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Interpolation strategies in repeated Richardson extrapolation

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Abstract. We discuss the interpolation strategies related to Richardson extrapolation and repeated Richardson extrapolation. We emphasize that, in most computations, the interest is in obtaining accurate solution on the current computational grid, not that on the coarse level grids on which the extrapolated solutions reside. We tackle the interpolation issue that has largely been overlooked in Richardson extrapolation related applications. We present numerical experiments to support our analysis.

Keywords: Computational grid, finite difference scheme, Richardson extrapolation, interpolation, high order solution

1. Introduction

The solution of a modeling and simulation problem is usually computed on a computational grid to a certain accuracy. Usually, a continuous problem, e.g., a partial differential equation, is first discretized using a discretization scheme, which results in a linear system to be solved. For a given discretization scheme, the accuracy of the computed solution is usually controlled by the relative distance or density of the discretization elements.

One approach to raising the order of accuracy of the computed solution is the utilization of Richardson extrapolation technique [6]. Although this approach is well-known, it is not considered to be cost-effective or convenient in many applications. The reason is that Richardson extrapolation needs solutions computed on at least two computational grids with different resolutions, and the extrapolated solution is obtained on the grid with the lower solution. Little attention has been paid to obtaining accurate solution on the grid with the higher resolution. More importantly, the generation of and the computation on a higher resolution grid are considered as the support to enhance the accuracy of the computed solution on the lower resolution grid in these cases. Often, accurate solutions are obtained only for a limited number of grid points or even a single grid point. This problem becomes severe in repeated Richardson extrapolation applications [1, 2].

This paper will discuss the issues related to obtaining accurate solution on the higher resolution grids in (repeated) Richardson extrapolation. We explore the concept of using solutions on the coarse level grids to improve the accuracy of the solutions on the finest (current) level grid. This concept has become popular in recently developed multiscale multilevel computational framework when solutions on multiple grids are computed without extra efforts [4, 8].

The remaining parts of this paper are arranged as follows. Section 2 presents a model problem with a discretization scheme and some extrapolation and interpolation strategies. Section 3 contains numerical results to compare the accuracy of the solutions obtained from different strategies and on various grids. Section 4 concludes this paper.

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2. Model problem and analysis

We use a one dimensional (1D) model Poisson equation to facilitate our discussions

$$\frac{d^2u(x)}{dx^2} = f(x), \qquad 0 \le x \le L. \tag{1}$$

Here $u(x)$ is the unknown function to be computed and $f(x)$ is the forcing function. x is the independent variable. When the context is clear, we will use u and f for simplicity. Without loss of generality, we restrict the computational domain to the interval $[0, L]$ for some positive number L, but the analysis is applicable to other domains.

2.1. The base discretization

Let $h = L/n$ be the mesh size of the uniform finite difference discretization over the interval $[0, L]$, where n is the number of uniform subintervals. For convenience, we assume that n is an even number and we use Ω^h to denote the discretized computational grid with the mesh size h. The nodal points are $x(j) = jh$ and $u^h(j) = u(x(j))$, $f^h(j) = f(x(j))$, where $j = 0, 1, \ldots, n$. When there is no doubt about the mesh size in question, we will drop the mesh size indicator h for simplicity. The discretization can also be done on some coarser level grids Ω^{2h}, Ω^{4h}, etc., with the corresponding mesh sizes $2h$, $4h$, respectively.

The 2nd order central difference scheme applied to Equation (1) results in a linear equation at a grid point j as

$$u(j-1) - 2u(j) + u(j+1) = h^2 f(j).$$

2.2. Richardson extrapolation for higher order accuracy

The well-known Richardson extrapolation technique [6] can be used to compute higher order solution, if the approximate solutions of 2nd order are available on a series of grids. Assume here that we already computed the approximate solutions of 2nd order on the fine grid Ω^h and on the coarse grid Ω^{2h}, the Richardson extrapolation computes

$$u^{2h}_{r(1)}(j/2) = \frac{4u^h(j) - u^{2h}(j/2)}{3}, \qquad j = 2, 4, \ldots, (n-2). \tag{2}$$

Here the subscript $r(1)$ indicates that the solution $u^{2h}_{r(1)}$ is obtained via the first Richardson extrapolation and is on the grid Ω^{2h}. The analysis in [5] shows that the solution $u^{2h}_{r(1)}$ is indeed of order $O(h^4)$ on the coarse grid Ω^{2h}.

