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Optimal Multigrid Methods with New Transfer Operators Based on Finite Difference Approximations

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Abstract We constructed new interpolation operator in multigrid methods, which is efficient to transfer residual error from coarse grid to fine grid. This operator used idea of solving local residual equation using the standard stencil and the skewed stencil of the centered difference approximation to the Laplacian operator. We also compared our new multigrid methods with traditional multigrid methods, and found that new method is optimal.

Keywords Multigrid · Interpolation · Finite difference

Mathematics Subject Classification (2000) 65N55 · 65N30 · 65N06

Multigrid methods is efficient to solve algebraic systems arising from the approximation to partial differential equations. But traditional multigrid methods seems inefficient when it is used to solve PDE such as elliptic equation with strongly discontinuous coefficients. There are several methods which can make up disadvantage of traditional multigrid methods. Alcouffe et al. [1], Dendy [6] derived Black-Box multigrid methods, in which the interpolation is defined solely from the nine-point stencils. The Black-Box multigrid approach exploits the underlying PDE, but is restricted to topological square grid problems. Crumpton et al. [5] given a new conservative cell centred finite volume scheme for the accurate solution of elliptic diffusion equations with strongly varying coefficients. Wan [7] and Briggs [3] compared kinds of elements of multigrid methods, analyzed how several cooperation influence convergence rate of multigrid methods. Good choices of interpolation and finding optimal cooperation in multigrid methods is significant. T.F. Chan surveyed the development of multigrid methods in his paper [4]. Xu [8–11], and J. Bramble [2] provided basic theory of multigrid methods.

In this paper, we constructed new interpolation operator in multigrid methods which is efficient to transfer residual error from coarse grid to fine grid. This operator used idea of

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solving local residual equation. We also compared our multigrid methods with traditional multigrid methods. Numerical experiments reflect that new method is optimal.

The remainder of the paper is organized as follows. In Sect. 1, we introduce convergence theory of multigrid method and elements which influence convergence rate. In Sect. 2, we give new interpolation stencil. And Sect. 3 is numerical results. The last section is conclusion.

1 What Influence Convergence of Multigrid Method

First, we introduce the standard V-cycle multigrid algorithm and some notations will be used in following theorems.

After the discretization of PDE, we always obtain a linear system

$$Au = F. \tag{1.1}$$

The basic iterative method to solve (1.1) is

$$u_l \leftarrow u_l + B_l(f_l - A_l u_l).$$

The operator $B_l : V_l \to V_l$ are recursively defined as follows:

Algorithm 1 (V-cycle) Let $B_0 = A_0^{-1}$, for $l > 0$, and $g \in V_l$, define $B_l g = \omega_3$.

(1) Presmoothing: $\omega_1 = R_l g$;

(2) Correction: $\omega_2 = \omega_1 + B_{l-1} Q_{l-1}(g - A_l \omega_1)$;

(3) Postsmoothing: $\omega_3 = \omega_2 + R_l^*(g - A_l \omega_2)$.

Here function space $V = \sum_{i=1}^{J} V_i$ and R_l is subspace solvers. $A_l : V_l \to V_l$ is a restriction operator of A on V_l.

Theorem 1.1 (Theorem 7.9 in [10])

(1) $(R_k v, v) \le \omega_1 (A_k^{-1} v, v), \forall v \in V_k$

(2) $\frac{C_0}{\lambda_k}(v, v) \le (R_k v, v) \le \frac{C_1}{\lambda_k}(v, v), \forall v \in V_k$

Assume that the smoothers R_k satisfy (1) *and* (2) *with* $\omega_1 < 2$, *then the backslash cycle* (*Algorithm* 7.11 *in* [10]) *satisfies*

$$\| I - B_J A_J \|_A^2 \le 1 - \frac{2 - \omega_1}{C}$$

for some positive constant C independent of mesh parameters.

