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Analytic approach to determine optimal conditions for maximizing altitude of sounding rocket

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ABSTRACT

An analytic approach to determine the optimal conditions for maximizing altitude of a sounding rocket at burn-out state or stationary state is suggested. The one-dimensional rocket momentum equation including thrust, gravitational force, and aerodynamic drag is solved. The typical case that has an analytic solution of the rocket equation is considered. Also, numerical calculations are conducted for comparisons. To build a standard approach to determine the optimal conditions, flights in a constant atmosphere where the air density is constant are considered. The analytic solutions agree well with the numerical ones. It is shown that the optimal condition for maximizing altitude at burn-out state or stationary state can be predicted with a characteristic equation.

1. Introduction

Sounding rockets carry scientific instruments to altitudes from 50 km to more than 300 km to explore the atmosphere at high altitude or near outer space. Scientific studies with a sounding rocket are simple, fast, and inexpensive compared to those with a satellite. The costs for a sounding rocket mission are much lower than those required for an orbiter mission, since sounding rocket missions do not need expensive boosters, extended telemetry or tracking coverage. Mission costs are also low because of the acceptance of a higher degree of risk in the mission compared to orbital missions [13]. Many countries are running sounding rocket programs and trying to develop technologies related to sounding rockets to exploit these advantages [1,17,10,5,12,7,3,8,18,2,16]. Most scientific measurements, observations, or experiments for sounding rocket missions are carried out near apogee. The low speed near apogee provides favorable chances to explore or observe space in a short time period. Furthermore, there are some important regions of space that are too low to be sampled by satellites; thus, sounding rockets provide platforms to carry out in-situ measurements in these regions [8]. Some missions are carried out during free-fall that provides microgravity environments [15,21] or good hypersonic condition at low cost [14,19].

Sounding rockets usually have parabolic trajectories during the whole flight range in order to secure safety of launching site or citizens and to collect used rocket bodies or parts. In parabolic motions, there are no accelerations or body forces in the lateral direction, which makes it simple to analyze the lateral motion of a rocket. Therefore, in the present study, we consider vertical direction only, for simplicity. The motion of a rocket can then be described with a one-dimensional momentum equation that includes thrust, gravitational force, and aerodynamic drag force.

The rocket mass varies with time, and the drag force is proportional to the square of the rocket velocity, which makes the governing equation nonlinear. Thus, we cannot obtain analytic solutions in a general form. Hence in most cases, numerical approaches are used to obtain solutions of the rocket equation because of easy implementation with less assumptions. Analytic solutions, on the contrary to numerical ones, are exact without numerical errors, give insights to understand the behavior of the system, show the critical parameters, and lead to ways to determine the optimal conditions. Therefore it is necessary to find out analytic solutions if possible. An approximate analytic solution can be obtained with neglecting drag force but it contains serious errors especially near ground. There are also approximate solutions with the Taylor series expansion, the perturbation method or the least square method [9], but they are complex and unhandy. An analytic exact solution of the rocket equation including drag force exists only in a typical situation where the three forces are well balanced. In the present study, we consider the typical cases where analytic exact solutions exist.

The design target of a sounding rocket is the altitude. The ejection conditions of propellant jet determine thrust force, rocket velocity or boost time that eventually change the rocket altitude at burn-out state or at stationary state. Therefore, it is necessary to determine an optimal jet condition for maximizing the altitude at

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Nomenclature

g	gravitational acceleration
h	altitude
K	drag parameter
m	rocket mass
$\dot{m}$	rate of rocket mass change or mass flow rate of propellant jet
q	velocity parameter for rocket velocity
t	time
u_e	velocity of propellant jet
v	rocket vertical velocity
p	static pressure
ρ	density

Subscripts

b	burn-out state
o	ground state
opt	optimal condition for maximizing altitude
s	stationary state

given launching conditions. Despite the wide usage of sounding rockets, as far as we know, there are no analytic methods that are simple and convenient enough to determine the optimal condition for maximizing altitude. Hence, the objective of the present study is to suggest an analytic method to determine the optimal condition for maximizing altitude at burn-out state or stationary state.

As a beginning study, we first consider the flight in a constant atmosphere where the air density is constant, since variable air density makes it impossible to obtain analytic solutions valid through whole flight range. In the standard atmosphere, the air density diminishes and thus the aerodynamic drag decreases as the altitude increases, which means that assuming constant air density leads to considerable underestimations of the altitude. However, even though limited to the flight in a constant atmosphere, the approach in the present study will show a standard procedure to obtain analytic solutions and to determine optimal conditions. Also, useful bases would be built for developing methods to solve the governing equation for the fights in a real atmosphere.

