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Notes on convergence of an algebraic multigrid method

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Abstract

The convergence theory for algebraic multigrid (AMG) algorithms proposed in Chang and Huang [Q.S. Chang, Z.H. Huang, Efficient algebraic multigrid algorithms and their convergence, SIAM J. Sci. Comput. 24 (2002) 597–618] is further discussed and a smaller and elegant upper bound is obtained. On the basis of element-free AMGe [V.E. Henson, P.S. Vassilevski, Element-free AMGe: General algorithms for computing interpolation weights in AMG, SIAM J. Sci. Comput. 23(2) (2001) 629–650] we rewrite the interpolation operator for the classical AMG (cAMG), present a uniform expression and then, by introducing a sparse approximate inverse in the Frobenius norm, give a general convergence theorem which is suited for not only cAMG but also AMG for finite elements and element-free AMGe.

Keywords: Algebraic multigrid; Interpolation operator; Convergence; Matrix analysis; Sparse approximate inverse

1. Introduction

Algebraic multigrid (AMG) has recently developed rapidly since Brandt et al. first introduced the classical AMG (cAMG) in the 1980s [1–4] due to its asymptotically optimal convergence and the increasing application need. AMG's key point is that the algebraically smooth error components after relaxation must be removed by determining a proper coarsening process and designing the appropriate prolongation operators. For cAMG, the smooth error is characterized by small r based on the defect equation $Ae = r$ [5]; the element-based AMGe [6] and ρAMGe [7] using spectral decomposition employed a similar criterion to produce the coarse grids and transfer operators. Element-free AMGe [8] gave a general matrix expression for computing AMG interpolation weights and described the results of cAMG and AMGe by proposing an extension operator. A new method of AMG coarse grid selection based on the element stiffness matrix and the technique of local relaxation for the smooth process are given in [9]. Here we would like to present a general framework for constructing the interpolating operator which is obtained by solving the defect equation approximately.

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2. The general interpolation operator

First, we start from the defect equation

$$a_{ii}^m e_i^m + \sum_{k\in C_i^m} a_{ik}^m e_k^m + \sum_{j\in D_i^m} a_{ij}^m e_j^m \approx 0, \quad i \in F^m \tag{2.1}$$

where m is the coarse grid level index which will be omitted while not causing any confusion, $C_i^m = C^m \bigcap S_i^m$, C^m denotes the coarse grid set, F^m the fine grid points, S_i^m the set of all strong connection points of the point i, and D_i^m is the complementary set of C_i^m in the neighborhood.

Let A have the block form

$$A = \begin{pmatrix} A_{FF} & A_{FC} \\ A_{CF} & A_{CC} \end{pmatrix}, \tag{2.2}$$

and A_{FF}, A_{CC} are square matrices. Evidently, $e_j^m (j \in D_i^m)$ must be approximated by e_i^m, $e_k^m (k \in C_i^m)$ or their (linear) combination in order to obtain the interpolation operator. When we let $e_j^m \approx pe_i^m + q\sum_{k\in C_i^m} g_{jk}^m e_k^m$ $(p, q \in \Re)$, $g_{jk}^m = \|a_{jk}^m\| / \sum_{k\in C_i^m} \|a_{jk}^m\|$, $j \in D_i^m$. Here, $e_j \approx pe_i + q\sum_{k\in C_i} g_{jk} e_k$ is equivalent to adding pa_{ij} to a_{ii} and $qa_{ij}g_{jk}$ to a_{ik}, i.e., we replace the j-th row of A_{FF} by the zero row and then place a 1 in the column j and p in the column i, and the block A_{FC} is modified to $\hat{A}_{FC}$ by zeroing the j-th row; we modify A_{FF} by zeroing out the off-diagonal entries, and replacing the diagonal entry a_{jj} with $\hat{a}_{jj} = -q\sum_{k\in C_i} a_{jk}$, keeping the off-diagonal entries. Then we have

$$\hat{A}_{FF} = \left[\left(a_{ii} + p\sum_{j\in D_i} a_{ij}\right)_{ii}\right], \qquad \hat{A}_{FC} = \left[\left(a_{ik} + q\sum_{j\in D_i} a_{ij} g_{jk}\right)_{ik}\right] \triangleq A_{FC} + B_{FC}, \quad i \in F, k \in C_i.$$

