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EVALUATION OF RICHARDSON EXTRAPOLATION IN COMPUTATIONAL FLUID DYNAMICS

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The concept of Richardson extrapolation is evaluated for improving the solution accuracy of two well-investigated two-dimensional flow problems: (1) laminar cavity flows with $\mathrm{Re} = 100$ ***and 1,000; and (2) the Reynolds-averaged backward-facing step turbulent flow with*** $\mathrm{Re} = 10^6$***, aided by the widely used*** k–ϵ ***two-equation model with wall function. Uniform grid systems are employed in all cases to facilitate unambiguous assessment. By systematically refining the grid, computational fluid dynamics (CFD) solutions with different resolutions are first obtained, then extrapolated from a finer to a coarser grid using Lagrangian interpolation with either a 9-point or a 16-point formula. For laminar flows, Richardson extrapolation does not exhibit consistent trends in order of accuracy. Furthermore, the relative performance of Richardson extrapolation, in comparison with solutions obtained directly from mesh refinement, is not sensitive to the level of residuals contained in each computation, the detailed interpolation formula between grids, the choice between second-order central difference and upwind convection schemes, and the selection of the error norms. For turbulent flow computations, large jumps in velocity profiles between the wall and the first node cause difficulties in interpolation, and Richardson extrapolation performs unsatisfactorily under such situations. The present study indicates that Richardson extrapolation does not work consistently in approaches typically employed for engineering CFD applications.***

1. INTRODUCTION

Accurate and reliable predictions of fluid flows are major goals in computational fluid dynamics (CFD). There are many issues relevant to quantification of uncertainty in such computations; see, e.g., [1–5]. In particular, issues about the uncertainties and mesh dependency of the solutions are of direct interest to both basic

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NOMENCLATURE

a_n, b_n	coefficient of Taylor series expansion	u_1	first-order numerical solution
E_1	error norm 1	u_2	second-order numerical solution
E_2	error norm 2	$\tilde{u}_{i,j}$	intermediate solution occurring at some point in the iterative process
E_∞	infinity error norm		
$\mathrm{Err}_{1,2}$	absolute value of the error between the exact solution and numerical solution	$\Delta u_{i,j}$	deviation of the intermediate solution from the converged solution
		U	exact solution of any flow variable
h, h_1, h_2	grid spacings	x, y, z	Cartesian coordinates
L	difference operator	x_p, y_p	points considered for interpolation
n, m	grid numbers	z_{ij}	interpolated value in the flow domain
N	order of accuracy		
Re	Reynolds number	**Subscripts**	
s, t	indices of the interpolated coordinates	i, j	indices of the grid
		p, q	indices of the coordinate considered for interpolation
u	numerical value of any flow variable		

research and practical applications. Grid refinement is often used to achieve accurate solutions; it also can help to perform sensitivity and uncertainty analysis of CFD predictions [3]. Gu et al. [4] proposed some measures on *a posteriori* estimate of the error in computed solutions based on the mesh, as well as on sensitivity, and uncertainty analysis for variable property flows has been researched by Turgeon et al. [5].

Richardson extrapolation has attracted interests in the CFD community because of its potential to improve the *quantitative accuracy* as well as *order of accuracy* of a given computational technique. It is built on the concept that by combining two separate discrete solutions, on two different grids, the leading order error term in the assumed error expansion can be eliminated. The extrapolation must be used with considerable caution, since it involves the additional assumption of monotone truncation-error convergence in the mesh spacing h [1]. The method has a major attractive feature: it is oblivious to the equations being discretized and to the dimensionality of the problem and can easily be applied as a postprocessor to solutions on two grids with no reference to the codes, algorithms, or governing equations that produced the solutions. However, this benefit can be realized only if the original solutions possess certain characteristics. For example, in practical computations, depending on the numerical schemes and fluid physics, the solution may not exhibit uniform degree of accuracy. In other words, noticeable difference may exist between a solution's global behavior in the entire domain, and that indicated by a local truncation error analysis based on the Taylor series. Furthermore, in engineering turbulent flow computations, with the use of a practical turbulence closure such as the k–ε two-equation model [6], the velocity profiles near a solid wall can vary significantly in a single mesh. These issues need to be clearly addressed before Richardson extrapolation can be adopted with confidence. There are also issues related to the effect of convection and other discretization operators, choice of error norms for extrapolation, interpolation formulas for information transfer between grids, and the residual contained in practical computations. Reference [1] offers an excellent review to address some of these points.

In the present work, we evaluate the applicability of the idea of Richardson extrapolation for helping achieve grid-independent CFD solutions. The basic approach is to combine the information obtained on two grid levels, and to project a more accurate solution using Richardson extrapolation. To facilitate the assessment, laminar cavity flows (with Re = 100 and 1,000) and turbulent backward-facing step flow (Re = 10^6) using different grids are considered in our present study. Lagrangian interpolation is used to transform a solution on higher grid resolution to that on a lower one, then Richardson extrapolation is used to combine the two and a finer solution is obtained. Based on these results, assessment of the performance of Richardson extrapolation is made. The laminar cavity has singularities at corners between moving and stationary walls. The finite-volume approach [7], such as that employed here, can treat such difficulties via cell-by-cell integration. The influences of these singularities and the large profile variations in turbulent wall-bounded flows on the performance of Richardson extrapolation are of major interest. In this work, we intentionally adopt uniform grid so that there is no ambiguity of the extrapolation procedure. For practical computations, nonuniform and curvilinear coordinates are often employed, which can make Richardson extrapolation more difficult to apply. Hence, the present study serves as a starting point to address the issues of employing Richardson extrapolation for practical flow computations.

2. APPROACH

2.1. Richardson Extrapolation

Consider a series of the form given below for a grid size h; then

$$U(x) = u(x; h) + a_2 h^2 + a_3 h^3 + a_4 h^4 + \cdots \tag{1}$$

where $U(x)$ is the exact solution and $u(x; h)$ is the numerical solution based on h.

The series with a lower grid size becomes

$$U(x) = u\left(x; \frac{h}{2}\right) + a_2 \frac{h^2}{4} + a_3 \frac{h^3}{8} + a_4 \frac{h^4}{16} + \cdots \tag{2}$$

Solving for $U(x)$, neglecting the higher-order terms will lead to the approximate value

$$U(x) = \frac{1}{3}\left[4u\left(x; \frac{h}{2}\right) - u(x; h)\right] + o(h^3) \tag{3}$$

Then, similarly using grid sizes $h/2$ and $h/4$, we arrive at

$$U(x) = \frac{1}{3}\left[4u\left(x; \frac{h}{4}\right) - u\left(x; \frac{h}{2}\right)\right] + o(h^3) \tag{4}$$

The second-order Richardson extrapolated value for $U(x)$ using grid sizes h and $h/2$ is given by

$$U(x) = \frac{1}{3}\left[4u\left(x;\frac{h}{2}\right) - u(x;h)\right] \tag{5}$$

By this extrapolation, the second-order error term can be removed and the final value will have only higher-order terms. Similarly, Richardson extrapolation for the first-order solution,

$$U(x) = u(x;h) + a_1 h + a_2 h^2 + \cdots \tag{6}$$

is given by the formula

$$U(x) = 2u\left(x;\frac{h}{2}\right) - u(h) \tag{7}$$

In this study, we apply both the first- and second-order Richardson extrapolation methods to the CFD solutions based on a given scheme. The order of accuracy for first- and second-order schemes is derived below. The reason for considering both extrapolations is that a practical CFD solution may not show a clear order of accuracy. Naturally, one wishes to find out what options there are to employ Richardson extrapolation under such an unclear situation.

To illustrate the situation, consider the different numerical solutions for an exact solution from two different schemes given below:

$$U_1(x) = u_1(x;h) + a_1 h + a_2 h^2 + a_3 h^3 + \cdots \tag{8}$$

$$U_2(x) = u_2(x;h) + b_2 h^2 + b_3 h^3 + b_4 h^4 + \cdots \tag{9}$$

Applying first-order Richardson extrapolation on the first-order solution, Eq. (8), we get

$$U_1(x) = 2u_1\left(x;\frac{h}{2}\right) - u_1(x;h) + o\left(h^2\right) \tag{10}$$

Now, using second-order Richardson extrapolation on Eq. (8), we get

$$U_1(x) = \frac{1}{3}\left[4u_1\left(x;\frac{h}{2}\right) - u_1(x;h)\right] + o(h) \tag{11}$$

We can see that the order of accuracy of the first-order solution improves upon applying first-order Richardson extrapolation, but does not improve using second-order Richardson extrapolation.

