plus sign occurring because the termini of integration cross one another at the origin. Now by (8) and (9)

$$\mathrm{A} = [h, \tfrac{1}{2}h]\,\theta = [\tfrac{1}{2}h, \tfrac{1}{2}h]\,(\mu + \tfrac{1}{2}\delta)\,\theta$$

And

$$a = [-h, -\tfrac{1}{2}h]\,\theta = [-\tfrac{1}{2}h, -\tfrac{1}{2}h]\,(\mu - \tfrac{1}{2}\delta)\,\theta.$$

Hence

$$\mathrm{A} + a = \{[\tfrac{1}{2}h, \tfrac{1}{2}h] + [-\tfrac{1}{2}h, -\tfrac{1}{2}h]\}\,\mu\theta + \tfrac{1}{2}\,\{[\tfrac{1}{2}h, \tfrac{1}{2}h] - [-\tfrac{1}{2}h, -\tfrac{1}{2}h]\}\,\delta\theta.$$

Therefore

$$[0]\,\mu\delta\Sigma = [0]\,\{\mu \,\|\, \mu\theta + \tfrac{1}{4}\delta \,\|\, \delta\theta\}. \qquad (10)$$

Next for the second difference in like manner

$$[0]\,\delta^2\Sigma = \mathrm{A} - a = [0]\,\{\delta \,\|\, \mu\,\theta + \mu \,\|\, \delta\theta\}. \qquad (11)$$

Now $\theta = \phi(y) \,.\, \kappa(x, y)$ in the ordinary notation; and $\phi(y)$ behaves as a constant relative to μ and δ, that is to say

$$\|\, \mu\theta = \phi(x) \,\|\, \mu\kappa, \qquad (12)$$

$$\|\, \delta\theta = \phi(x) \,\|\, \delta\kappa. \qquad (13)$$

We may now write ϕ simply for $\phi(x)$.

Next by the rules of § 6 for differencing and averaging products

$$\mu \,\|\, \mu\theta = \mu\phi \,\pmb{|}\, \mu \,\|\, \mu\kappa + \tfrac{1}{4}\,\delta\phi \,\pmb{|}\, \delta \,\|\, \mu\kappa, \qquad (14)$$

$$\delta \,\|\, \mu\theta = \mu\phi \,\pmb{|}\, \delta \,\|\, \mu\kappa + \delta\phi \,\pmb{|}\, \mu \,\|\, \mu\kappa, \qquad (15)$$

$$\mu \,\|\, \delta\theta = \mu\phi \,\pmb{|}\, \mu \,\|\, \delta\kappa + \tfrac{1}{4}\,\delta\phi \,\pmb{|}\, \delta \,\|\, \delta\kappa, \qquad (16)$$

$$\delta \,\|\, \delta\theta = \mu\phi \,\pmb{|}\, \delta \,\|\, \delta\kappa + \delta\phi \,\pmb{|}\, \mu \,\|\, \delta\kappa. \qquad (17)$$

Thus the "sum-equation" (3) yields for the first difference

$$0 = [0]\,\mu\delta h^{-1}p + [0]\,\{\mu\phi \,\pmb{|}\, (\mu \,\|\, \mu\kappa + \tfrac{1}{4}\delta \,\|\, \delta\kappa) + \tfrac{1}{4}\delta\phi \,\pmb{|}\, (\delta \,\|\, \mu\kappa + \mu \,\|\, \delta\kappa)\}. \qquad (18)$$

Again, for the second difference, the sum equation yields

$$0 = [0]\,\delta^2 h^{-2} p + h^{-1}[0]\,\{\mu\phi \,|\, (\delta \,\|\, \mu\kappa + \mu \,\|\, \delta\kappa) + \delta\phi \,|\, (\mu \,\|\, \mu\kappa + \tfrac{1}{4}\delta \,\|\, \delta\kappa)\}. \quad . \quad (19)$$

For insertion in the NEWTON-BESSEL series § 7 (7) we need $\mu\phi$ and $\delta\phi$ separately. They are obtained from the pair of equations (18), (19) just as in the marching problem the differences are in general found in pairs (§ 9.3).

$$[0]\,\mu\phi = [0]\,\frac{-\mu\delta h^{-1} p \,|\, \mathrm{A} + \frac{1}{4}h^2\delta^2 h^{-2} p \,|\, \mathrm{B}}{\mathrm{A}^2 - \frac{1}{4}h^2\mathrm{B}^2}, \quad \ldots\ldots \quad (20)$$

$$[0]\,h^{-1}\delta\phi = [0]\,\frac{\mu\delta h^{-1} p \,|\, \mathrm{B} - \delta^2 h^{-2} p \,|\, \mathrm{A}}{\mathrm{A}^2 - \frac{1}{4}h^2\mathrm{B}^2}, \quad \ldots\ldots \quad (21)$$

in which

$$\mathrm{A} = \mu \,\|\, \mu\kappa + \tfrac{1}{4}h^2 \frac{\delta}{h} \,\|\, \frac{\delta}{h}\kappa, \quad \ldots\ldots \quad (22)$$

$$\mathrm{B} = \frac{\delta}{h} \,\|\, \mu\kappa + \mu \,\|\, \frac{\delta}{h}\kappa. \quad \ldots\ldots \quad (23)$$

The limits of $\mu\phi$ and $h^{-1}\delta\phi$ when $h \to 0$ are accordingly

$$[0]\,\phi = [0]\,\frac{-\mathrm{D}p}{\|\,\kappa}, \quad \ldots\ldots \quad (24)$$

$$[0]\,\mathrm{D}\phi = [0]\,\frac{\mathrm{D}p \,|\, \{\mathrm{D} \,\|\, \kappa + \|\, \mathrm{D}\kappa\} - \mathrm{D}^2 p \,|\, \|\, \kappa}{(\|\,\kappa)^2}, \quad \ldots\ldots \quad (25)$$

The infinitesimal calculus gives, in the analogous problem, equations § 11.2 (3A, 5A), which, on being solved for $[0]f$ and $[0]\,\mathrm{D}f$, become identical with (24) and (25) provided that $f = \underset{h\to 0}{\mathrm{Limit}}\ \phi$ as we should expect. It is here that the tabulation of fig. 5 succeeds, while that of fig. 4 fails.