2. One-dimensional rocket equation

2.1. Equation in boost phase

The motion of a rocket in boost phase climbing in the vertical direction can be described with the following one-dimensional rocket equation [20,6].

$$m\frac{dv}{dt} = F - D - mg. \tag{2.1}$$

The mass of a rocket decreases with the mass flow of propellant.

$$m = m_o + \int_o^t \dot{m}dt, \tag{2.2a}$$

$$\dot{m} = \frac{dm}{dt}. \tag{2.2b}$$

The mass flow rate $\dot{m}$ is equal to the rate of the rocket mass and has a negative sign by definition.

The terms F and D stand for thrust force and drag force, respectively. The rocket thrust is composed of two parts:

$$F = \dot{m}u_e + A_e(p_e - p_a). \tag{2.3}$$

The term A_e is the area at the exit plane of a rocket nozzle and the term p_a is the ambient air pressure. For an adiabatic nozzle flow, the total enthalpy is constant, and then we can assume that the jet velocity u_e is constant. The jet velocity has the negative sign since its direction is opposite to the rocket velocity; thus, the thrust term $\dot{m}u_e$ has the positive sign. If the nozzle flow has a perfect expansion, the second term of the thrust vanishes. Hereafter, we ignore the second term of the thrust for simplicity.

The aerodynamic drag force D can be represented with the drag parameter K as follows:

$$D = Kv^2, \tag{2.4a}$$

$$K = \frac{S}{2}C_d\rho_a. \tag{2.4b}$$

The terms S, C_d and ρ_a denote cross-section area of a rocket, aerodynamic drag coefficient and ambient air density, respectively. In the present study, we consider the aerodynamic drag coefficient as a constant for simplicity. Then, for a constant atmosphere where the ambient air density is constant, we can treat the drag parameter K as a constant.

Then the governing equation becomes

$$m\frac{dv}{dt} = \dot{m}u_e - Kv^2 - mg. \tag{2.5}$$

The mass is variable with time, and the square of the solution appears in the drag force, which makes the governing equation nonlinear. Thus, we cannot obtain analytic solutions in a general form. However, there is a typical case where an analytic solution exists. We introduce a velocity parameter as follows:

$$q = \sqrt{\frac{\dot{m}u_e - mg}{K}}. \tag{2.6}$$

The governing equation can then be reduced as

$$m\frac{dv}{dt} = K\left(q^2 - v^2\right). \tag{2.7a}$$

Separating variables leads to

$$\frac{dv}{q^2 - v^2} = \frac{K}{m}dt. \tag{2.7b}$$

This equation can be represented according to the mass instead of the time as follows:

$$\frac{dv}{q^2 - v^2} = \frac{K}{m}\frac{dm}{\dot{m}}. \tag{2.7c}$$

If the velocity parameter is constant, the left hand side can be analytically integrated, yielding an analytic solution of the governing equation if the term $K/m\dot{m}$ is analytically integrated.

2.2. Equation in coast phase

After the propellant is totally consumed, the flight phase turns into coast phase, where a rocket climbs with inert force until stationary state or apogee. The mass flow of propellant jet disappears in the equation and the rocket mass is kept constant. Also the velocity parameter shown in boost phase is irrelevant in coast phase. Then the rocket equation becomes

$$m_b\frac{dv}{dt} = -Kv^2 - m_b g. \tag{2.8}$$

Separating variables in the above equation yields

$$\frac{dv}{Kv^2+m_b g}=-\frac{1}{m_b}dt, \quad \text{or} \tag{2.9a}$$

$$\frac{v\,dv}{Kv^2+m_b g}=-\frac{1}{m_b}dh. \tag{2.9b}$$

In coast phase, the initial velocity is the velocity at burn-out state and the final velocity is zero at stationary state.