Alternatively, applying first-order Richardson extrapolation to the second-order solution, Eq. (9), we get

$$U_2(x) = 2u_2\left(x; \frac{h}{2}\right) - u_2(x; h) + o(h^2) \tag{12}$$

and second-order Richardson extrapolation on (9) gives

$$U_2(x) = \frac{1}{3}\left[4u_2\left(x; \frac{h}{2}\right) - u_2(x; h)\right] + o(h^3) \tag{13}$$

In this case we see that order of accuracy of the second-order solution is improved only upon applying second-order Richardson extrapolation.

Assuming that coefficients associated with each term in Eqs. (8) and (9) are constants, then Eq. (11) suggests that using second-order extrapolation for a first-order solution, although the order of accuracy does not improve, will also do no harm. Not only does the coefficient of the leading truncation error term decrease in magnitude, but also the next truncation error term is eliminated. Similar observation can be made for Eq. (9). In summary, while only the "correct" extrapolation can improve the *order* of accuracy of a given scheme, either a first-order or a second-order extrapolation can be employed with no apparent damage to the *quantitative* accuracy of either a first-order or a second-order solution. Of course, such an analysis is based strictly on the Taylor series expansion, which may not be applicable for global analysis. Such observations can be of much relevance in practical computations, since one may not be able to identify the order of accuracy of CFD solutions over several grid systems uniformly.

2.2. Error Norms and Residuals

Assuming the error norms and residue of Richardson extrapolation are of Nth-order accuracy, then the values and exact value satisfy the following:

$$\text{Err}_1 = |u_1 - U| = a_1 h_1^N + o(h_1^{N+1}) \tag{14}$$

$$\text{Err}_2 = |U - u_2| = a_1 h_2^N + o(h_2^{N+1}) \tag{15}$$

where h_1 and h_2 are the intervals of the grid. Neglecting the higher-order terms and combine the two equations we can get the order as

$$N = \frac{\lg(\text{Err}_2) - \lg(\text{Err}_1)}{\lg h_2 - \lg h_1} \tag{16}$$

The norms used here to represent the error are

$$\text{1-Norm:} \qquad E_1 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}|U_{i,j} - u_{i,j}|}{n \times m} \tag{17}$$

$$\text{2-Norm:} \qquad E_2 = \frac{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}(U_{i,j} - u_{i,j})^2}}{n \times m} \tag{18}$$

$$\text{Infinity norm:} \qquad E_\infty = \max\left|U_{i,j} - u_{i,j}\right| \tag{19}$$

where $U_{i,j}$ is the actual solution and in our case it is the velocity interpolated from the finest mesh, and $u_{i,j}$ is the interpolated values from coarse grids; $n \times m$ is the grid number.

In this work, the residual level is monitored in the CFD calculation. The residual is calculated in the following way. We can substitute $\tilde{u}_{i,j} + \Delta u_{i,j}$ for $u_{i,j}$, where $\tilde{u}_{i,j}$ represents a provisional solution such as might occur at some point in an iterative process before convergence, in the difference equation $\mathbf{L}u_{i,j} = 0$, and obtain [8]

$$\mathbf{L}\tilde{u}_{i,j} + \mathbf{L}\Delta u_{i,j} = 0 \tag{20}$$

where $\mathbf{L}$ is the difference operator. The residual is defined as the the number that results when the difference equation, written in a form giving zero on the right-hand side, is evaluated for an intermediate or provisional solution. If the provisional solution satisfies the difference equation exactly, the residual vanishes.

With this definition, Eq. (20) can be written as

$$\mathbf{L}\Delta u_{i,j} = -R_{i,j} \tag{21}$$

where $R_{i,j}$ is the residual at the point (i, j). In our computation, the total residual is used to monitor computations; it is estimated by summing the absolute value of the residual in each computational cell.

2.3. Lagrangian Interpolation

In the present scheme, the unknown variable is located at the center of each computational cell, the so-called cell-centered arrangement. In order to apply Richardson extrapolation, it is necessary to have values of the two different grids at the same points. This is done by projecting the values of one of the grids onto the other while retaining the accuracy of the original CFD solutions. This is achieved by using an interpolation scheme with higher-order accuracy than the existing CFD solutions. Specifically, Lagrangian interpolation using 9 and 16 points is used to determine the values at a desired location in the flow domain.

The coordinates in the base field that contains $n \times m$ given points are

$$x_1 < x_2 < \cdots < x_{n-1} < x_n$$
$$y_1 < y_2 < \cdots < y_{m-1} < y_m$$

and the corresponding values at the points are

$$z_{ij} = z(x_i, y_i) \qquad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m \tag{22}$$

Now a given point has the coordinate (s, t), and nine points in the base field are taken to interpolate the value at (s, t). These points satisfy the conditions

$$x_p < x_{p+1} < x_{p+2}$$
$$y_q < y_{q+1} < y_{q+2}$$

Then the 9-point interpolation equation is

$$z(s, t) = \sum_{i=p}^{p+2} \sum_{j=q}^{q+2} \left(\prod_{\substack{k=p \\ k \neq i}}^{p+2} \frac{s - x_k}{x_i - x_k} \right) \left(\prod_{\substack{l=q \\ l \neq j}}^{q+2} \frac{t - y_l}{y_j - y_l} \right) z_{ij} \tag{23}$$

The interpolated value using 16 points is found in a similar way.

2.4. Convection Schemes

The convection schemes for CFD calculation employed here include second-order upwind and second-order centered schemes. For second-order upwind scheme, some special treatments are needed at the boundary. In the present study, the so-called hybrid scheme, chosen between the first-order upwind and second-order central difference schemes according to the local cell Reynolds number, is employed [8]. All other terms are treated by second-order central difference schemes. For detailed descriptions of the computational procedures, refer to Shyy [9] and Shyy et al. [10].

3. RESULTS AND DISCUSSION

CFD calculations are performed on grid sizes h and $h/2$, and the values of grid $h/2$ are projected on grid h using Lagrangian interpolation. Interpolations using 9 and 16 points in space are considered in this report. While interpolating for the boundary points, more points are taken from the main domain. After getting the two grid values at the same points, Richardson extrapolation is done by using Eq. (5) or (7), and the results are compared with the benchmark values. While the goal of Richardson extrapolation is to project solutions on coarser grids to improve accuracy, in this work we conduct the interpolation from a fine to a coarse grid. Since a coarse-grid solution is less accurate, and a fine-grid solution has more information to support interpolation procedures, we decide to project the fine-grid solutions to a coarse grid. Such a procedure does not affect the fundamental issues involved, and offers an upper bound for the performance of Richardson extrapolation.

3.1. Description of Problems

To help investigate the issues involved, both analytically defined mathematical expressions and well-investigated fluid dynamics problems will be used. In the latter aspect, the 2-D laminar lid-driven cavity and the 2-D turbulent flow over a backward-facing step will be considered. The test cases are run until the residual from the start to the end reaches less than or equal to 10^{-4}. The laminar flow of an incompressible, constant-property, Newtonian fluid in a square cavity is analyzed for Reynolds numbers 100 and 1,000 separately. As shown in Figure 1*a*, the flow has velocity only in the x-direction in the top of the rectangular cavity. It enters the cavity with a constant velocity with a magnitude of U. Velocity and pressure fields of different grids (11×11), (21×21), (41×41), (81×81), and (161×161) are computed using the second-order upwind scheme. Richardson extrapolation is done on grids of lower sizes and the results are compared with the values computed using the finest mesh (161×161), and also compared with those reported by Ghia et al. [11]. Figures 2 and 3 show the results for Re = 100 and 1,000, respectively. Figure 4 shows the differences in velocity profiles between the finest grid and other grid solutions. Figure 5 shows the order-of-accuracy contours. The asymptotic order of the scheme is calculated by the formula given in Eq. (16). It is clear that the solutions do not exhibit a uniform order of accuracy.

The geometry of the backward-facing step is given in Figure 1*b*. In this case, calculations are performed for a flow at Re = 10^6 using a model with an aspect ratio (height to length of the channel) of 3/7. The step height h is taken to be 1.0 unit in our study. Four different grids considered are (36×16), (71×31), (141×61), and (281×121). Richardsons extrapolation is done combining the coarser grids (36×16), (71×31) and (71×31), (141×61). The solution obtained is compared with that of the results of the finest grid (281×121). The k–ε two-equation turbulence model with the wall function treatment is adopted [10]. The results from the grid (281×121) are used as a benchmark for comparison; see Figure 6.