That is to say, the interpolation operator I_{m+1}^m can be obtained by approximately solving the equation

$$A_{FF}e_F + A_{FC}e_C = 0, \tag{2.3}$$

and has the operator form

$$e_F = \hat{A}_{FF}^{-1}(A_{FC} + B_{FC})e_C = (I_{m+1}^m)_{FC} e_C \triangleq I_{FC} e_C. \tag{2.4}$$

As for the cAMG, the interpolation weights are

$$w_{ik}^m = -\frac{a_{ik}^m + q\sum\limits_{k\in D_i^m} a_{ij}^m g_{jk}^m}{a_{ii}^m + p\sum\limits_{j\in D_i^m} a_{ij}^m}, \quad k \in C_i^m. \tag{2.5}$$

It is easy to see that the formula (2.5) is general and it may include Ruge and Stübe, Chang–Wong–Fu [10] and Chang–Huang [11] forms, etc., even any interpolation operator which is constructed based on matrix elements. As we choose $p = 1$ for the weak connection points and $q = 1$ for the strong connection points in D_i, we will get Ruge and Stüben interpolation formulae. Furthermore, by introducing two geometric assumption, and $p = 0, \pm 1$ for all points in D_i, $q = 1/2, 1, 2$ for D_i^s, $q = 1, 2$ for D_i^w, we can deduce easily Chang–Wong–Fu interpolation formulae, and an additional Jacobi or Gauss–Seidel iteration will get Chang–Huang interpolation formulae, Gauss–Seidel-type AMG interpolation operator [12].

Eq. (2.5) is more flexible and reasonable because it will sacrifice the accuracy that the set D_i^m is empirically divided into strong and weak, in particular, when the size of the matrix elements is almost the same order. Finally, we hope to emphasize that the set D_i^m is cut partly to save the computing work and memory in the practical computation, as also shown in the following theoretical analysis.

3. Convergence analysis

Let $(x, y)_E$ (or (x, y) for simplicity) and $\|\cdot\|$ be the Euclidean inner product and the associated norm respectively. If the matrix A is symmetric positive definite (i.e. $A > 0$), we also use the following three inner products:

$$(u, v)_0 = (Du, v), \qquad (u, v)_1 = (Au, v), \qquad (u, v)_2 = (D^{-1}Au, v),$$

along with their associated norms $\|\cdot\|_i$ $(i = 0, 1, 2)$, in which $D = \text{diag}(A)$. Denote by $\|\cdot\|_{Fr}$ the Frobenius norm of the matrix. To prove the convergence of the AMG method, by the theory of AMG [3,8,10–20], we need only to demonstrate the interpolation operator satisfying

$$\min_{e^{m+1}} \|e^m - I^m_{m+1} e^{m+1}\|_0^2 \le \beta \|e^m\|_1^2, \tag{3.1}$$

where β is independent of e^m and m.

First, we give the lemma:

Lemma 3.1. *Let $A > 0$, $B > 0$, then we have $(Ae, e)_E \le c(Be, e)_E$ with the constant c, if and only if $\rho(B^{-1}A) \le c$, where ρ is the spectral radius of the matrix.*

Proof. Because $A > 0$, $B > 0$, the following equality holds.

$$\rho(B^{-1}A) = \rho(B^{-1/2}AB^{-1/2}) = \sup_{y \in \mathbb{R}^n} \frac{(B^{-1/2}AB^{-1/2}y, y)_E}{(y, y)_E}.$$

When it is assumed that $B^{-1/2}y = e$, we have

$$\rho(B^{-1}A) = \sup_{e \in \mathbb{R}^n} \frac{(Ae, e)_E}{(Be, e)_E}.$$

Therefore we obtain the equivalence

$$(Ae, e)_E \le c(Be, e)_E \Leftrightarrow \rho(B^{-1}A) \le c. \quad \square \tag{3.2}$$

Lemma 3.2. *Let A_{FF} be strongly diagonally dominant, that is,*

$$a_{ii} - \sum_{j \in F, j \neq i} |a_{ij}| \ge \delta a_{ii} \quad (i \in F) \tag{3.3}$$

with some fixed, pre-defined $\delta > 0$. Then the following inequality holds

$$\rho(A_{FF}^{-1} D_{FF}) \le 1/\delta.$$

Proof. As (3.3), then

$$-\lambda + 1 - \sum_{j \in F, j \neq i} \frac{|a_{ij}|}{a_{ii}} \ge \delta - \lambda, \tag{3.4}$$

where λ is any eigenvalue of the matrix $D_{FF}^{-1} A_{FF}$.