As with differential equations, we have next to make h fully explicit in (20), (21), (22), (23) by expressing the means and differences of the known functions p and κ in terms of their derivatives by the aid of the series of § 8. For p it suffices to refer to § 8 (3, 4). The case of κ is complicated by $\|$. We have by § 8 (1, 2) or SHEPPARD's expansions of them

$$\mu\kappa = (1 + \tfrac{1}{8}h^2\mathrm{D}^2 + \tfrac{1}{384}h^4\mathrm{D}^4 + \ldots)\,\kappa \quad \ldots\ldots \quad (26)$$

$$\frac{\delta}{h}\kappa = \mathrm{D}\,(1 + \tfrac{1}{24}h^2\mathrm{D}^2 + \tfrac{1}{1920}h^4\mathrm{D}^4 + \ldots)\,\kappa \quad \ldots\ldots \quad (27)$$

Thus

$$\|\,\mu\kappa = \|\,\kappa + \tfrac{1}{8}h^2 \,\|\, \mathrm{D}^2\kappa + \tfrac{1}{384}h^4 \,\|\, \mathrm{D}^4\kappa + \ldots \quad \ldots\ldots \quad (28)$$

$$\|\,\delta h^{-1}\kappa = \|\,\mathrm{D}\kappa + \tfrac{1}{24}h^2 \,\|\, \mathrm{D}^3\kappa + \tfrac{1}{1920}h^4 \,\|\, \mathrm{D}^5\kappa + \ldots \quad \ldots\ldots \quad (29)$$

And then, by putting $\|\, \mu\kappa$ in the place of κ in (26),

$$\mu \,\|\, \mu\kappa = \|\, \kappa + \tfrac{1}{8} h^2 \,\|\, D^2\kappa + \tfrac{1}{384} h^4 \,\|\, D^4\kappa + \ldots$$
$$+ \tfrac{1}{8} h^2 D^2 \,\|\, \kappa + \tfrac{1}{64} h^4 D^2 \,\|\, D^2\kappa + \ldots$$
$$+ \tfrac{1}{384} h^4 D^4 \,\|\, \kappa + \ldots \qquad (30)$$

in which h is fully explicit as far as h^4. In like manner $\delta h^{-1} \,\|\, \delta h^{-1} \kappa$, $\delta h^{-1} \,\|\, \mu\kappa$, $\mu \,\|\, \delta h^{-1} \kappa$ can be obtained, and so h made fully explicit in the first two terms of the NEWTON–BESSEL series. It is seen that *only even powers of h occur*. Also when p and κ are specified we have the formulæ ready to settle how small h must be, for these terms.

The higher pairs of terms can be treated similarly. For instance, referring to Fig. 6 on p. 346,

$$[0]\, \mu\delta^3\Sigma = \tfrac{1}{2}(B + C) - A - a + \tfrac{1}{2}(b + c)$$
$$[0]\, \delta^4\Sigma = (B + C) - 4\,A + 4a - (b + c)$$

Now all the values A, B, C, a, b, c can be transferred to the sloping line, $x = y$ by means of the advancing and retarding operators. When on the sloping line, values can be expressed by the operator $\|$, and so can be brought into comparison with the corresponding terms in the infinitesimal calculus of § 9.1.

§ 12. SUMMARY AND ABSTRACT.

(1) This is an investigation of the validity of an arithmetical process, here called the "h^2-extrapolation," which has previously been used for solving differential and integral equations. We obtain by arithmetic, often easily, $\phi(x, h)$ the solution of the analogous problem in centered differences, made with step $h \equiv \delta x$. If it is possible to expand thus

$$\phi(x, h) = f(x) + h f_1(x) + h^2 f_2(x) + h^3 f_3(x) + h^4 f_4(x) \ldots \text{ to inf.},$$

then $f(x)$ the limit of $\phi(x, h)$ as $h \to 0$ is usually the desired solution of the problem in the infinitesimal calculus. Now if the function f_1, vanishes, and if further h can be made so small that $h^2 f_2(x)$ is much larger than the sum to infinity of the higher terms of the series, then after solving the difference-problem for two unequal steps h_1, h_2, the unknown $f_2(x)$ can be eliminated and $f(x)$ found. This elimination is called the "h^2-extrapolation."

(2) The method of investigating its validity is to obtain $\phi(x, h)$ as a fully explicit function of h. This is done by a study of the properties of the difference-operator δ and of SHEPPARD's averaging operator μ, combined with the NEWTON-STIRLING and NEWTON-BESSEL expansions in differences of ϕ. There is a general resemblance to corresponding operations in the infinitesimal calculus, but also a number of remarkable contrasts, see, for example, § 9.4.4.

(3) A particular arrangement of the arithmetic, which fits with the properties of μ and δ is used throughout. See § 2, § 11.3.

(4) The investigation is restricted, except in § 4.1, to functions $f(x)$ which can be expanded by TAYLOR's theorem.

(5) The definition of $\phi(x, h)$ is extended by interpolation in § 7 so as to make $\phi(x, h)$ a continuous function of x, to which an h^2-extrapolation can be applied at any value of x.

(6) No exceptions have been found to the rule that odd powers of h are absent from the expansion of $\phi(x, h)$.

(7) General methods for finding how small h must be, in order to make the h^2-extrapolation valid, have been indicated, and have been applied in detail to some simple examples in § 8, § 9.4, § 10, § 11.

(8) In § 4.1 and § 10 cases have been found in which it is not possible to choose h small enough to make an h^2-extrapolation valid for $f(x)$, but this occurred only where $f(x)$ became indeterminate. Some of the branches of the function in § 9.4.4 become more divergent as h decreases.