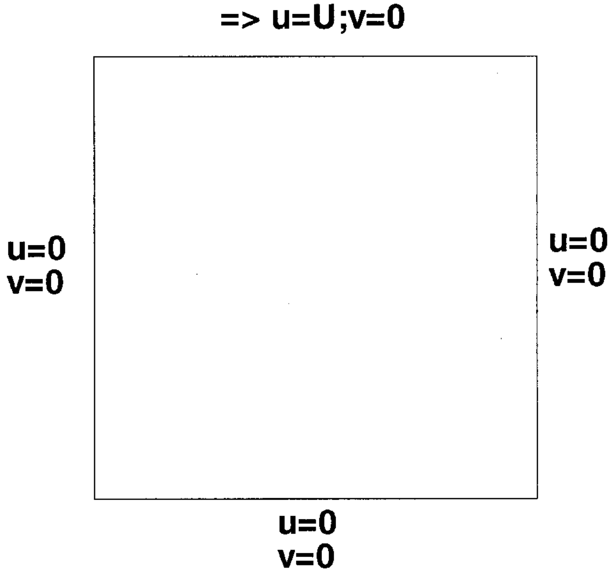
3.2. Evaluation of Lagrangian Interpolation

First, the following polynomials and trigonometric functions test the accuracy of the interpolation schemes mentioned in Section 2.3:

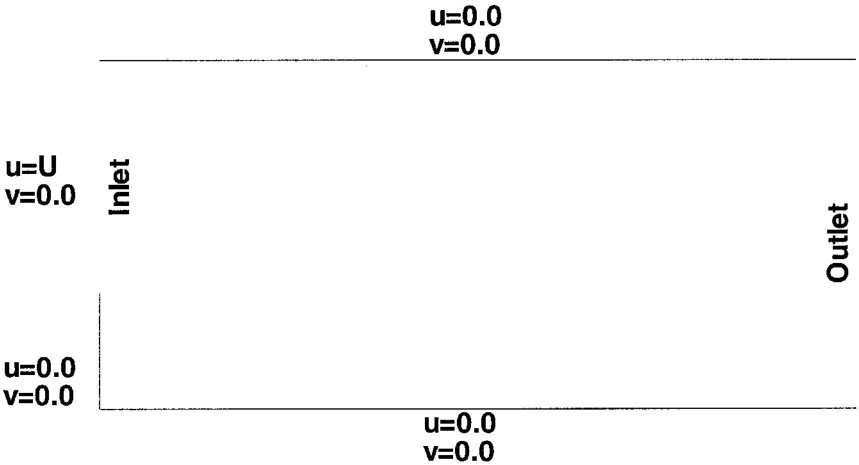
$$ax^2 + by^2$$
$$ax^3 + by^3$$
$$ax^4 + by^4$$
$$\sin(10x) + \cos(11y)$$
$$\sin^2(10x) + \cos^2(11y)$$

where a and b can be assigned freely; here a is given the values 1 and 0.5, and b the same.

For the cavity flow, the various grid sizes considered are (161×161), (81×81), (41×41), and (21×21). The x and y dimensions are taken as 1.0 unit. The correct values are first assigned at the center of each computational cell. Then the interpolation scheme is used to map the values from a fine grid to a coarse grid. The accuracy and error norm in that process for various orders of polynomials are tabulated. From Table 1, it can be seen that using 9 points, a polynomial of order up to 2 can be mapped



(a) Two-dimensional lid-driven cavity

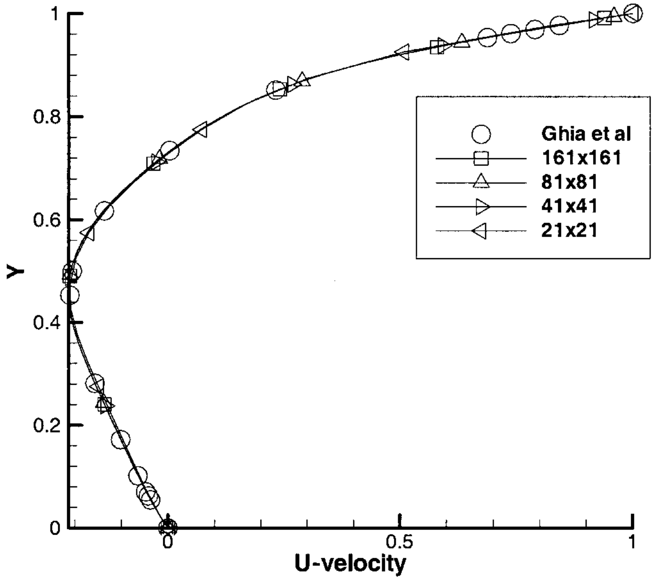


(b) Backward-Facing step

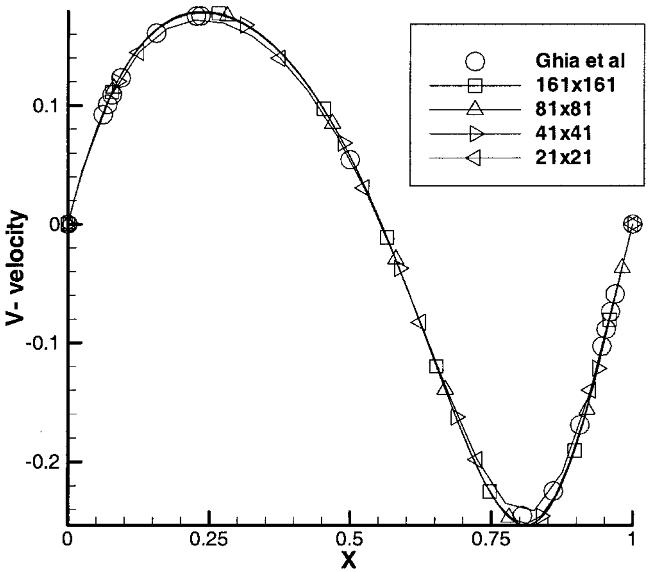
Figure 1. Two-dimensional lid-driven cavity and backward-facing step flows.

with the highest accuracy, and further higher-order polynomials can be mapped with an order of accuracy about 3 for 9-point, and about 4 for 16-point interpolation. These results demonstrate that the interpolation formulas work as expected.

For cavity flow the results from Re = 1,000 are used to interpolate from a fine mesh (161×161) to a coarse mesh (21×21) using both 16 and 9 points. As shown in Figure 7, results from the two interpolation formulas are close to each other. Ob-

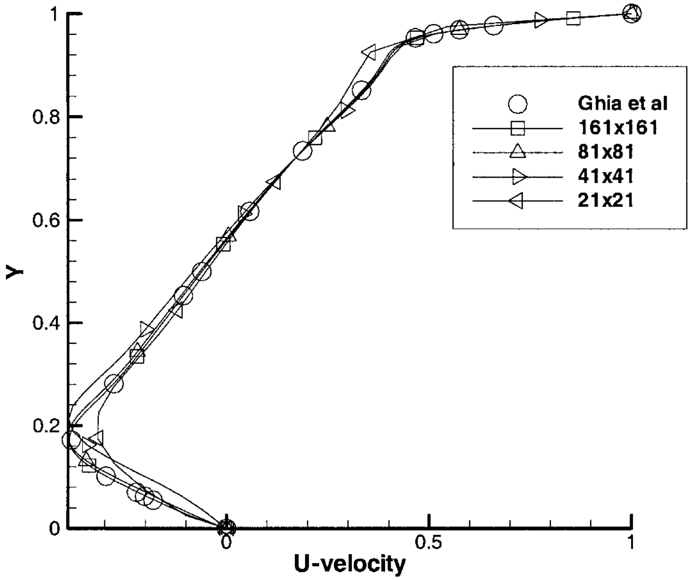


(a) U-velocity profile

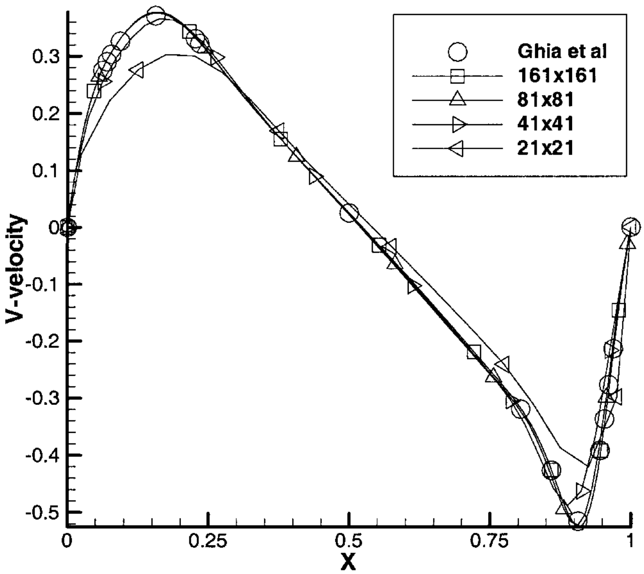


(b) V-velocity profile

Figure 2. Comparsion of u-and v-velocity profiles at geometric center, Re = 100, cavity flow.