3. Analytic solutions

3.1. Solutions of the governing equation

3.1.1. Solutions in boost phase

Since, in the present study, the velocity parameter is constant, the mass flow rate is variable and can be expressed with the mass and the velocity parameter.

$$\dot{m}=\frac{mg+Kq^2}{u_e}. \tag{3.1.1a}$$

Inserting this relation to the governing equation leads to

$$\frac{dv}{q^2-v^2}=\frac{Ku_e}{mg+Kq^2}\frac{dm}{m}. \tag{3.1.1b}$$

Integrating the governing equation from the ground state yields

$$\frac{1}{2q}\ln\left(\frac{q+v}{q-v}\right)_o^v=\frac{u_e}{q^2}\ln\left(\frac{m}{mg+Kq^2}\right)_{m_o}^m. \tag{3.1.2a}$$

Rearranging this equation leads to

$$\ln\left(\frac{q+v}{q-v}\right)=\frac{2u_e}{q}\ln\left(\frac{m}{m_o}\frac{m_o g+Kq^2}{mg+Kq^2}\right). \tag{3.1.2b}$$

Hence, we can express the rocket velocity as a function of the mass as follows:

$$v=q\frac{x-1}{x+1}, \tag{3.1.3a}$$

$$x=\left(\frac{m}{m_o}\frac{m_o g+Kq^2}{mg+Kq^2}\right)^\sigma=\left(\frac{m}{m_o}\frac{G_o+q^2}{G+q^2}\right)^\sigma, \tag{3.1.3b}$$

$$G=\frac{mg}{K}, \tag{3.1.3c}$$

$$\sigma=\frac{2u_e}{q}. \tag{3.1.3d}$$

The rocket mass ratio is defined as the ratio between the masses at ground state and at burn-out state as follows:

$$\Omega=\frac{m_o}{m_b}>1. \tag{3.1.4}$$

The term x at ground state or at burn-out state becomes

$$x_o=\left(\frac{m_o}{m_o}\frac{G_o+q^2}{G_o+q^2}\right)^\sigma=1, \tag{3.1.5a}$$

$$x_b=\left(\frac{m_b}{m_o}\frac{G_o+q^2}{G_b+q^2}\right)^\sigma=\left(\frac{1}{\Omega}\frac{G_o+q^2}{G_b+q^2}\right)^\sigma. \tag{3.1.5b}$$

The time derivative of the mass flow rate expressed in Eq. (3.1.1a) becomes

$$\frac{d\dot{m}}{dt}=\frac{g}{u_e}\dot{m}. \tag{3.1.6a}$$

Separating variables leads to

$$\frac{d\dot{m}}{\dot{m}}=\frac{g}{u_e}dt. \tag{3.1.6b}$$

Integrating this equation and inserting the mass flow rate in Eq. (3.1.1a) yields

$$\ln\frac{G+q^2}{G_o+q^2}=\frac{g}{u_e}t. \tag{3.1.6c}$$

Then, we have

$$m=\left(m_o+\frac{Kq^2}{g}\right)\exp\left(\frac{g}{u_e}t\right)-\frac{Kq^2}{g}. \tag{3.1.6d}$$

The time can be represented as a function of the mass by rearranging the above equation.

$$t=\frac{u_e}{g}\ln\left(\frac{mg+Kq^2}{m_o g+Kq^2}\right)=\frac{u_e}{g}\ln\left(\frac{G+q^2}{G_o+q^2}\right). \tag{3.1.7a}$$

The burn-out time when the propellant is totally consumed can be determined as

$$t_b=\frac{u_e}{g}\ln\left(\frac{G_b+q^2}{G_o+q^2}\right). \tag{3.1.7b}$$

The altitude of a rocket at burn-out state can be obtained with the time integration of the velocity.

$$h_b=\int_o^{t_b}v\,dt=\int_{m_o}^{m_b}q\frac{x-1}{x+1}\frac{u_e}{mg+Kq^2}dm. \tag{3.1.8}$$

This integral cannot be analytically obtained, thus should be calculated with numerical integration. In the present study, numerical integration with the Simpson rule [4] is used to calculate altitude.

3.1.2. Solutions in coast phase

In coast phase, the governing equation can be analytically integrated. Eq. (2.9a) can be expressed as follows:

$$dt=-m_b\frac{dv}{m_b g+Kv^2}=-\frac{m_b}{K}\frac{dv}{G_b+v^2}. \tag{3.1.9a}$$

Integrating this equation from the burn-out state to the stationary state yields

$$t_{bs}=t_s-t_b=-\frac{m_b}{K\sqrt{G_b}}\tan^{-1}\left(\frac{v}{\sqrt{G_b}}\right)_{v_b}^o$$
$$=\frac{m_b}{K\sqrt{G_b}}\tan^{-1}\left(\frac{v_b}{\sqrt{G_b}}\right). \tag{3.1.9b}$$

The altitude can be analytically obtained with Eq. (2.9b) as follows:

$$dh=-m_b\frac{v\,dv}{m_b g+Kv^2}=-\frac{m_b}{K}\frac{v\,dv}{G_b+v^2}. \tag{3.1.10a}$$