Theorem 1.2 (Theorem 7.10 in [10]) *If the full elliptic regularity is valid, then for the iteration* (7.12) (*in* [10]) *with B_J given by Algorithm* 7.11 *with one smoothing on each level*,

$$\| I - B_J A_J \|_A^2 \le 1 - \frac{C_0}{C_1}$$

Theorem 1.3 *Let M_{mg} be the iteration matrix given by the V-cycle multirid* (*Algorithm* 1), *Then*

$$\| M_{mg} \|_A^2 \le 1 - \frac{2 - \omega_1}{K_0(1 + K_1)^2}$$

where $\omega_1 = \max_{1\le i\le J} \rho(R_i A_i)$.

$$K_0 = \sup_{\|v\|_A=1} \inf_{V=\sum V_i} \sum_i (R_i^{-1} v_i, v_i),$$

and K_1 *satisfies* ($T_i = R_i Q_i A$, Q_i *is defined in following remark*)

$$\sum_{i>j} (T_i u_i, T_j v_j)_A \le \omega_1 (K_1 - 1) \left(\sum_{i=0}^{J} (T_i u_i, u_i)_A \right)^{1/2} \left(\sum_{j=0}^{J} (T_j v_j, v_j)_A \right)^{1/2}$$

Proof See [9]. □

Theorem 1.4 $K_1 \le \omega_1 J$, $K_0 \le \frac{C_0}{\omega_0}$ *where* $\omega_0 = \min_{1\le k\le J} \lambda_{min}(R_k^{-1} A_k)$.

Proof See [9]. □

Remark (1) By Theorem 1.3, the convergence rate can be improved by producing a small K_0 or K_1.

(2) By Theorem 1.4, for V-cycle multigrid, small ω_1 give a small K_1, and small C_0 give a small K_0, where C_0 satisfies:

$$\|Q_1 v\|_A^2 + \sum_{k=2}^{J} \|(Q_k - Q_{k-1}) v\|_A^2 \le C_0 \|v\|_A^2 \tag{1.2}$$

$$\|(Q_k - Q_{k-1}) v\| \le C_0 h_{k-1} \|Q_k v\|_A, \quad k > 1 \tag{1.3}$$

$Q_k : V \longrightarrow V_k$ is the L^2 projection.

(3) How to let K_1 be small:

$\omega_1 = \max_{1\le i\le J} \rho(R_i A_i)$, R_i is a solver of linear system in subspace. So, choosing a good solver R_i can finish this work. Gauss-Seidel iteration is a good solver in multigrid algorithm.

(4) How to let K_0 be small:

The inequality (1.2) plays an essential role in the convergence analysis of multigrid methods. It requires that for any given $v \in V$. We must be able to decompose it into $v_k \in V_k$ such that the total energy of all the pieces v_k is bounded by a small constant factor of original energy of v. In the multigrid context, it can be translated into the following: the coarse grid basis functions must have small energy.

The inequality (1.3) requires that the functions on the coarse grids approximate the fine grid functions to at least first-order accuracy. A sufficient condition is that the coarse subspace contains constant functions.

So, good interpolation operator can satisfy (1.2), (1.3). Next section, we will provide such an operator. They are new stencils, and are first try in multigrid methods.

2 Explain of New Interpolation Operator

We explain new interpolation operator through a model problem.

For model problem:

$$-\Delta u + u = f \quad \text{in } (0,1) \times (0,1)$$

$$u = 0 \quad \text{on } \partial\Omega$$

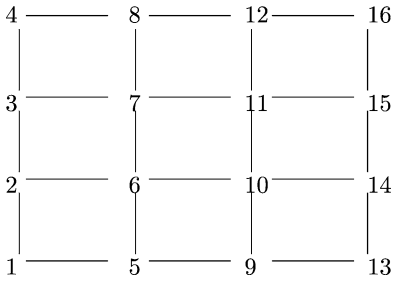

Fig. 1 6, 8, 14, 16 are points in coarse grid

Discrete this equation using finite difference method (the standard stencil)

$$\frac{-u_{i+1,j}+2u_{i,j}-u_{i-1,j}}{h^2}+\frac{-u_{i,j+1}+2u_{i,j}-u_{i,j-1}}{h^2}+u_{i,j}=f_{i,j}. \tag{2.1}$$

we obtain linear system (1.1). Of course, we also have the skewed stencil:

$$\frac{-u_{i+1,j+1}+2u_{i,j}-u_{i-1,j-1}}{2h^2}+\frac{-u_{i-1,j+1}+2u_{i,j}-u_{i+1,j-1}}{2h^2}+u_{i,j}=f_{i,j}. \tag{2.2}$$

If U is solution of (1.1), then we can compute residual $r=f-AU$. Then, we can give new interpolation stencils:

Figure 1 is a nested grid which has two levels: fine grid and coarse grid. Where 6, 8, 14, 16 are points in coarse grid, and others are new added points in fine grid. Let h be meshsize of fine grid and k, $k-1$ be levels of fine grid and coarse grid respectively.