Now we can draw the conclusion that $\lambda \ge \delta$.

Otherwise, by (3.4) we will deduce $\det(-\lambda I + D_{FF}^{-1} A_{FF}) \neq 0$. This will contradict that λ is the eigenvalue of $D_{FF}^{-1} A_{FF}$.

Thus, $\rho(A_{FF}^{-1} D_{FF}) \le 1/\delta$ is straightforward. $\square$

Lemma 3.3. *Let $A > 0$ and A_{FF} be strongly diagonally dominant. $(u, v)_{E,F}$ is the Euclidean inner product for the F component, and $(u, v)_{1,F} = (A_{FF}u, v)_{E,F}$. Then the following estimate holds:*

$$\|I_{FF} - D_{FF}^{-1} A_{FF}\|_{1,F} < 1.$$

Proof. The positive definiteness of A ensures the following computation.

$$\begin{aligned} \|I_{FF} - D_{FF}^{-1} A_{FF}\|_{1,F}^2 &= \max_{\|x\|_{1,F}=1} (A_{FF}(I_{FF} - D_{FF}^{-1} A_{FF})x, (I_{FF} - D_{FF}^{-1} A_{FF})x)_{E,F} \\ &= \max_{\|x\|_{1,F}=1} (A_{FF}^{-1/2} (I_{FF} - D_{FF}^{-1} A_{FF})^T A_{FF} (I_{FF} - D_{FF}^{-1} A_{FF})x, A_{FF}^{1/2} x) \\ &\le \rho((I_{FF} - D_{FF}^{-1} A_{FF})^2). \end{aligned}$$

And $\rho((I_{FF} - D_{FF}^{-1}A_{FF})^2) < 1$ then follows from the diagonal dominance of A_{FF}.
Hence, we have $\|I_{FF} - D_{FF}^{-1}A_{FF}\|_{1,F} < 1$. □

Now we give the main convergence theorem.

Theorem 3.1. *Assume $A > 0$, and $\rho(D^{-1}A) \le \eta$. Let A_{FF} be a strongly diagonally dominant matrix, and the interpolation operator $\bar{I}_{FC}$ satisfy*

$$\|e_F - \bar{I}_{FC}e_C\|_{0,F}^2 \le \beta_1\|e\|_1^2.$$

Then we have the following estimate for $\tilde{I}_{FC}$ which is one step of the fully relaxed Jacobi interpolation of $\bar{I}_{FC}$:

$$\|e_F - \tilde{I}_{FC}e_C\|_{0,F}^2 \le \beta_2\|e\|_1^2.$$

Proof.

$$\begin{aligned}\|e_F - \tilde{I}_{FC}e_C\|_{0,F}^2 &\le \rho(A_{FF}^{-1}D_{FF})\|e_F - \tilde{I}_{FC}e_C\|_{1,F}^2\\ &\le \frac{2}{\delta}(\|e_F + A_{FF}^{-1}A_{FC}e_C\|_{1,F}^2 + \|\tilde{I}_{FC} + A_{FF}^{-1}A_{FC}e_C\|_{1,F}^2)\\ &\le \frac{2}{\delta}(\|e_F + A_{FF}^{-1}A_{FC}e_C\|_{1,F}^2 + \|I_{FF} - D_{FF}^{-1}A_{FF}\|_{1,F}^2\|I_{FC} + A_{FF}^{-1}A_{FC}e_C\|_{1,F}^2)\\ &\le \frac{2(1+\eta\beta_1)}{\delta}\|e\|_1^2 = \beta_2\|e\|_1^2. \quad \square\end{aligned}$$

Remark. The general AMG convergence result is given in [11]. Obviously, the above main theorem presents a smaller and elegant upper bound.