(9) But it is not in general possible to fix h so that an h^2-extrapolation may remain valid for $d^n f(x)/dx^n$ as $n \to \infty$. This remark applies both when $f(x)$ is a given function as in § 8, and when $f(x)$ is the solution of a differential equation as in § 7, § 9.

(10) Isolated discontinuities are less inconvenient than frills (§ 4.2).

(11) In order to prevent ambiguity some new operators have been introduced in § 6, § 11. One of them is useful in solving integral equations by the infinitesimal calculus and a non-commutative algebra.

(12) The laborious expansions in μ and δ of the present paper are not intended for obtaining numerical results, but only for testing the validity of results obtained by arithmetic in the simple way illustrated in § 3.

§ 13. PLACES WHERE RECURRING SYMBOLS ARE DEFINED.

x, h, ϕ, f, § 1; δ, μ, § 2; G, § 5.2; $\|$, § 6; λ, k, κ, L_{2s}, § 7; D, § 8; ψ, § 9.2; κ, $||$, [0], $[a, b]$, § 11.

Part II.—Interpenetrating Lattices.

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1. Introduction.

This paper is supplementary to a paper by L. F. Richardson,* in which he describes arithmetical methods of solving differential equations by means of centred differences. It will be best to explain one of these methods by an example.

We will take the simple differential equation, $\frac{dy}{dx} = -y$, with $y = 1$ at $x = 0$, and find the value of y at $x = 1$. The interval $(0 \leq x \leq 1)$ is divided into a number of equal steps, say 10, and the following table calculated :—

x.	y.	$0 \cdot 2\frac{dy}{dx}$.	Error.
0·0	1·00000		0·00000
0·1	0·90499	—0·18100	0·00015
0·2	0·81900	—0·16380	0·00027
0·3	0·74119	—0·14824	0·00037
0·4	0·67076	—0·13415	0·00044
0·5	0·60704	—0·12141	0·00051
0·6	0·54935	—0·10987	0·00054
0·7	0·49717	—0·09943	0·00058
0·8	0·44992	—0·08998	0·00059
0·9	0·40719	—0·08144	0·00062
1·0	0·36848		0·00060

The method of obtaining the value of y at $x = 0 \cdot 1$ will be discussed later (§§ 7 and 9). For the next step, we replace dy/dx in the differential equation by $\frac{y(0 \cdot 2) - y(0)}{0 \cdot 2}$, and consider this value to hold at $x = 0 \cdot 1$, the centre of the interval $(0 \leq x \leq 0 \cdot 2)$. Note that it is essential always to use *centred* differences. Our equation then gives : $y(0 \cdot 2) = y(0) - 0 \cdot 2y(0 \cdot 1)$. The second term on the right-hand side is tabulated in the third column, opposite $x = 0 \cdot 1$, and added to the value of y at $x = 0$, to give the value of y at $x = 0 \cdot 2$. This process is continued, until $x = 1$ is reached. This type of solution, in which each value of y is obtained from the value two steps back, used to be called the " step-over " method ; but lately Dr. Richardson has named it " the method of the interpenetrating lattices " (*see* Part I § 2).

* L. F. Richardson, 'Phil. Trans.,' A, vol. 210, pp. 307 to 357. See also a summary in the 'Mathematical Gazette,' July, 1925, pp. 415 to 421.

The difference between our approximate solution and the correct result, viz., $y = e^{-x}$, is tabulated in the last column. The errors would have been greater if we had not centered our differences. Still greater accuracy, however, can be achieved if we can assume that the approximate solution has the form

$$A + Bh^2 + Ch^4 + \dots, \qquad (1.0)$$

where h is the length of the step. We calculate the value of y at $x = 1$, using steps of twice the length, obtaining $0 \cdot 37029$. The error of this result should be four times the error of the original result, if the terms of (1.0) decrease rapidly in importance.

Thus the value with large steps was	0·37029
the value with small steps was	0·36848
The difference	0·00181
should be three times the error with small steps, which is therefore	0·00060
and the corrected value is	0·36788

This is accurate to five places of decimals.

There are two outstanding questions which this paper tries to answer. First, how is the table to be started? Secondly, can the approximate solution be expanded in a power series in h, with odd powers missing, as in (1.0)? In other words: how should we take the first step? and, is our final correction justified?

We shall assume that an expansion in powers of h is possible, and prove that the odd powers do not appear, and that the first step must be taken in a definite manner. §§ 2 to 8 give the detailed analysis for ordinary differential equations. § 9 contains two examples. § 10 gives an outline of the work for simultaneous differential equations and an example. In each case the method of solution is that of the interpenetrating lattices, which has just been explained.

2. The Range- and Difference-Equations.

We will consider an ordinary differential equation in two variables, resolved with respect to the highest derivative which appears:

$$\frac{d^n y}{dx^n} = F\left(x, y, \frac{dy}{dx}, \dots \frac{d^{n-1}y}{dx^{n-1}}\right), \qquad (2.0)$$

with initial conditions:

$$y, \frac{dy}{dx}, \dots \frac{d^{n-1}y}{dx^{n-1}} \qquad \text{are given as } x = 0. \qquad (2.1)$$

We suppose that F is differentiable, to an order which will be specified later (§ 8).

Owing to the presence of $\frac{d^{n-1}y}{dx^{n-1}}$, this equation is best solved by the interpenetrating lattices.* The functions in the various columns of the table, headed by y, dy/dx, d^2y/dx^2, ... d^ny/dx^n, are only approximations. We will denote them by $f_0(x)$, $f_1(x)$,... $f_n(x)$, respectively. f_0; f_1, ... f_n, may be called the approximate solution and its difference ratios. They depend on the parameter h as well as x, and are defined only at those points where x/h is an integer.