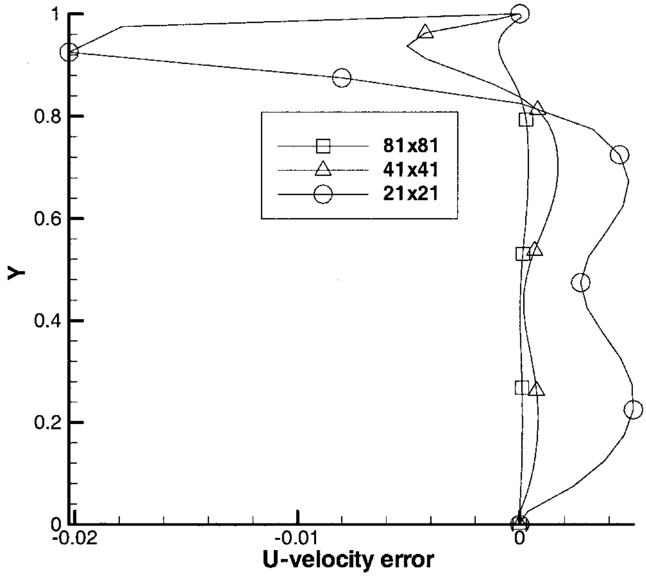


(a) U-velocity profile

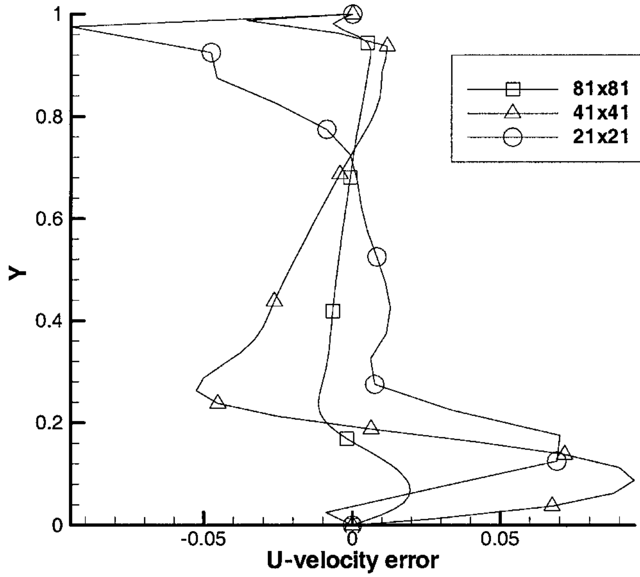


(b) V-velocity profile

Figure 3. Comparsion of u- and v-velocity profiles at geometric center, Re = 1,000, cavity flow.

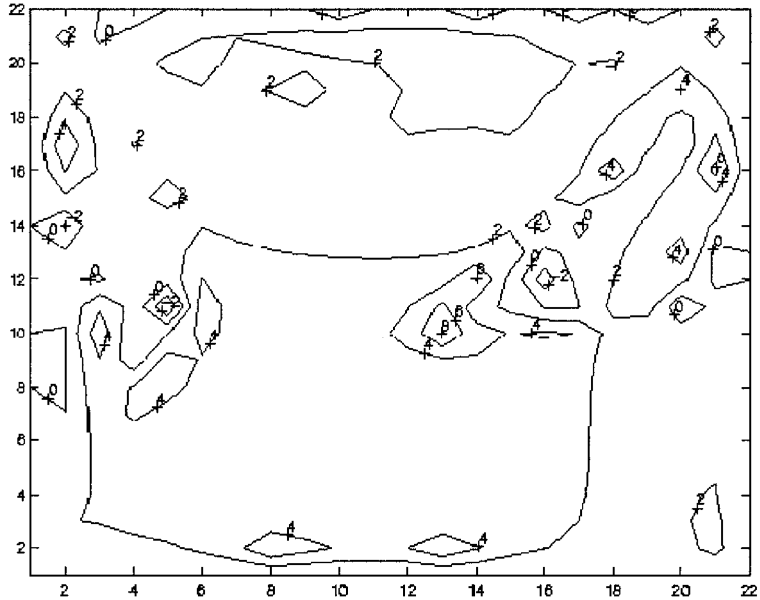


(a) U-velocity errors based on 161×161, Re=100

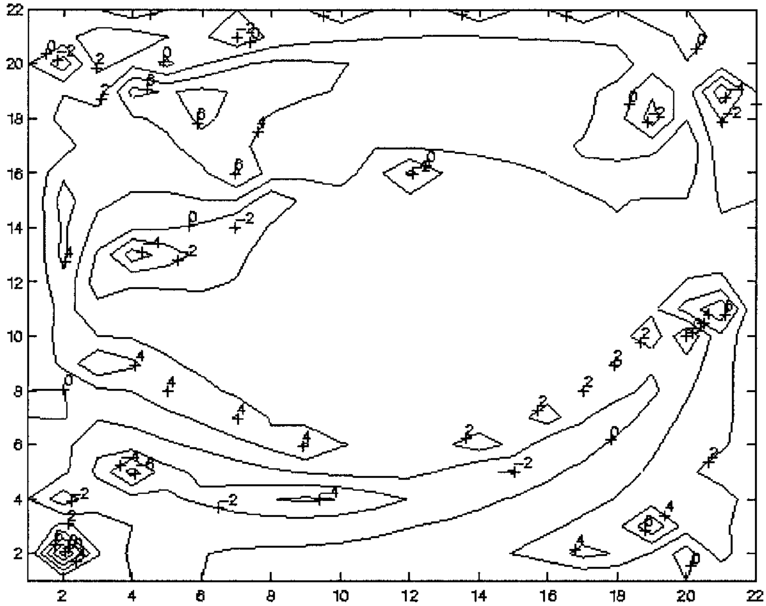


(b) U-velocity errors based on 161×161, Re=1000

Figure 4. Velocity errors of different grids and Reynolds numbers, cavity flow.



(a) Re = 100, u –velocity(16–points)



(b) Re = 1000, u –velocity(16–points)

Figure 5. Contour of the u-velocity asymptotic order, cavity flow.

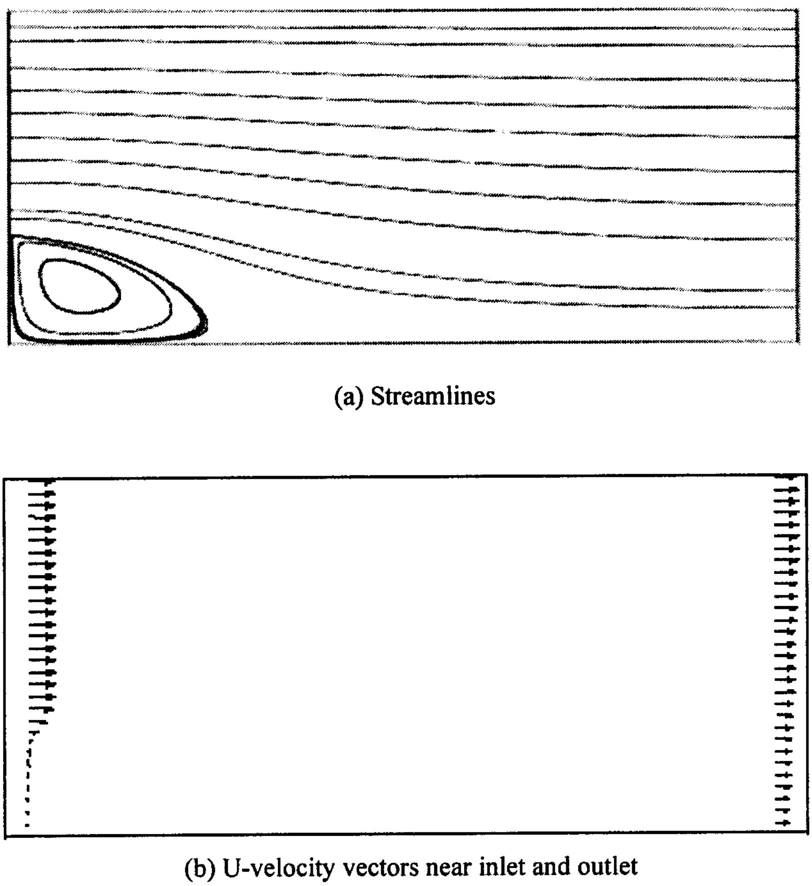


Figure 6. Streamlines and velocity vectors, backward-facing step flow.

viously, it is important to employ an interpolation formula of higher accuracy than the numerical solution so that no significant loss in accuracy occurs in the course of data interpolation.