Define FULL-LOCAL interpolation as following:

$$I_{k-1}^{k}(u^{k-1})_{11}=\frac{1}{4+2h^2}(u_6^{k-1}+u_8^{k-1}+u_{14}^{k-1}+u_{16}^{k-1}+2h^2r_{11}^{k}) \tag{2.3}$$

$$I_{k-1}^{k}(u^{k-1})_{7}=\frac{1}{4+h^2}(u_3^{k}+u_6^{k-1}+u_8^{k-1}+u_{11}^{k}+h^2r_{7}^{k}) \tag{2.4}$$

where r_{11}^{k} indicates residual at point 11 on k level, and r_7^k is residual at point 7 on k level.

Indeed, our stencils (2.3), (2.4) are finite difference method to solve $-\Delta u+u=r$ on k level. Equation (2.3) (parallel with (2.2)) used values of u^{k-1} at points 6, 8, 14, 16 to compute value of u^k at point 11. Then (2.4) (parallel with (2.1)) used values of u^{k-1} at points 6, 8 and values of u^k at points 3, 11 (are computed by (2.3)) to compute u^k at point 7. Finally, other values of u^k can be computed analogously.

Remark Equations (2.3) and (2.4) are new interpolation for model problem. We can use them as transfer operator in multigrid method. Difference from traditional multigrid method is that we used new interpolation operator instead of bilinear interpolation. Of course, we still choose restriction and smoothing like in traditional multigrid. For example, restriction operators can be INJECTION, FULL-WEIGHTING, and HALF-WEIGHTING, smoothing operator can be GAUSS-SEIDEL, WEIGHTED JACOBI, and RED-BLACK GAUSS-SEIDEL.

In next section, we will use new interpolation method, only change their stencils according to underlying problems.

3 Numerical Examples

In this section, we study implementation of new interpolation methods in several numerical examples. These examples contain Poisson equation, elliptic equation with discontinuous coefficients. For each problem, we will give stencils of new interpolation (we call them as FULL-LOCAL interpolation).

In each experiment, we use RED-BLACK GAUSS-SEIDEL, HALF-WEIGHTING operator as smoothing and restriction respectively. When compare convergence factor, let stop tolerance be $\frac{\|r_k\|}{\|r_0\|} < 10^{-16}$ and the max number of iteration be 12. In first example, we also compare the number of required iterations when increasing the matrix dimension (here we let tolerance $\frac{\|r_k\|}{\|r_0\|} < 10^{-13}$).

Here the convergence factor of the k-th iteration is $(\frac{\|r_k\|}{\|r_0\|})^{1/k}$. Small value of convergence factor give a high convergence rate.

Example 1 Poisson equation

$$-\Delta u = f \quad \text{in } (0,1) \times (0,1)$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Let $f = 2((1-6x^2)y^2(1-y^2) + (1-6y^2)x^2(1-x^2))$, and $u = (x^2 - x^4)(y^4 - y^2)$ is solution of this problem.

The FULL-LOCAL interpolation for this problem is as following:

$$I_{k-1}^k(u)_{11} = \frac{1}{4}(u_6^{k-1} + u_8^{k-1} + u_{14}^{k-1} + u_{16}^{k-1} + 2h^2 \times r_{k11}) \tag{3.1}$$

$$I_{k-1}^k(u)_7 = \frac{1}{4}(u_6^{k-1} + u_8^{k-1} + u_{11}^{k} + u_3^{k} + h^2 \times r_{k7}). \tag{3.2}$$

Here r_{k11} is residual at point 11 on k level. r_{k7} is residual at point 7 on k level.