Lemma 3.4. *Let $A = (a_{ij})_{n\times n}$ be a weakly diagonally dominant matrix, that is, $t_i = a_{ii} - \sum_{i\ne j}|a_{ij}| \ge 0$, $i = 1, 2, \ldots, n$. Then for an arbitrary $e = (e_F, e_C)^{\mathsf{T}}$ we have*

$$\|A_{FC}e_C\|^2 \le M(A_{CC}e_C, e_C), \tag{3.5}$$

in which $M = \max_{i\in F}\sum_{j\in C}|a_{ij}|$.

Proof. Let $s_i = \sum_{j=1}^n a_{ij} = t_i + 2\sum_{j\ne i}a_{ij}^+$ and

$$a_{ij}^- = \begin{cases} a_{ij}, & \text{if } a_{ij} < 0,\\ 0, & \text{if } a_{ij} > 0,\end{cases} \qquad a_{ij}^+ = \begin{cases} a_{ij}, & \text{if } a_{ij} < 0,\\ 0, & \text{if } a_{ij} > 0.\end{cases} \tag{3.6}$$

For any $e = (e_F, e_C)^{\mathsf{T}}$, we have

$$\begin{aligned}(Ae, e) &= \sum_{i,j}a_{ij}e_ie_j = \frac{1}{2}\sum_{i,j}(-a_{ij})(e_i - e_j)^2 + \sum_i s_ie_i^2\\ &= \frac{1}{2}\sum_{i,j}|a_{ij}^-|(e_i - e_j)^2 - \frac{1}{2}\sum_{i,j}a_{ij}^+(e_i - e_j)^2 + \sum_i s_ie_i^2\\ &= \frac{1}{2}\sum_{i,j}|a_{ij}^-|(e_i - e_j)^2 + \sum_i\sum_{j\ne i}a_{ij}^+(2e_i - \frac{1}{2}(e_i - e_j)^2) + \sum_i t_ie_i^2\\ &\ge \frac{1}{2}\sum_{i,j}|a_{ij}^-|(e_i - e_j)^2 + \frac{1}{2}\sum_i\sum_{j\ne i}a_{ij}^+(2e_i + 2e_j^2 - (e_i - e_j)^2) + \sum_i t_ie_i^2\\ &= \frac{1}{2}\sum_i\left(\sum_{j\ne i}|a_{ij}^-|(e_i - e_j)^2 + \sum_{j\ne i}a_{ij}^+(e_i + e_j)^2\right) + \sum_i t_ie_i^2\\ &\ge \sum_{i\in F}\left(\sum_{j\in C}|a_{ij}^-|(e_i - e_j)^2 + \sum_{j\in C}a_{ij}^+(e_i + e_j)^2\right) + \sum_{i\in F}t_ie_i^2.\end{aligned}$$

Let $e_F = 0$ in the above equation. Then we have

$$(A_{CC}e_C,\ e_C) = (Ae,\ e) \geq \sum_{i\in F}\left(\sum_{j\in C}|a_{ij}^-|e_j^2 + \sum_{j\in C}a_{ij}^+e_j^2\right) = \sum_{i\in F}\sum_{j\in C}|a_{ij}|e_j^2. \tag{3.7}$$

On the other hand, employing Schwarz's inequality, we can estimate

$$\|A_{FC}e_C\|^2 = \sum_{i\in F}\left(\sum_{j\in C}a_{ij}e_j\right)^2 \leq \sum_{i\in F}\left(\sum_{j\in C}|a_{ij}^-|e_j^2 + \sum_{j\in C}a_{ij}^+e_j^2\right) \leq M(A_{CC}e_C, e_C),$$

where $M = \max_{i\in F}\sum_{j\in C}|a_{ij}|$. □

Then we present the other main result.