The rules for making the table are expressed mathematically by the following equations and conditions :

$$f_{s-1}(\overline{p+1}\,h) - f_{s-1}(\overline{p-1}\,h) = 2hf_s(ph) \quad (s = 1, 2, \ldots n\,;\ p \text{ is a positive integer}). \quad (2.2)$$

$$f_n = \mathrm{F}(x, f_0, f_1, \ldots f_{n-1}). \quad (2.3)$$

At $x = 0$: $f_0, f_1, \ldots f_{n-1}$ have the initial values of

$$y, \frac{dy}{dx}, \ldots \frac{d^{n-1}y}{dx^{n-1}}. \quad (2.4)$$

At $x = h$: $f_0, f_1, \ldots f_{n-1}$ have values to be settled later, when we consider the first step (§ 7).

(2.2) will be referred to as the " difference-equation," and (2.3) as the " range-equation."

3. The Problem.

We propose to find necessary conditions that

$$f_s(x) = y_s(x) + h\dot{y}_s(x) + \frac{h^2}{2\,!}\ddot{y}_s(x) + \frac{h^3}{3\,!}\dddot{y}_s(x) + \frac{h^4}{4\,!}\{\ddddot{y}_s(x) + \varepsilon_s(x, h)\} \quad (3.0)$$

for $s = 0, 1, \ldots n$, where y_s, $\dot{y}_s$, $\ddot{y}_s$, $\dddot{y}_s$, $\ddddot{y}_s$ are independent of h, and differentiable to an order to be specified later (§ 8) ; and, as $h \to 0$, $\varepsilon_s(x, h) \to 0$ uniformly in x.

In other words, we shall see under what conditions the approximations to y and its derivatives can be expanded in power series in h, with remainders ; and we shall determine the coefficients in the expansion. In practice, y_0 would be the exact solution of the problem ; and some knowledge of the size of $\dot{y}_0$, $\ddot{y}_0$, etc., would give an estimate of the accuracy of our approximation.

To shorten the lengthy expressions which follow, dashes and upper suffixes will be used to denote differentiation with respect to x. Thus

$$\dot{y}'_s \equiv \frac{d\dot{y}_s}{dx}\,; \qquad y^{\text{iv}}_s \equiv \frac{d^4y_s}{dx^4}\,; \qquad y^{(n)} \equiv \frac{d^ny}{dx^n}.$$

* The single lattice can be used ; but it involves, at each line of the table, the solution of a pair of simultaneous equations, one of which is not in general linear (*cf.* Part I § 9.3 and § 9.4.4).

Also for partial differential coefficients we shall use :

$$F_{y_s} \equiv \frac{\partial F}{\partial y_s}; \qquad F_{y_s y_t} \equiv \frac{\partial^2 F}{\partial y_s \, \partial y_t}; \qquad \text{etc.}$$

4. Substitution in the Difference-Equation.

By (2.2)

$$2hf_s(ph) = f_{s-1}(\overline{p+1}\,h) - f_{s-1}(\overline{p-1}\,h) \qquad (s = 1 \ldots n,\ p = 1, 2, \ldots).$$

Substitute from (3.0) :

$$2h\left[y_s(ph) + h\dot{y}_s(ph) + \frac{h^2}{2!}\ddot{y}_s(ph) + \frac{h^3}{3!}\dddot{y}_s(ph) + \frac{h^4}{4!}\{\ddddot{y}_s(ph) + \varepsilon_s(ph, h)\}\right]$$

$$= [y_{s-1}(\overline{p+1}\,h) - y_{s-1}(\overline{p-1}\,h)] + h[\dot{y}_{s-1}(\overline{p+1}\,h) - \dot{y}_{s-1}(\overline{p-1}\,h)]$$

$$+ \frac{h^2}{2!}[\ddot{y}_{s-1}(\overline{p+1}\,h) - \ddot{y}_{s-1}(\overline{p-1}\,h)] + \frac{h^3}{3!}[\dddot{y}_{s-1}(\overline{p+1}\,h) - \dddot{y}_{s-1}(\overline{p-1}\,h)]$$

$$+ \frac{h^4}{4!}[\ddddot{y}_{s-1}(\overline{p+1}\,h) - \ddddot{y}_{s-1}(\overline{p-1}\,h) + \varepsilon_{s-1}(\overline{p+1}\,h, h) - \varepsilon_{s-1}(\overline{p-1}\,h, h)].$$

In order to make the arguments of most of the functions the same on both sides of the equation, expand the right-hand side by Taylor's theorem, at the point $x = ph$, with remainders after terms in h^4. It becomes

$$2[hy'_{s-1}(ph) + \frac{h^3}{3!}y'''_{s-1}(ph) + \frac{h^4}{4!}\eta_{s-1}(ph, h)]$$

$$+ 2h[h\dot{y}'_{s-1}(ph) + \frac{h^3}{3!}\{\dot{y}'''_{s-1}(ph) + \dot{\eta}_{s-1}(ph, h)\}]$$

$$+ \frac{2h^2}{2!}[h\ddot{y}'_{s-1}(ph) + \frac{h^2}{2!}\ddot{\eta}_{s-1}(ph, h)]$$

$$+ \frac{2h^3}{3!}h\{\dddot{y}'_{s-1}(ph) + \dddot{\eta}_{s-1}(ph, h)\}$$

$$+ \frac{h^4}{4!}[\ddddot{\eta}_{s-1}(ph, h) + \varepsilon_{s-1}(\overline{p+1}\,h, h) - \varepsilon_{s-1}(\overline{p-1}\,h, h)] \quad . \ . \ . \quad (4.0)$$

where $\eta_{s-1}(x, h)$, $\dot{\eta}_{s-1}(x, h)$, $\ddot{\eta}_{s-1}(x, h)$, $\dddot{\eta}_{s-1}(x, h)$, $\ddddot{\eta}_{s-1}(x, h)$, all $\rightarrow 0$ as $h \rightarrow 0$ for fixed x.