Integrating this equation from the burn-out state to the stationary state yields

$$h_{bs}=h_s-h_b=-\frac{1}{2}\frac{m_b}{K}\ln\left(G_b+v^2\right)_{v_b}^o$$
$$=\frac{m_b}{2K}\ln\left(\frac{G_b+v_b^2}{G_b}\right). \tag{3.1.10b}$$

3.2. Optimal condition for maximum altitude at burn-out state

The rocket altitude changes according to the velocity parameter, since the rocket velocity also changes. Now, we find an analytic way to determine the optimal velocity parameter for maximizing altitude at burn-out state. The governing equation (2.7b) can be rewritten according to altitude instead of time as follows:

$$\frac{dv}{q^2 - v^2} = \frac{K}{m}\frac{dh}{v}. \tag{3.2.1a}$$

Separating variables and integrating the equation from ground state to burn-out state yields

$$\int_o^{v_b} \frac{v dv}{q^2 - v^2} = \int_o^{h_b} \frac{K}{m} dh. \tag{3.2.1b}$$

The left hand side of the above equation is reduced as

$$-\frac{1}{2}\ln\left(q^2 - v^2\right)_o^{v_b} = -\frac{1}{2}\ln\left(1 - \frac{v_b^2}{q^2}\right) = -\frac{1}{2}\ln\left[\frac{4x_b}{(x_b+1)^2}\right]. \tag{3.2.2a}$$

Differentiating this term with respect to the velocity parameter yields

$$\left(-\frac{1}{2x_b} + \frac{1}{x_b+1}\right)\frac{dx_b}{dq} = \frac{1}{2x_b}\frac{x_b-1}{x_b+1}\frac{dx_b}{dq}. \tag{3.2.2b}$$

Differentiating the right hand side in Eq. (3.2.1b) with respect to the velocity parameter leads to

$$\int_o^{h_b} \frac{d}{dq}\frac{K}{m} dh + \frac{K}{m_b}\frac{dh_b}{dq} - \frac{K}{m_o}\frac{dh_o}{dq}. \tag{3.2.3}$$

The Leibniz rule [11] is applied to the right hand side. The derivative of the altitude at ground state is zero. Also, for the maximum altitude, the derivative of the altitude at burn-out state should be zero. The mass at a given altitude would change with the velocity parameter and thus the derivative of the mass with respect to the velocity parameter could not vanish. Thus the following equation must be satisfied for the maximum altitude at the burn-out state.

$$\frac{1}{2x_b}\frac{x_b-1}{x_b+1}\frac{dx_b}{dq} = -\int_o^{h_b} \frac{K}{m^2}\frac{dm}{dq} dh. \tag{3.2.4}$$

The logarithm of the term x_b becomes

$$\ln(x_b) = \frac{2u_e}{q}\ln\left(\frac{1}{\Omega}\frac{G_o + q^2}{G_b + q^2}\right). \tag{3.2.5a}$$

Differentiating this equation with respect to the velocity parameter yields

$$\frac{1}{x_b}\frac{dx_b}{dq} = -\frac{2u_e}{q^2}\left[\frac{q}{2u_e}\ln(x_b)\right] + \frac{2u_e}{q}\left[\frac{2q}{G_o + q^2} - \frac{2q}{G_b + q^2}\right]. \tag{3.2.5b}$$

Thus, we have

$$\frac{1}{x_b}\frac{dx_b}{dq} = -\frac{1}{q}\left[\ln(x_b) - \frac{4qu_e G_b(1-\Omega)}{(G_o + q^2)(G_b + q^2)}\right]. \tag{3.2.5c}$$

The rocket mass can be expressed as a function of the time and the velocity parameter.

$$m = m(t, q). \tag{3.2.6a}$$

The derivative of the mass with respect to the velocity parameter becomes

$$\frac{dm}{dq} = \left.\frac{\partial m}{\partial t}\right)_q \frac{dt}{dq} + \left.\frac{\partial m}{\partial q}\right)_t. \tag{3.2.6b}$$