As shown in Figure 8, for the backward-facing step flow, the interpolation from finer grid to coarser one results in numerical oscillations, which are caused by the high gradient in velocity in the wall region. One can eliminate such oscillations by concentrating only on the nodes away from the wall, as shown in Figures 8*b* and 8*d*. However, such practices will leave the near-wall flow structure unresolved, which is undesirable.

3.3. Richardson Extrapolation of CFD Results

For cavity flow, Figures 9–14 give the comparison of the Richardson extrapolation results using different interpolations (9 and 16 points), different Reynolds numbers (Re = 100 and 1,000), and different CFD convection schemes (central difference and second-order upwind). All figures are drawn to logarithm scale, and h_{fine} is the gird spacing of grid (81×81).

Table 1. Evaluation of Lagrangai n interpolation for analytical functions

$ax^2 + by^2, \quad a = 0.5, b = 1.0$

Grids	(81, 81) ⇒ (41, 41)		(41, 41) ⇒ (21, 21)	
	16 points	9 points	16 points	9 points
$\mathrm{Log}(E_1)$	− 16.0	− 16.4	− 16.1	− 16.6

$ax^3 + by^3, \quad a = 0.5, b = 1.0$

Grids	(81, 81) ⇒ (41, 41)		(41, 41) ⇒ (21, 21)	
	16 points	9 points	16 points	9 points
$\mathrm{Log}(E_1)$	− 16.1	− 6.2	− 16.2	− 5.3
Order of accuracy	9 points		2.9	
	16 points	Interpolated results equal to exact solutions		

$ax^4 + by^4 \quad a = 0.5, b = 1.0$

Grids	(81, 81) ⇒ (41, 41)		(41, 41) ⇒ (21, 21)	
	16 points	9 points	16 points	9 points
$\mathrm{Log}(E_1)$	− 7.6	− 5.9	− 6.4	− 5.0
Order of accuracy	9 points		2.9	
	16 points		3.9	

$\sin(10x) + \cos(11y), x = 1.0$ and $y = 0.5$

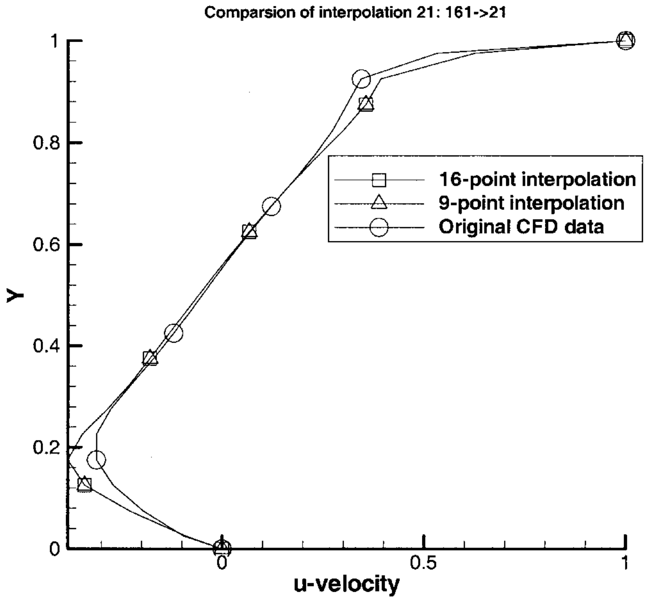
Grids	(81, 81) ⇒ (41, 41) 16 points	(41, 41) ⇒ (21, 21) 16 points
$\mathrm{Log}(E_1)$	− 5.5	− 4.3
Order of accuracy	4.3	

$\sin^2(10x) + \cos^2(11y), x = 1.0, y = 0.5$

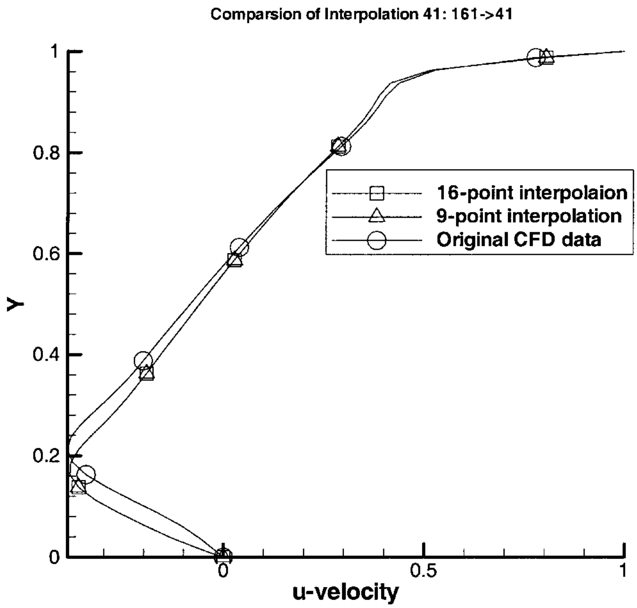
Grids	(81, 81) ⇒ (41, 41) 16 points	(41, 41) ⇒ (21, 21) 16 points
$\mathrm{Log}(E_1)$	− 4.6	− 3.4
Order of accuracy	3.9	

From Figure 9, using second-order Richardson extrapolation combining 16-point Lagrangian interpolation, it can be inferred that Richardson extrapolation improved the order of accuracy for only Re = 1,000 and reduced the absolute error in most of the cases. The second-order upwind scheme is used to treat convection.

Figure 10 demonstrates that first-order Richardson extrapolation exhibits qualitatively similar behavior to second-order Richardson extrapolation. Together, Figures 9 and 10 point out that for smooth solutions such as those presented in the

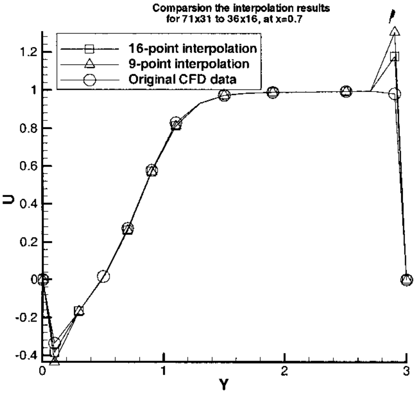


(a) Interpolation of 161×161 to 21×21, Re=1000

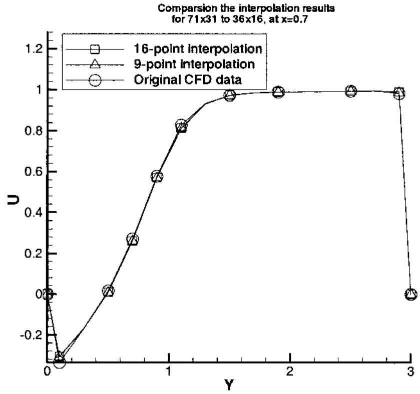


(b) Interpolation of 161×161 to 41×41, Re=1000

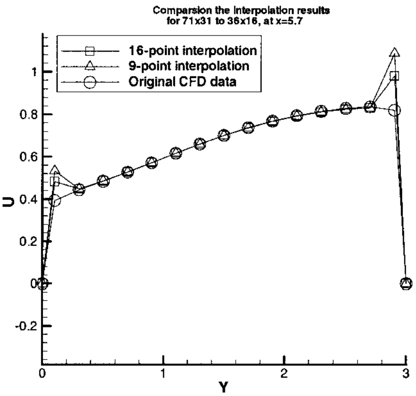
Figure 7. Different interpolations of u velocity at geometric center, cavity flow.