Coefficients of the same powers of h on the two sides of the equation are now equal. For, dividing by h and making $h \rightarrow 0$ (keeping ph constant) we see that the coefficients of h are equal, as all the other terms $\rightarrow 0$. So for the coefficients of h^2, h^3, h^4, and finally the odds and ends. (The fact that the ε's are evaluated at $\overline{p \pm 1}h$, and not at ph, presents no difficulty, since the ε's $\rightarrow 0$ uniformly.)

By equating coefficients we obtain the following equations:

$$\left.\begin{aligned} y_s &= y'_{s-1} \\ \dot{y}_s &= \dot{y}'_{s-1} \\ \ddot{y}_s &= \tfrac{1}{3}y'''_{s-1} + \ddot{y}'_{s-1} \\ \dddot{y}_s &= \dot{y}'''_{s-1} + \dddot{y}'_{s-1} \end{aligned}\right\} (s = 1, 2, \ldots n) \quad \ldots \ldots \quad (4.1)$$

Transforming slightly:

$$\left.\begin{aligned} y'_{s-1} &= y_s \\ \dot{y}'_{s-1} &= \dot{y}_s \\ \ddot{y}'_{s-1} &= \ddot{y}_s - \tfrac{1}{3}y''_s \\ \dddot{y}'_{s-1} &= \dddot{y}_s - \dot{y}''_s \end{aligned}\right\} (s = 1, 2, \ldots n) \quad \ldots \ldots \quad (4.2)$$

5. Substitution in the Range-Equation.

By (2.3)

$$f_n = F(x, f_0, f_1, \ldots f_{n-1}).$$

Expand by Taylor's theorem at the point $(x, y_0, y_1, \ldots y_{n-1})$, which is independent of h:

$$f_n = F + \sum_{s=0}^{n-1} F_{y_s}(f_s - y_s) + \frac{1}{2!}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1} F_{y_s y_t}(f_s - y_s)(f_t - y_t)$$
$$+ \frac{1}{3!}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}\sum_{u=0}^{n-1}(F_{y_s y_t y_u} + \eta_{stu})(f_s - y_s)(f_t - y_t)(f_u - y_u), \quad . \quad (5.0)$$

where

$$\eta_{stu} \to 0 \text{ as } f_s - y_s, \text{ etc.} \to 0; \ i.e., \text{ as } h \to 0.$$

Substitute for $f_s - y_s$ from (3.0) and equate coefficients of h^0, h^1, h^2, h^3, as in the last section.

$$\left.\begin{aligned} y_n &= F \\ \dot{y}_n &= \sum_{s=0}^{n-1} F_{y_s}\dot{y}_s \\ \ddot{y}_n &= \sum_{s=0}^{n-1} F_{y_s}\ddot{y}_s + \sum_{s=0}^{n-1}\sum_{t=0}^{n-1} F_{y_s y_t}\dot{y}_s\dot{y}_t \\ \dddot{y}_n &= \sum_{s=0}^{n-1} F_{y_s}\dddot{y}_s + \tfrac{3}{2}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1} F_{y_s y_t}(\dot{y}_s\ddot{y}_t + \ddot{y}_s\dot{y}_t) \\ &\quad + \sum_{s=0}^{n-1}\sum_{t=0}^{n-1}\sum_{u=0}^{n-1} F_{y_s y_t y_u}\dot{y}_s\dot{y}_t\dot{y}_u \end{aligned}\right\}, \quad \ldots \ldots \quad (5.1)$$

where F, F_{y_s}, etc., are evaluated at $(x, y_0, y_1, \ldots y_n)$.

6. Determination of y_s, $\dot{y}_s$, $\ddot{y}_s$, $\dddot{y}_s$.

(4.2) and (5.1) are differential equations in the unknown functions of (3.0). Also we know their initial values at $x = 0$. For by (2.4), with (2.0) and (2.3),

$$f_s = y^{(s)} \quad \text{at } x = 0 \quad (s = 0, 1, \dots n).$$

Substitute in (3.0) and " equate coefficients " as before :
At $x = 0$:

$$\left.\begin{array}{l} y_s = y^{(s)} \\ \dot{y}_s = \ddot{y}_s = \dddot{y}_s = \ddddot{y}_s = \varepsilon_s = 0 \end{array}\right\} (s = 0, 1, \dots n)\ (x = 0) \quad . \quad . \quad . \quad . \quad (6.0)$$

By (4.2), for any x :

$$y_s = y'_{s-1} = y''_{s-2} = \dots = y_0^{(s)}.$$

Therefore by (5.1)

$$y_0^{(n)} = \mathrm{F}(x, y_0, y'_0, \dots y_0^{(n-1)}).$$

This is our original differential equation (2.0). The initial conditions are the same, by (6.0). Therefore

$$y_0 = y\,;\ y_s = y^{(s)} \quad (s = 0, 1, \dots n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

The equations in $\dot{y}_s$'s, and their initial conditions evidently have the solution

$$\dot{y}_s = 0 \quad (s = 0, 1, \dots n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.2)$$

and the form of the equations is such that the solution is unique.*

The same now applies to $\ddot{y}_s$'s :

$$\ddot{y}_s = 0 \quad (s = 0, 1, \dots n). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

On substituting in (4.2) for y_s by (6.1), and in (5.1) for $\dot{y}_s$ by (6.2), and by using the initial condition (6.0), $\dddot{y}_s$ can be determined.

7. The First Step.

In the solution of differential equations by interpenetrating lattices, it is generally difficult to see a priori how to take the first step. Various empirical suggestions have been made, such as the " algebraic first step," and the use of smaller steps at the beginning.† Our work so far, however, has been independent of any explicit method of taking the first step. Yet it has sufficed to determine most of the functions on the right-hand side of (3.0). This equation, if it is to hold for all (sufficiently small) x, determines the way in which the first step must be taken ; for we have only to put $x = h$, to obtain $f_s\,(h)$.

* See Ch. J. de la Vallée Poussin, 'Cours d'Analyse Infinitésimale,' tome II, chap. 5, § 4 (5th edition).