Table 1 gives average convergence factor of two methods on levels: 64×64, 128×128, 256×256.

Table 2 gives L^2 norm of residual after each V-cycle and convergence factor of each V-cycle on level 256×256.

From Table 1, we can observe that new method has high convergence rate. For example, when use $V(1,1)$ cycle we have high convergence rate 0.05, but it is 0.08 when use bilinear interpolation.

Table 1 Average convergence factor

R-B GAUSS	LEVEL	BILINEAR	FULL-LOCAL
$V(1,0)$	64×64	0.69	0.41
	128×128	0.71	0.38
	256×256	0.71	0.37
$V(1,1)$	64×64	0.08	0.06
	128×128	0.08	0.05
	256×256	0.08	0.05

Table 2 Residual vs. number of iteration

	L^2 norm of residual $\|r\|$		Convergence factor	
	BILINEAR	FULL-LOCAL	BILINEAR	FULL-LOCAL
1	7.97E−05	0.00017	0.01	0.04
2	6.77E−07	1.10E−05	0.03	0.05
3	7.94E−07	6.39E−07	0.05	0.05
4	1.05E−07	2.87E−08	0.07	0.05
5	1.41E−08	1.07E−09	0.08	0.04
6	1.97E−09	5.06E−11	0.08	0.04
7	2.80E−10	3.21E−12	0.09	0.04
8	4.05E−11	2.05E−13	0.09	0.05
9	5.93E−12	1.26E−14	0.10	0.05
10	8.76E−13	2.20E−15	0.10	0.05
11	1.30E−13	–	0.11	0.07
12	1.97E−14	–	0.11	0.09

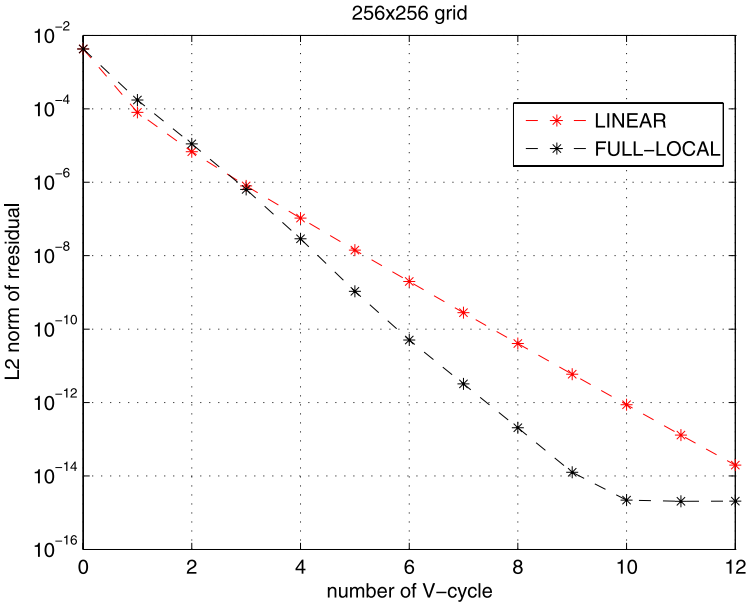

Fig. 2 Residual vs. number of iteration

From Table 2, the advantages of FULL-LOCAL interpolation is more clear. For FULL-LOCAL method, 10 V-cycles can reduce $\|r\|_2$ by 2.20E−15, and more iterations are unnecessary. But when use bilinear interpolation, 12 V-cycles only reduce $\|r\|_2$ by 1.97E−14. This can be showed in Fig. 2. Where horizontal axis indicates number of iteration and longitudinal axis indicates L^2 norm of residual.

Table 3 Required iterations when increasing the matrix dimension

LEVEL	BILINEAR	FULL-LOCAL
64×64	15	11
128×128	>50	36
256×256	>50	>50

Table 4 Convergence factor vs. discontinuous coefficient

a in $(0.4, 0.6)^2$	LINEAR	FULL-LOCAL
10	0.67	0.56

In Table 3, we compare number of iterations of two methods when tolerance is 10^{-13}. Clearly, new multigrid method needs few iterations than traditional method. But iterations will increase when matrix dimension increase for both methods.