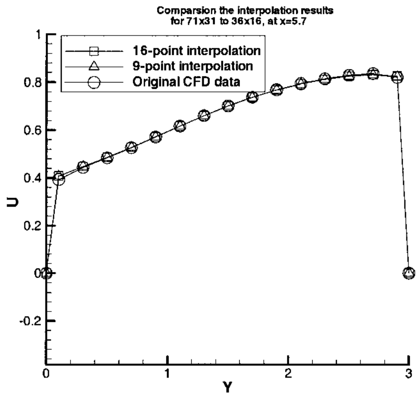
(a) Using the values at wall, at x/*h* =0.7



(b) Neglecting the values at wall, at x/*h* =0.7



(c) Using the values at wall, at x/*h* =5.7

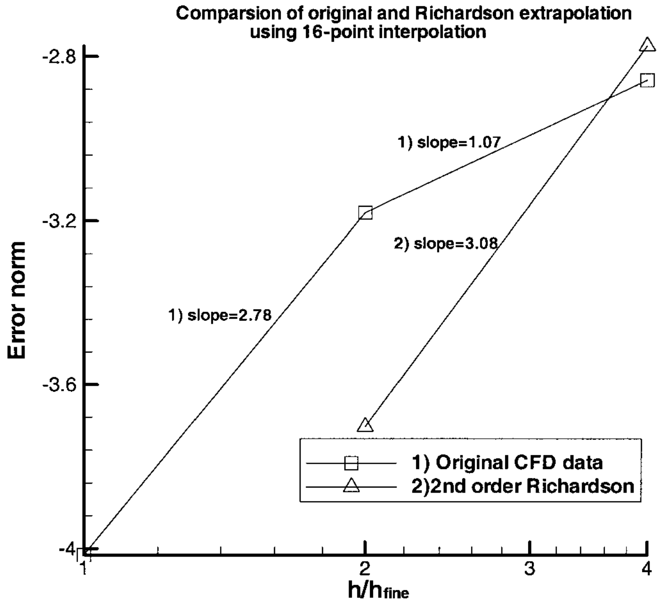


(d) Neglecting the values at wall, at x/*h* =5.7

Figure 8. Different interpolations of u velocity at the pointed-out section, backward-facing step flow.

laminar cavity flows, Richardson extrapolation can improve solution accuracy. However, Figure 9 shows that with second-order Richardson extrapolation, the grid resolution cannot be too coarse. Figure 10 shows that the first-order Richardson extrapolation may be less effective quantitatively. As discussed in relation to the analysis presented in Section 2, such an observation is not surprising. It seems clear that in order to benefit from Richardson extrapolation, even for smooth CFD profiles, care needs to be exercised.

Figure 11 shows that different error norms affect the trends of Richardson extrapolation. On the other hand, if we compare the performance between Richardson extrapolation and solutions obtained directly from mesh refinement, then the choice of the error norm does not have an impact. Figures 9 and 10 are indicative of the relative performance between Richardson extrapolation and mesh refinement exercises. This observation is reasonable from the viewpoint that Richardson ex-



(a)Using 2nd order upwind scheme CFD data, Re=1000

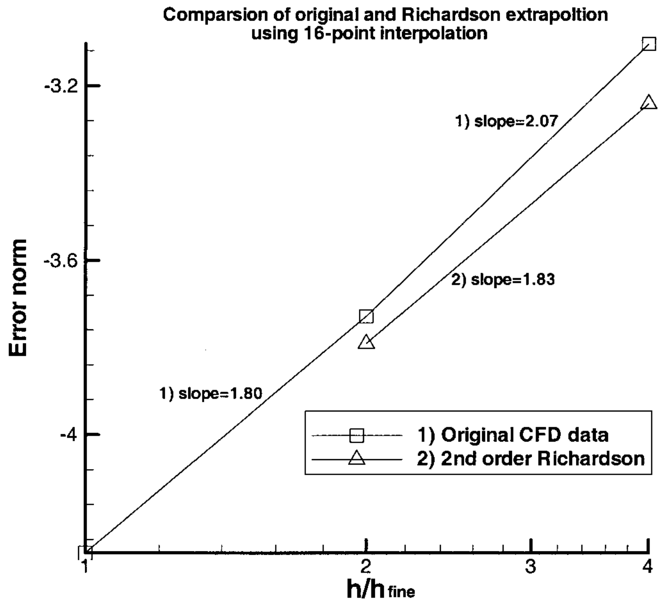
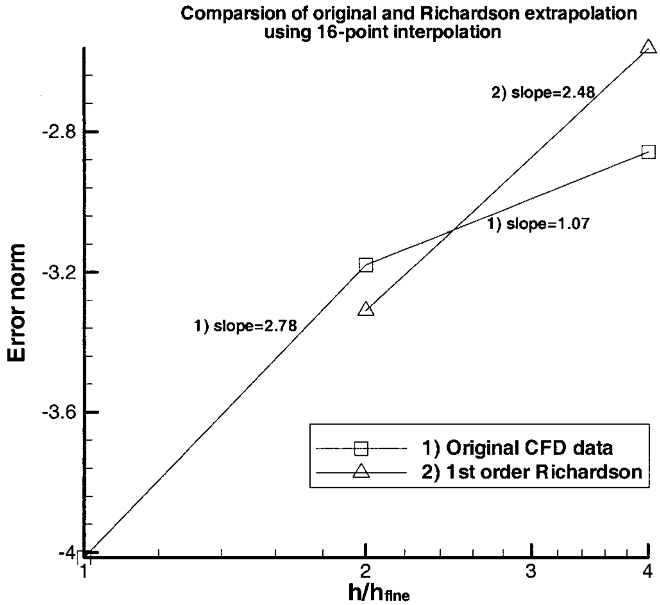
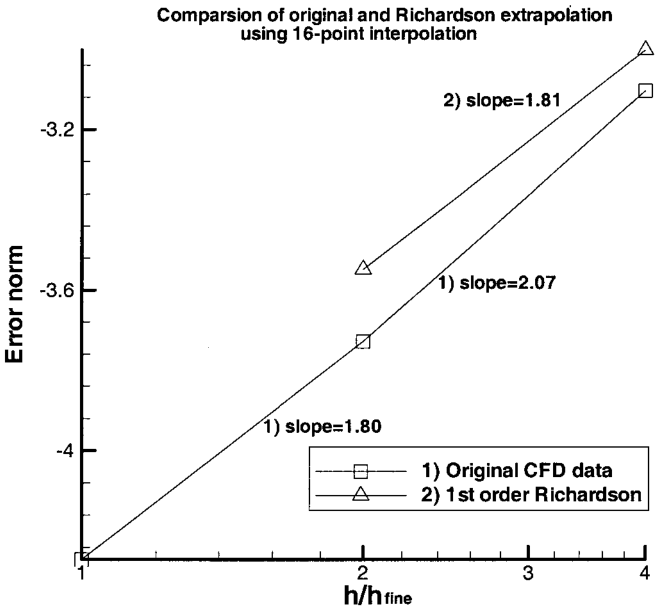


Figure 9. 2-norm of second-order Richardson extrapolation and original CFD data, cavity flow.



(a)Using 2[nd] order upwind scheme CFD data, Re=1000



(b)Using 2[nd] order upwind scheme CFD data, Re=100

Figure 10. 2-norm of first-order Richardson extrapolation and original CFD data, cavity flow.