If computed solutions are available on multiple levels of grids, the Richardson extrapolation can be performed on different levels. Furthermore, the extrapolated high order solutions can be recursively extrapolated to obtain solutions of even higher orders [2, 3]

$$u^{2h}_{r(k)}(j/2) = \frac{2^k u^h_{r(k-1)}(j) - u^{2h}_{r(k-1)}(j/2)}{2^k - 1}, \qquad j = 2, 4, \ldots, (n-2). \tag{3}$$

Here k is the number of times that the Richardson extrapolation is applied. We note that the highest order solution is only available on the coarsest grid, see Table 1 for an illustration of the orders of the computed solutions from repeated Richardson extrapolation on grids of different levels.

2.3. Interpolation to the fine grid

There are applications that may compute solutions on multiple grids with different resolutions and repeated Richardson extrapolation can produce very high order solutions with virtually no additional cost. A well-known example is the Romberg algorithm in numerical integration [7].

Other applications, for example, in (certain) multigrid computations, solutions can be obtained on different grids, but Richardson extrapolation has rarely been discussed as a means to obtain higher order solutions, until very recently [8]. The main reason, as noted above, is that the higher order solutions from the Richardson extrapolation are only available on the coarse level grids. In many applications, we are interested in the solutions on the current (finest level) computational grid.

The key question here is how to bring the high order solution extrapolated on the coarse grid to the fine grid. The obvious answer is to use interpolation techniques of some kinds.

Table 1
Orders of the solutions of the repeated Richardson extrapolation

u^{8h}	$u^{8h}_{r(1)}$	$u^{8h}_{r(2)}$	$u^{8h}_{r(3)}$
u^{4h}	$u^{4h}_{r(1)}$	$u^{4h}_{r(2)}$	
u^{2h}	$u^{2h}_{r(1)}$		
u^{h}			
2nd order	4th order	6th order	8th order

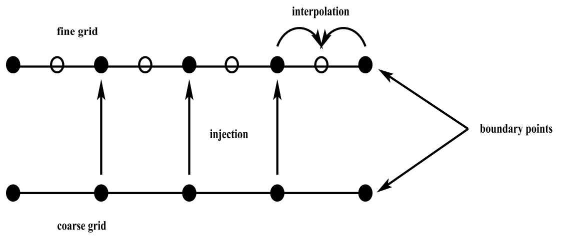

Fig. 1. Illustration of the compact interpolation process.

Note that the Richardson extrapolation technique computes the 4th order solution $u^{2h}_{r(1)}(j)$ on the coarse grid Ω^{2h}. This coarse grid solution can be directly injected to the fine grid Ω^h at their common grid points, i.e., $u^h(j) = u^{2h}(j/2)$, for even-numbered grid points, see Fig. 1. For the odd-numbered grid points on the Ω^h grid, we can use an operator induced interpolation formula as

$$u^h(j) = \frac{u^h(j-1)+u^h(j+1)-h^2 f^h(j)}{2} + O(h^4), \quad j = 1, 3, \ldots, (n-1), \tag{4}$$

which has a truncation error of 4th order. We point out that the standard linear interpolation is 2nd order and cannot preserve the 4th order solution from the neighboring grid points.

A 6th order interpolation operator is also available as [5]

$$u^h(j) = \frac{u^h(j-1)+u^h(j+1)}{2} - \frac{h^2(f^h(j-1)+10f^h(j)+f^h(j+1))}{24} + O(h^6), \tag{5}$$

for $j = 1, 3, \ldots, (n-1)$.

3. Numerical results and discussions

Numerical experiments were conducted to solve the 1D model problem (1) with the forcing function and boundary conditions satisfying the exact solution $u(x) = \sin(\pi^2 x)$. The domain of computation is [0, 1]. The code was written in MATLAB and run in double precision. The errors reported in Table 2 are the maximum absolute errors of the computed solution against the exact solution on the given grid. The solutions were computed on a series of four grids. The finest grid has a mesh size $h = 1/256$, and the coarsest grid has a mesh size $8h = 1/32$. The 2nd order base solutions were computed on the four levels of grids Ω^h, Ω^{2h}, Ω^{4h}, and Ω^{8h}, see Row 1 in Table 2.

With the computed 2nd order solutions on four levels of grids, we can compute 4th order solutions using the first Richardson extrapolation on three levels of grids Ω^{2h}, Ω^{4h}, and Ω^{8h}, see Row 2 in Table 2. Interpolated solutions on Ω^h, Ω^{2h}, and Ω^{4h} were obtained by using the 1st extrapolated solutions on Ω^{2h}, Ω^{4h}, and Ω^{8h}, with the 4th order interpolation formula (4) and the 6th order interpolation formula (5), respectively. These results are shown in Table 2 in Row 3 (with the 4th order interpolation) and Row 4 (with the 6th order interpolation), respectively. It can be verified that these interpolated solutions are 4th order, although the solutions interpolated by the 6th order formula are slightly better than those interpolated by the 4th order formula. It comes as no surprise that both the 4th order and the 6th order interpolation formulas can interpolate the 4th order solutions accurately.