† L. F. Richardson, 'Mathematical Gazette,' July, 1925, pp. 417 to 418 ; also 'Weather Prediction by Numerical Process,' Camb. Univ. Press, chap. 7/2.

The expression found for $f_s(h)$ can be put in the form of a power series in h, with a remainder after the term in h^4. This remainder is indeterminate, because ε_s is indeterminate, and for convenience it may be chosen equal to zero. It is not necessary, however, that the remainder should vanish.

We must first evaluate various functions at $x = 0$.

By (6.1) $\quad y_s = y^{(s)},\ y'_s = y^{(s+1)}$, etc. $\quad (s = 0, 1, \dots n)$

By (6.0) $\quad \ddot{y}_s = 0$ at $x = 0$. $\quad (s = 0, 1, \dots n)$

By (4.2) $\quad \ddot{y}'_s = \ddot{y}_{s+1} - \frac{1}{3}y''_{s+1} \quad (s = 0, 1, \dots n-1)$

$\quad = -\frac{1}{3}y^{(s+3)}$ at $x = 0$. $\quad (s = 0, 1, \dots n-1)$

By (4.2) $\quad \ddot{y}''_s = \ddot{y}'_{s+1} - \frac{1}{3}y'''_{s+1} \quad (s = 0, 1, \dots n-1)$

$\quad = -\frac{2}{3}y^{(s+4)}$ at $x = 0$. $\quad (s = 0, 1, \dots n-2)$

By (6.0) $\quad \ddddot{y}_s = 0$ at $x = 0$. $\quad (s = 0, 1, \dots n)$.

Putting $x = h$ in (3.0), and using (6.2) and (6.3):

$$f_s(h) = y_s(h) + \frac{h^2}{2\,!}\ddot{y}_s(h) + \frac{h^4}{4\,!}[\ddddot{y}_s(h) + \varepsilon_s(h, h)].$$

Therefore for $s = 0, 1, \dots n-2$, by TAYLOR's expansion

$$f_s(h) = y^{(s)}(0) + hy^{(s+1)}(0) + \frac{h^2}{2\,!}y^{(s+2)}(0) + \frac{h^3}{3\,!}y^{(s+3)}(0) + \frac{h^4}{4\,!}\{y^{(s+4)}(0) + \delta_s\}$$
$$+ \frac{h^2}{2\,!}\left[0 - h\,.\,\tfrac{1}{3}y^{(s+3)}(0) + \frac{h^2}{2\,!}\{-\tfrac{2}{3}y^{(s+4)}(0) + \ddot{\delta}_s\}\right] + \frac{h^4}{4\,!}\{\ddddot{\delta}_s + \varepsilon_s(h, h)\},$$

where $\delta_s, \ddot{\delta}_s, \ddddot{\delta}_s, \to 0$ as $h \to 0$. Therefore

$$f_s(h) = y^{(s)}(0) + hy^{(s+1)}(0) + \frac{h^2}{2}y^{(s+2)}(0) - \frac{h^4}{8}y^{(s+4)}(0) \qquad (s = 0, 1, \dots n-2) \quad (7.0)$$

if we choose $\varepsilon_s(h, h) = -\delta_s - 6\ddot{\delta}_s - \ddddot{\delta}_s$.

For $s = n - 1$, we require $\ddot{y}''_{n-1}(0)$.

By (4.2) and (5.1)

$$\ddot{y}'_{n-1} = -\tfrac{1}{3}y''_n + \sum_{s=0}^{n-1} F_{y_s}\ddot{y}_s.$$

Therefore at $x = 0$

$$\ddot{y}''_{n-1} = -\tfrac{1}{3}y^{(n+3)} + \sum_{s=0}^{n-1} F_{y_s}(-\tfrac{1}{3}y^{(s+3)}).$$

Expanding as before:

$$f_{n-1}(h) = y^{(n-1)}(0) + hy^{(n)}(0) + \frac{h^2}{2}y^{(n+1)}(0) - \frac{h^4}{24}\left[y^{(n+3)}(0) + 2\sum_{s=0}^{n-1} F_{y_s}y^{(s+3)}(0)\right], \quad (7.1)$$

if we choose

$$\varepsilon_{n-1}(h, h) = -\delta_{n-1} - 6\ddot{\delta}_{n-1} - \ddddot{\delta}_{n-1}.$$

By the analysis of § 5, $f_n(h)$ is given, to the same order, by equation (2.3).

Equations (7.0) and (7.1) are important; they differ from all the empirical rules mentioned above. Their application is not so complicated as might appear at first sight. (See the example of § 9.)

8. Precise Assumptions as to Differentiability.

On looking through the work up to this point, it will be seen that we have assumed that

F is 3 times differentiable with respect to $y_0, y_1, \ldots y_n$				(§ 5)
F_{y_s} is once	,,	,, x	$(s = 0, 1, \ldots n-1)$	(§ 7)
y_s is 4 times	,,	,, x	...	(§ 4)
$\dot{y}_s$ is 3 times	,,	,, x	...	(§ 4)
$\ddot{y}_s$ is twice	,,	,, x	...	(§ 4)
$\dddot{y}_s$ is once	,,	,, x	...	(§ 4)

also that $\ddddot{y}_s$ is continuous in x $(s = 0, 1, \ldots n-1)$ (§ 4).

9. Examples.

$$\text{(i)} \quad \frac{dy}{dx} = -y; \quad y = 1 \text{ when } x = 0.$$

This is the example of § 1. By comparison with (2.0), $n = 1$ and $F = -y$.

(7.0) does not apply.

(7.1) requires the values of y, y', y'', y''' and y^{iv} at $x = 0$; $y = 1$ is given; $y' = -y = -1$, by the differential equation.

Differentiating: $y'' = -y' = 1$.

Similarly, $y''' = -1$, $y^{iv} = 1$.

Also $F_y = -1$.