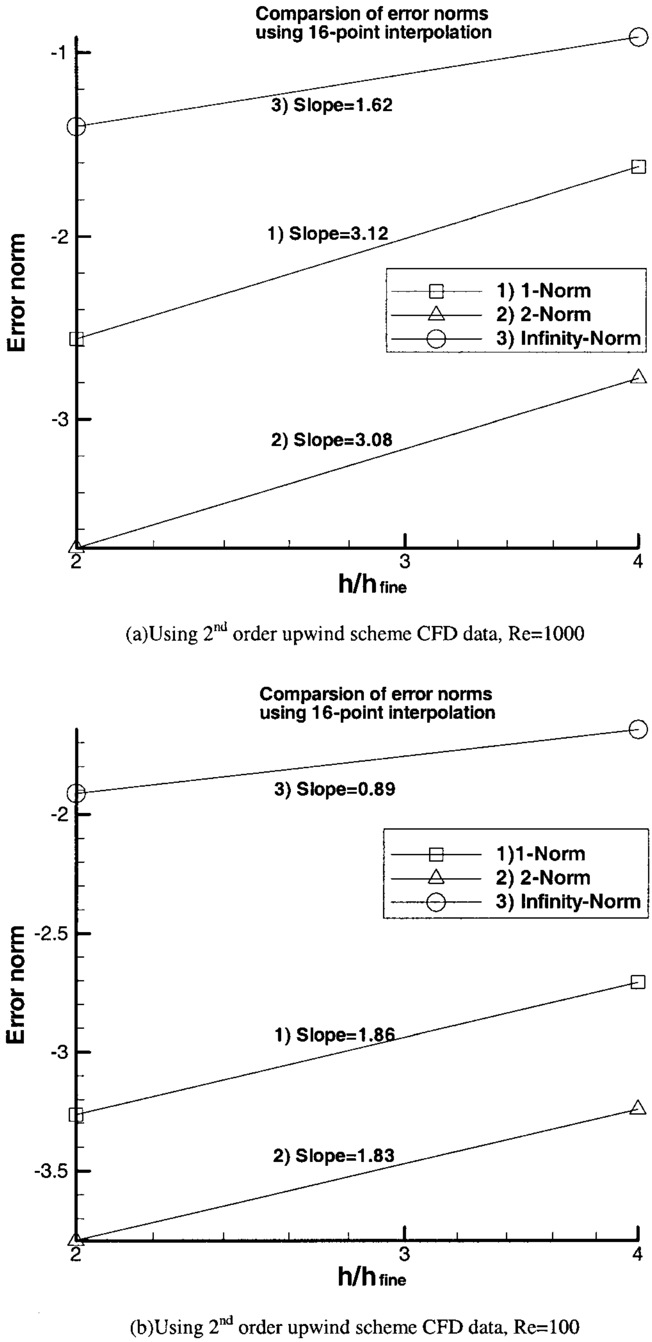
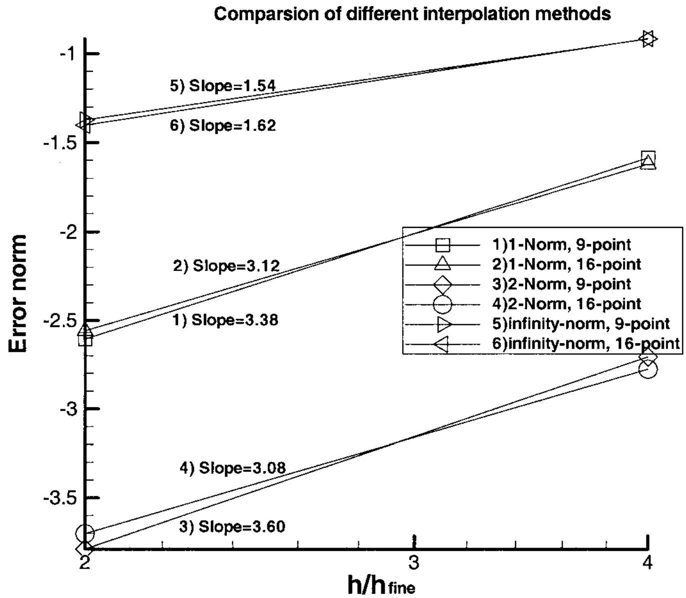
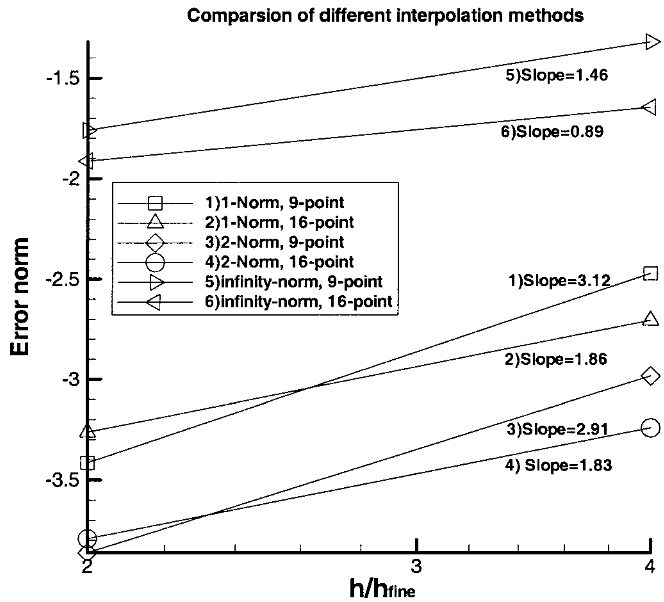


Figure 11. Error norms of second-order Richardson extrapolation, cavity flow.

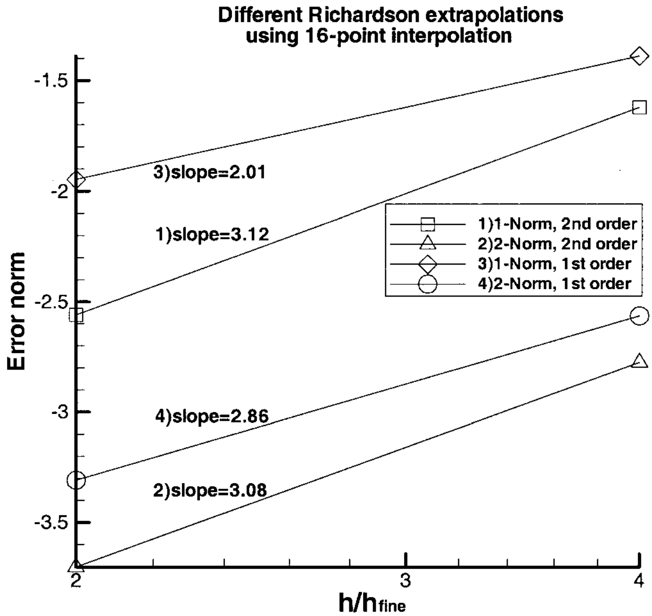


(a)Using 2nd order upwind scheme CFD data, Re=1000

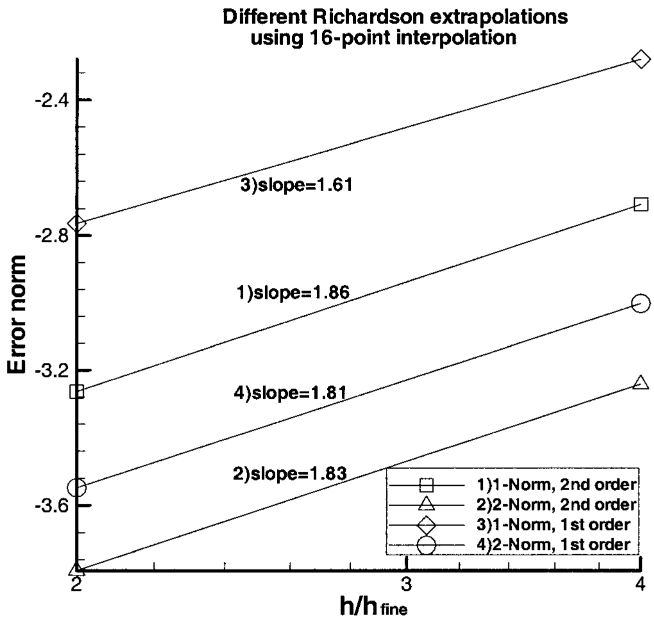


(b)Using 2nd order upwind scheme CFD data, Re=100

Figure 12. Error norms of second-order Richardson extrapolation using different interpolations, cavity flow.