Example 2 elliptic equation with discontinuous coefficient

$$-\nabla(a\nabla u) = f \quad \text{in } (0,1)\times(0,1)$$
$$u = 0 \quad \text{on } \partial\Omega$$

Let $f = 1$, and

$$a = \begin{cases} 1 & (0,1)\times(0,1)\setminus(0.4,0.6); \\ 10 & (0.4,0.6)\times(0.4,0.6) \end{cases}$$

is a discontinuous coefficient of ∇u in this problem.

The FULL-LOCAL interpolation for this problem is as following:

If we remark point 11 is (k, j), then a_{11} indicates $a(k, j)$, $a_{1/2,1/2}$ indicates $a(k+1/2, j+1/2)$, and $a_{0,1/2}$ indicates $a(k, j+1/2)$.

$$I_{k-1}^{k}(u)_{11} = (a_{-1/2,-1/2}u_6^{k-1} + a_{-1/2,1/2}u_8^{k-1} + a_{1/2,-1/2}u_{14}^{k-1} + a_{1/2,1/2}u_{16}^{k-1} + 2h^2 \times r_{k11}) \times \frac{1}{a_{1/2,1/2} + a_{1/2,-1/2} + a_{-1/2,1/2} + a_{-1/2,-1/2}} \tag{3.3}$$

$$I_{k-1}^{k}(u)_{7} = (a_{0,-1/2}u_6^{k-1} + a_{0,1/2}u_8^{k-1} + a_{1/2,0}u_{11}^{k} + a_{-1/2,0}u_3^{k} + h^2 \times r_{k11}) \times \frac{1}{a_{0,1/2} + a_{0,-1/2} + a_{-1/2,0} + a_{-1/2,0}} \tag{3.4}$$

Here r_{k11} is residual at point 11 on k level. r_{k7} is residual at point 7 on k level.

Table 4 shows average convergence factor of two methods.

From Table 4, we can observe that even for elliptic problem with discontinuous coefficient, FULL-LOCAL interpolation is still efficient. When use $V(1, 1)$ cycle, convergence factor of new method is 0.56, but it is 0.67 for traditional multigrid method.

Figure 3 shows relationship between L^2 norm of residual and number of iteration. It is similar with Example 1.

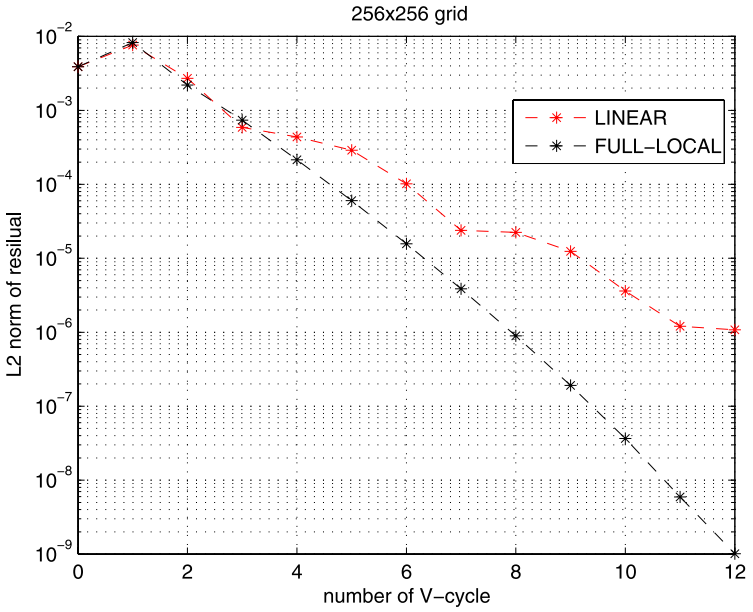

Fig. 3 Residual vs. number of iteration

4 Conclusion and Future Works

In this paper, we constructed new interpolation operator (we call FULL-LOCAL) in multigrid method. New multigrid method with FULL-LOCAL interpolation is optimal. But we still need to research this new operator about its theory and application to more difficult problems.

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