Therefore

$$f_0(h) = 1 - h + \frac{h^2}{2} - \frac{h^4}{24}[1 + 2(-1)(-1)].$$

Putting $h = 0 \cdot 1$ we obtain $0 \cdot 90499$ to five places.

It is interesting to compare the results of § 1, in which the above value is used for y at $x = 0 \cdot 1$, with a similar solution using an incorrect first step. Suppose we take at $x = 0 \cdot 1$ the correct analytical value for y, in violation of equation (7.1). We might expect greater accuracy, but the actual errors are as follows:—

x	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
Error $\times 10^5$	0	0	30	21	50	34	64	39	72	40	78

The error is seen to oscillate, and is greater than that in § 1 at alternate points. The result at $x=1$ is $y=0\cdot36866$; the similar result, using steps of double the length, is $y=0\cdot36868$. The method of approximation explained in § 1 gives a corrected result $y=0\cdot36865$. The final error is $10^{-5}\times 77$, as compared with zero in § 1.

$$\text{(ii)}\ \frac{d^2y}{dx^2}-y\frac{dy}{dx}=0;\ \text{at}\ x=0,\ y=-1\cdot09262,\ y'=2\cdot5969.$$

In this case $n=2$; $\mathrm{F}\equiv yy'$.

$f_0(h)$ is determined by (7.0); $f_1(h)$ by (7.1).

We require various derivatives of y at $x=0$ (some of them only roughly)

$$\begin{aligned} y'' &= yy' &&= -\ 2\cdot838\\ y''' &= yy''+y'^2 &&= 9\cdot845\\ y^{\text{iv}} &= 3y'y''+yy''' &&= -\ 32\cdot87\\ y^{\text{v}} &= 3y''^2+4y'y'''+yy^{\text{iv}} &&= 162 \end{aligned}$$

Also

$$\begin{aligned} \mathrm{F}_y &= y' &&= 2\cdot5969\\ \mathrm{F}_{y'} &= y &&= -\ 1\cdot09262 \end{aligned}$$

Suppose we take $h=0\cdot05$, and substitute in

$$f_0(h)=y+hy'+\frac{h^2}{2}y''-\frac{h^4}{8}y^{\text{iv}}$$

$$f_1(h)=y'+hy''+\frac{h^2}{2}y'''-\frac{h^4}{24}[y^{\text{v}}+2(\mathrm{F}_yy'''+\mathrm{F}_{y'}y^{\text{iv}})],$$

which are (7.0) and (7.1).
We obtain at $x=0\cdot05$, $f_0=-0\cdot96629$, $f_1=2\cdot4672$.

We enter these in our table under y and dy/dx, and proceed happily ever after. In the last column of the table is entered the difference between the approximate solution, and the analytical solution, viz., $y=2\tan(x-\frac{1}{2})$.

x.	y.	dy/dx.	d^2y/dx^2.	Error $\times 10^5$.
0·00	−1·09262	2·5969	−2·838	0
0·05	−0·96629	2·4672	−2·384	−17
0·10	−0·84590	2·3585	−1·995	−32
0·15	−0·73044	2·2677	−1·656	−38
0·20	−0·61913	2·1929	−1·358	−49
0·25	−0·51115	2·1319		−47
0·30	−0·40594			−54

The value of y at $x = 0 \cdot 3$, using steps of double the length, is $- 0 \cdot 40729$. The corrected result is $- 0 \cdot 40549$, with an error of $- 10^{-5} \times 9$.

10. Simultaneous Differential Equations.

Let t be the independent variable ; $x, y, z, \ldots$ dependent variables.

Resolve the equations with respect to the highest derivatives

$$\left.\begin{aligned} x^{(m)} &= F(t\,; x, x', \ldots, x^{(m-1)}\,;\ y, y', \ldots, y^{(n-1)}\,; \ldots) \\ y^{(n)} &= G(t\,; x, x', \ldots, x^{(m-1)}\,;\ y, y', \ldots, y^{(n-1)}\,; \ldots) \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \quad (10.0)$$

where dashes and upper suffixes denote differentiation with respect to t.

Use the interpenetrating lattices. Let $f_0, f_1, \ldots$, be the approximations to $x, x', \ldots$; $g_0, g_1, \ldots$, to $y, y', \ldots$; and so on.

The difference-equations are (compare (2.2))

$$\left.\begin{aligned} f_{s-1}(\overline{p+1}\,h) - f_{s-1}(\overline{p-1}\,h) &= 2hf_s(ph) \quad (s = 1, 2, \ldots m) \\ g_{s-1}(\overline{p+1}\,h) - g_{s-1}(\overline{p-1}\,h) &= 2hg_s(ph) \quad (s = 1, 2, \ldots n) \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \quad (10.1)$$

and the range-equations (compare (2.3))

$$\left.\begin{aligned} f_m &= F(t\,;\ f_0, f_1, \cdots f_{m-1}\,;\ g_0, g_1, \cdots g_{n-1}\,; \cdots) \\ g_n &= G(t\,;\ f_0, f_1, \cdots f_{m-1}\,; g_0, g_1, \cdots g_{n-1}\,; \ldots) \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \ldots \quad (10.2)$$

We assume (compare (3.0))

$$\left.\begin{aligned} f_s &= x_s + \frac{h^2}{2\,!}\ddot{x}_s + \frac{h^4}{4\,!}\ddddot{x}_s + \ldots \quad (s = 0, 1, \ldots m) \\ g_s &= y_s + \frac{h^2}{2\,!}\ddot{y}_s + \frac{h^4}{4\,!}\ddddot{y}_s + \ldots \quad (s = 0, 1, \ldots n) \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \ldots \quad (10.3)$$

The analysis of § 4 holds good.

$$\left.\begin{aligned} x'_{s-1} &= x_s\,; \quad \ddot{x}'_{s-1} = \ddot{x}_s - \tfrac{1}{3}x''_s. \qquad (s = 1, 2, \ldots m) \\ y'_{s-1} &= y_s\,; \quad \ddot{y}'_{s-1} = \ddot{y}_s - \tfrac{1}{3}y''_s. \qquad (s = 1, 2, \ldots n) \\ &\text{etc.} \end{aligned}\right\} \quad \ldots \ldots \quad (10.4)$$