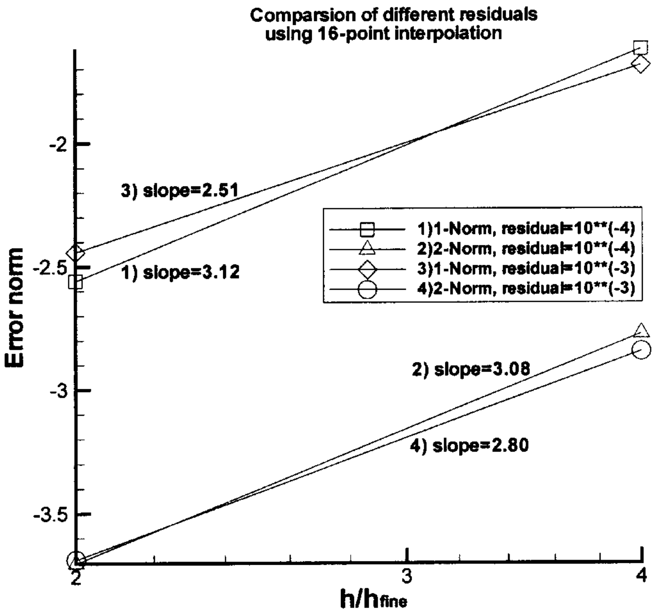


(a)Using 2[nd] order upwind scheme CFD data, Re=1000

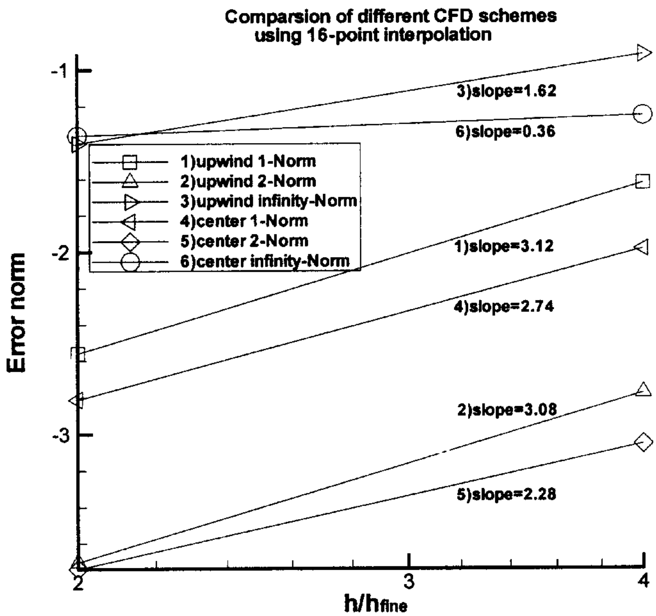


(b)Using 2[nd] order upwind scheme CFD data, Re=100

Figure 13. Comparsion of first- and second-order Richardson extrapolation, cavity flow.

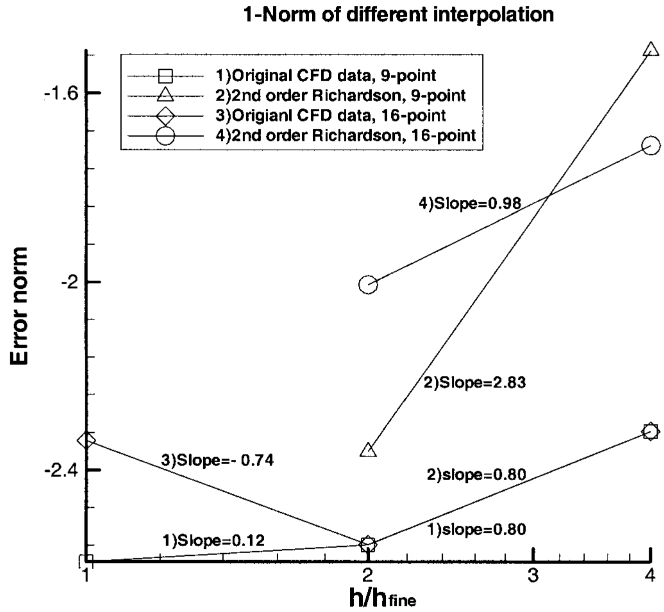


(a) Different residuals of 2^{nd} order upwind scheme, Re=1000

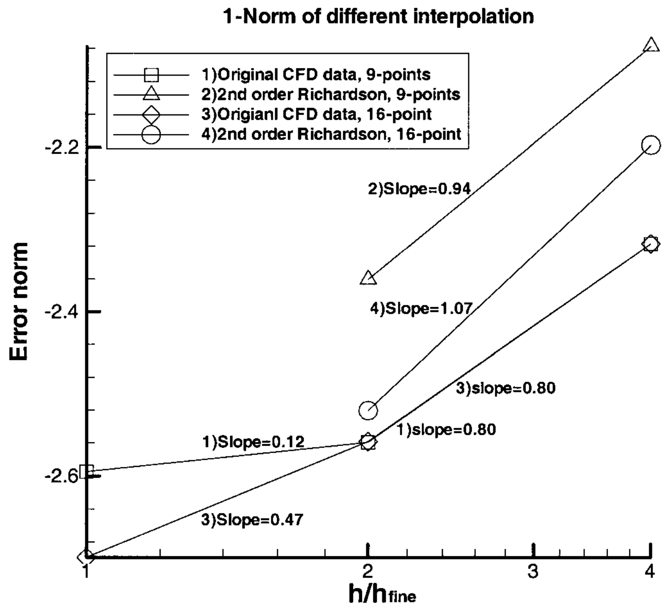


(b) Using different CFD schemes, Re=1000

Figure 14. Comparsion of second-order Richardson extrapolation using different residuals and CFD schemes, cavity flow.



(a) Using the values of wall



(b) Neglecting the values of wall

Figure 15. Different treatments for interpolation near wall, backward-facing step flow.

trapolation cannot be considered effective if its performance based on *any* error norms is less satisfactory.

Figure 12 shows that different interpolations (9- and 16-point) for Re = 1,000 have no uniform or consistent impact on the effectiveness of Richardson extrapolation.

Figure 13 compares first- and second-order Richardson extrapolation results. As expected, second-order Richardson extrapolation gives a higher order of accuracy, and lower quantitative error, when compared to first-order extrapolation. These are consistent with Figures 9 and 10 as well as the analysis in Section 2.1. The difference is less prominent for Re = 100.

Figure 14 shows that tighter residual results in marginally better performance of the Richardson extrapolation. Of course, this observation can be justified only when the level of residual does not adversely affect the quality of the CFD solutions in noticeable ways.

For backward-facing step flow, from Figure 15*a*, when interpolation errors exist, Richardson extrapolation depicts higher slopes but lower quantitative accuracy than the original CFD solution. When some of the extra errors are removed, the accuracy is improved, and the slopes of 9-point and 16-point interpolation became close. Figure 15*b* shows the changes after the wall value (zero) is neglected in the interpolation procedure. Again, no improvement from Richardson extrapolation is observed.

4. CONCLUSIONS

In this work, we intentionally adopt a uniform grid so that there is no ambiguity in the extrapolation procedure. For practical computations, nonuniform and curvilinear coordinates are often employed, which can make Richardson extrapolation more difficult to apply. It is demonstrated that the order of accuracy of numerical solutions does not follow the guidance offered by the local truncation error analysis. While different CFD codes and numerical schemes are likely to affect the quantitative details of the investigation, it is expected that no qualitative differences will appear for the commonly adopted approaches. In the following, we summarize the main findings reached in the present study.

For cavity flow with Re = 100 and 1,000, the flow fields are smooth, and Richardson extrapolation can improve the slope of convergence. Even with corner singularities, the present finite-volume formulation is capable of producing satisfactory results.

Both theoretical analysis and CFD results show that the second-order Richardson extrapolation performs better than the first-order Richardson extrapolation. Both first- and second-order Richardson extrapolation can affect the accuracy, but not according to the local error analysis. The trends among different cases are also not uniform.

For a flow field with a high gradient in properties (near the walls in backward-facing step flow), Lagrangian interpolation using a 9-point or 16-point formula creates an overshoot in the velocity profile. Consequently, Richardson extrapolation does not work effectively. In fact, it worsens the accuracy. In other words, Richardson extrapolation can improve accuracy only if there are no high gradient values to degrade the accuracy of Lagrangian interpolation.

From the results using both 9 and 16 points for interpolating, it can be summarized that an interpolation scheme with higher-order accuracy than the original CFD results retains the accuracy of the original solution. The results are very close when the order of the interpolation scheme is increased.

One can employ different formulas to interpolate. However, the need for maintaining adequate accuracy from interpolation means that one cannot resort to low-order methods, which restrict our ability to eliminate oscillations near sharp gradients.

Richardson extrapolation behaves consistently for any reasonable choice of error norms and residuals.

Different interpolation schemes and numerical treatments may result in different outcomes with regard to Richardson extrapolation. Suffice it to say that in the context of commonly adopted CFD techniques for engineering computations, such as those considered here, the findings reported in this study are expected to be representative. Other concepts, such as those discussed in [12–14], are needed for systematic accuracy improvement using extrapolation procedures.

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