We performed the 2nd Richardson extrapolation using the 1st extrapolated solutions on Ω^{2h}, Ω^{4h}, and Ω^{8h}, and the results are listed in Row 5 of Table 2 on Ω^{4h} and Ω^{8h}, respectively. These solutions can be verified to be 6th order. Again, we interpolated these

Table 2
The maximum absolute errors on different grids with various extrapolation and interpolation strategies

Row	$h = 1/256$	Ω^{8h}	Ω^{4h}	Ω^{2h}	Ω^h
1	Base 2nd order scheme	1.06e-2	2.66e-3	6.66e-4	1.66e-4
2	1st extrapolation	1.27e-5	7.92e-7	4.95e-8	
3	4th order interpolation		1.61e-5	1.01e-6	6.28e-8
4	6th order interpolation		1.26e-5	7.92e-7	4.95e-8
5	2nd extrapolation	2.98e-9	4.67e-11		
6	4th order interpolation		2.35e-5	1.47e-6	1.47e-6
7	6th order interpolation		3.10e-8	4.83e-10	4.83e-10
8	3rd extrapolation	1.41e-13			
9	4th order interpolation		2.35e-5	2.35e-5	2.35e-5
10	6th order interpolation		2.80e-8	2.80e-8	2.80e-8
Column	1	2	3	4	5

6th order solutions to finer grids using the 4th and the 6th order interpolation formulas. The Ω^{4h} results in Column 3 of Rows 6 and 7 used the 2nd extrapolated solution on Ω^{8h}. The Ω^{2h} results in Column 4 of Rows 6 and 7 used the 2nd extrapolated solution on Ω^{4h}. However, since there is no 2nd extrapolated solution on Ω^{2h}, the Ω^{h} results in Column 5 of Rows 6 and 7 were obtained by using the interpolated solutions in Column 4 of Rows 6 and 7, respectively. Due to this reason, there is no change of accuracy for solutions on the Ω^{h} and Ω^{2h} grids. However, we can easily observe that the solutions interpolated by the 6th order interpolation formula are much more accurate than those interpolated by the 4th order interpolation formula. In fact, it is easily verified that the former solutions are 6th order, and the latter solutions are 4th order, i.e., the order of the interpolated solution is limited by the order of the interpolation formula.

Using the 2nd extrapolated solutions on the Ω^{4h} and Ω^{8h} grids, we computed 3rd Richardson extrapolation solution on the Ω^{8h} grid, which is in Row 8 of Table 2. This solution is very accurate and almost reaches the limit of the double precision arithmetic that we used. We verified that this solution is 8th order (additional data is not shown). We again interpolated the 3rd extrapolated solution to Ω^{4h}, which was further interpolated to Ω^{2h} and Ω^{h}, respectively, using the 4th order and 6th order interpolation formulas. The results are shown in the last two rows of Table 2. We can see that the interpolated solutions are not accurate to the 8th order, as one might have wished. In fact, comparing these solutions to those interpolated from the 2nd extrapolated solutions, we can see that they are only 4th and 6th order solutions on the Ω^{4h} grid, respectively.

For a reference purpose only, according to Table 2, the most accurate solution on Ω^{h} is that (twice) interpolated from the 2nd Richardson extrapolation solution on Ω^{4h}, in Row 7 and Column 5 of Table 2.

The conclusion from our experiments and analyses is that the order of the interpolation formula limits the order of the solution that is interpolated to the fine grid. In fact, the order of the interpolated extrapolated solution is determined by the orders of the interpolation formula and the Richardson extrapolation solution, which one is lower. Therefore, without high order interpolation techniques, high order solutions computed from the repeated Richardson extrapolation technique cannot be propagated to the finest computational grid. On the other hand, repeated Richardson extrapolation may be useful to obtain more accurate solutions on the finest grid in certain applications, if suitable interpolation techniques are available. Unfortunately, to the best of our knowledge, there is no compact (3 point) interpolation formula that is higher than the 6th order formula (5) that we presented in this paper.

4. Concluding remarks

The main purpose of this paper is to draw attention to the usefulness of the repeated Richardson extrapolation technique in some computational applications. It is no doubt that repeated Richardson extrapolation can produce highly accurate solutions, but only on very coarse grids, compared to the finest computational grid on which the solutions are of main interests. Future efforts should be invested to develop techniques to bring the highly accurate solutions from the coarse grids to the finest grid. Highly accurate solutions are much more difficult to interpolate in higher dimensions.

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