By the method of § 5

$$\left.\begin{aligned} x_m &= \mathrm{F}\,; \quad \ddot{x}_m = \sum_{s=0}^{m-1} \mathrm{F}_{x_s}\ddot{x}_s + \sum_{s=0}^{n-1} \mathrm{F}_{y_s}\ddot{y}_s + \dots \\ y_n &= \mathrm{G}\,; \quad \ddot{y}_n = \sum_{s=0}^{m-1} \mathrm{G}_{x_s}\ddot{x}_s + \sum_{s=0}^{n-1} \mathrm{G}_{y_s}\ddot{y}_s + \dots \\ &\text{etc.} \end{aligned}\right\} \quad \dots\dots\dots \quad (10.5)$$

(If odd powers of h had been inserted in (10.3) their coefficients would have disappeared exactly as in § 6.)

The analysis leading up to (7.0) holds good.

$$\left.\begin{aligned} f_s(h) &= x^{(s)} + hx^{(s+1)} + \frac{h^2}{2}x^{(s+2)} - \frac{h^4}{8}x^{(s+4)} \quad (s = 0, 1, \dots m-2) \\ g_s(h) &= y^{(s)} + hy^{(s+1)} + \frac{h^2}{2}y^{(s+2)} - \frac{h^4}{8}y^{(s+4)} \quad (s = 0, 1, \dots n-2) \\ &\text{etc.} \end{aligned}\right\} \quad \dots \quad (10.6)$$

Finally, corresponding to (7.1)

$$\left.\begin{aligned} f_{m-1}(h) &= x^{(m-1)} + hx^{(m)} + \frac{h^2}{2}x^{(m+1)} - \frac{h^4}{24}\left[x^{(m+3)} + 2\sum_{s=0}^{m-1}\mathrm{F}_{x_s}x^{(s+3)} + 2\sum_{s=0}^{n-1}\mathrm{F}_{y_s}y^{(s+3)} + \dots\right] \\ g_{n-1}(h) &= y^{(n-1)} + hy^{(n)} + \frac{h^2}{2}y^{(n+1)} - \frac{h^4}{24}\left[y^{(n+3)} + 2\sum_{s=0}^{m-1}\mathrm{G}_{x_s}x^{(s+3)} + 2\sum_{s=0}^{n-1}\mathrm{G}_{y_s}y^{(s+3)} + \dots\right] \\ &\text{etc.} \end{aligned}\right\}. \quad (10.7)$$

*Example**—

$$x' = -x\sqrt{x^2+y^2} - 1\,; \quad y' = -y\sqrt{x^2+y^2}. \quad \dots\dots \quad (10.8)$$

At $t = 0$, suppose $x = 0$, $y = 1$

Then by successive differentiation of (10.8) we have at $t = 0$

$x' = -1$	$y' = -1$
$x'' = 1$	$y'' = 2$
$x''' = -3$	$y''' = -7$
$x^{\text{iv}} = 15$	$y^{\text{iv}} = 29$
$\mathrm{F}_x = -1$	$\mathrm{F}_y = 0$
$\mathrm{G}_x = 0$	$\mathrm{G}_y = -2$

Substituting in (10.7), we get formulæ for the first step

$$\left.\begin{aligned} f(h) &= 0 - h + \frac{h^2}{2} - \frac{h^4}{24}[15 + 2(-1)(-3) + 0] \\ g(h) &= 1 - h + \frac{h^2}{2}\,.\,2 - \frac{h^4}{24}[29 + 0 + 2(-2)(-7)] \end{aligned}\right\}$$

* Taken from L. F. RICHARDSON, "Theory of the Measurement of the Wind by Shooting Spheres Upward," 'Phil. Trans.,' A, vol. 223, p. 376, equations (6) and (7).

If, for instance, we choose a step of length $0 \cdot 1$, we have

$$\left.\begin{aligned} f(0\cdot 1) &= -0\cdot 0951 \\ g(0\cdot 1) &= 0\cdot 9098 \end{aligned}\right\}.$$

11. Summary.

We have been considering the arithmetical solution, by means of centred differences, of a certain class of differential equation. (Marching problems.)*

Assuming that the approximate solution can be expanded in powers of the length of the step (with a remainder after the fourth power), and that the coefficients in the expansion are conveniently differentiable (§ 8), we have shown that the odd powers must be absent ((6.2) and (6.3)). This is the basis of a useful method of diminishing the error (§ 1), named in Part I the h^2-extrapolation.

It is necessary, however, for the existence of such an expansion that the first step shall be taken in accordance with certain formulæ ((7.0), (7.1), (10.6), (10.7)). Strictly speaking, these formulæ may be varied by adding to the right-hand side any term that $\rightarrow 0$ more rapidly than h^4 (§ 7).

We may say that they are necessary "to the order of h^4." If they are violated, either the expansion in powers of h is impossible, or the coefficients are not differentiable as in § 8 and their behaviour may be inconvenient. In practice, it is often found that if the solution is started wrongly, the error oscillates with increasing violence (see § 9, example (i)).

Incidentally, we have found equations determining the coefficients $\ddot{y}_s$ in the expansion. (§ 6, (10.4), (10.5)). It will often be possible to form from them an estimate of the size of these coefficients, and hence of the error of the solution.

There would be no essential difficulty in extending the expansion beyond the fourth power of h; but such a refinement would have little practical value.

The rules for taking the first step in the two types of solution are :—

Simple Differential Equation : Interpenetrating Lattices—(7.0) and (7.1).

Simultaneous Differential Equations : Interpenetrating Lattices—(10.6) and (10.7).

My thanks are due to Dr. Richardson for inspiration, encouragement and advice.

* See Part I, § 1.