Theorem 3.2. *Let A^m be symmetric positive definite and weakly diagonally dominant matrices. Suppose that there are constants M_1 and M_2 such that $(A^m e^m, e^m) \leq M_1(e^m, e^m)$ and $(A_{FF}^m e_F^m, e_F^m) \geq M_2(e_F^m, e_F^m)$ for $e^m = (e_F^m, e_C^m)^{\mathsf{T}}$. If the approximate inverse $(\hat{A}_{FF}^m)^{-1}$ satisfies*

$$\|I - (\hat{A}_{FF}^m)^{-1}A_{FF}^m\|_{Fr} \leq \mu,$$

in which μ is a constant, then the inequality (3.5) holds for the interpolation and $\beta_3 = \frac{1}{M_2}(\sqrt{M_1\Theta} + \mu\Theta\sqrt{\Phi})$, where $\Phi = \rho(A_{CC}^m)\rho((A^m)^{-1})$, $\Theta = \rho(\mathrm{diag}(A^m))$.

Proof. For any $e^m = (e_F^m, e_C^m)^{\mathsf{T}}$, we have

$$\begin{aligned}\|e^m - I_{m+1}^m e^{m+1}\|_0 &= \|e_F^m + (\tilde{A}_{FF}^m)^{-1}A_{FC}^m e_C^m\|_{0,F}\\ &\leq \|e_F^m + (A_{FF}^m)^{-1}A_{FC}^m e_C^m\|_{0,F} + \|(A_{FF}^m)^{-1}A_{FC}^m e_C^m - (\tilde{A}_{FF}^m)^{-1}A_{FC}^m e_C^m\|_{0,F}.\end{aligned} \tag{3.8}$$

By the hypotheses and Lemma 3.4, we can obtain

$$\begin{aligned}\|e_F^m + (A_{FF}^m)^{-1}A_{FC}^m e_C^m\|_{0,F}^2 &= (D_{FF}^m(e_F^m + (A_{FF}^m)^{-1}A_{FC}^m e_C^m), (e_F^m + (A_{FF}^m)^{-1}A_{FC}^m e_C^m))\\ &\leq \Theta((A_{FF}^m)^{-1}(A_{FF}^m e_F^m + A_{FC}^m e_C^m), (A_{FF}^m)^{-1}(A_{FF}^m e_F^m + A_{FC}^m e_C^m))\\ &\leq \frac{\Theta}{M_2^2}\|Ae\|^2 \leq \frac{\Theta M_1}{M_2^2}\|e\|_1^2;\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\|(A_{FF}^m)^{-1}A_{FC}^m e_C^m - (\tilde{A}_{FF}^m)^{-1}A_{FC}^m e_C^m\|_{0,F}^2 &= (D_{FF}^m(((A_{FF}^m)^{-1} - (\tilde{A}_{FF}^m)^{-1})A_{FC}^m e_C^m), ((A_{FF}^m)^{-1}\\ &\quad - (\tilde{A}_{FF}^m)^{-1})A_{FC}^m e_C^m)\\ &\leq \Theta\|(((A_{FF}^m)^{-1} - (\tilde{A}_{FF}^m)^{-1})A_{FC}^m e_C^m)\|^2\\ &\leq \Theta\|I_F - (\tilde{A}_{FF}^m)^{-1}A_{FF}^m\|_{Fr}^2\|(A_{FF}^m)^{-1}A_{FC}^m e_C^m\|^2\\ &\leq \Theta\mu^2\|(A_{FF}^m)^{-1}A_{FC}^m e_C^m\|^2 \leq \frac{\Theta\mu^2}{M_2^2}\|A_{FC}^m e_C^m\|^2\\ &\leq \frac{\Theta^2\mu^2}{M_2^2}(A_{CC}^m e_C^m, e_C^m) \leq \frac{\Phi\Theta^2\mu^2}{M_2^2}(A^m e^m, e^m)\\ &= \frac{\Phi\Theta^2\mu^2}{M_2^2}\|e^m\|_1^2.\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10), we arrive at

$$\|e^m - I_{m+1}^m e^{m+1}\|_0^2 \leq \frac{1}{M_2^2}\left(\sqrt{\Theta M_1} + \Theta\mu\sqrt{\Phi}\right)^2\|e^m\|_1^2,$$

and this completes the proof. □

Remark. This theorem gives the AMG convergence from the approximate inverse angle, which will help for constructing more practical interpolation operators.

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