According to the expression of the mass in Eq. (3.1.6d), we have

$$\frac{\partial m}{\partial t} = \frac{g}{u_e}\left(m_o + \frac{Kq^2}{g}\right)\exp\left(\frac{g}{u_e}t\right) = \frac{m_o g + Kq^2}{u_e}\frac{mg + Kq^2}{m_o g + Kq^2} = \frac{mg + Kq^2}{u_e}, \tag{3.2.6c}$$

$$\frac{\partial m}{\partial q} = \frac{2Kq}{g}\left[\exp\left(\frac{g}{u_e}t\right) - 1\right] = \frac{2Kq}{g}\left[\frac{mg + Kq^2}{m_o g + Kq^2} - 1\right] = \frac{2q(m - m_o)}{G_o + q^2}. \tag{3.2.6d}$$

According to the expression of the time in Eq. (3.1.7a), the first term on the right hand side in Eq. (3.2.5b) becomes

$$\frac{\partial m}{\partial t}\frac{dt}{dq} = \frac{mg + Kq^2}{u_e}\frac{u_e}{g} \times \frac{(\frac{dm}{dq}g + 2Kq)(m_o g + Kq^2) - 2Kq(mg + Kq^2)}{(m_o g + Kq^2)(mg + Kq^2)}. \tag{3.2.6e}$$

Hence, Eq. (3.2.5b) is reduced as

$$\frac{dm}{dq} = \frac{dm}{dq} + \frac{2q(m_o - m)}{G_o + q^2} + \frac{2q(m - m_o)}{G_o + q^2}. \tag{3.2.6f}$$

This equation is trivial and does not give an explicit expression of the derivative of the mass with respect to the velocity parameter. However, it suggests that the derivative could be expressed in the following form.

$$\frac{dm}{dq} = \frac{2q(m - m_{cr})}{G_o + q^2}, \tag{3.2.7a}$$

$$m_{cr} = \beta m_b. \tag{3.2.7b}$$

If β is Ω, then the critical mass m_{cr} becomes m_o. The coefficient β would change with the rocket mass or rocket mass ratio.

Hence, the characteristic equation to determine the optimal condition for maximizing altitude at burn-out state becomes

$$\ln(x_b) - \frac{4qu_e G_b(1-\Omega)}{(G_o + q^2)(G_b + q^2)} = S_b, \tag{3.2.8a}$$

$$S_b = \frac{x_b + 1}{x_b - 1}\frac{4Kq^2}{G_o + q^2}\int_o^{h_b} \frac{m - \beta m_b}{m^2} dh. \tag{3.2.8b}$$

It is necessary to calculate the integral in the above equation by numerical integration since it could not be analytically integrated.

It is impossible to obtain explicit analytic solution of the above characteristic equation, since the term x_b is a function of the velocity parameter. Thus, it is inevitable to calculate with iterations. The above equation can be reduced to the fourth order one as follows:

$$\left(\frac{q^2}{G_b}\right)^2 + B\left(\frac{q^2}{G_b}\right) + \Omega = 0, \tag{3.2.9a}$$

$$B = \Omega + 1 - \frac{4u_e(1-\Omega)}{[\ln(x_b) - S_b]q}. \tag{3.2.9b}$$

Then, we can obtain a converged solution of the characteristic equation with iterations:

$$q_{b,k+1} = \sqrt{G_b \frac{\sqrt{B_k^2 - 4\Omega} - B_k}{2}}. \tag{3.2.9c}$$

However, if the coefficient B is not negative, it is impossible to obtain a solution since the term inside the square root becomes negative, which means we cannot start iteration with a wild guess. Hence, to avoid such serious situations, it is necessary to change the order of the equation and build a reduced order equation in the following form.

$$q^3\left[\ln(x_b) - S_b\right] - \frac{4u_e G_b(1-\Omega)}{(\frac{G_o}{q^2}+1)(\frac{G_b}{q^2}+1)} = 0, \quad \text{or} \tag{3.2.10a}$$

$$q\left[\ln(x_b) - S_b\right] - \frac{4u_e G_b(1-\Omega)}{(\frac{G_o}{q}+q)(\frac{G_b}{q}+q)} = 0. \tag{3.2.10b}$$

The iterative procedure to solve the above characteristic equations can be represented as

$$q_{k+1} = \sqrt[3]{\frac{4u_e G_b(1-\Omega)}{[\ln(x_{b,k}) - S_{b,k}](\frac{G_o}{q_k^2}+1)(\frac{G_b}{q_k^2}+1)}}, \quad \text{or} \tag{3.2.11a}$$

$$q_{k+1} = \frac{4u_e G_b(1-\Omega)}{[\ln(x_{b,k}) - S_{b,k}](\frac{G_o}{q_k}+q_k)(\frac{G_b}{q_k}+q_k)}. \tag{3.2.11b}$$

The term S_b in the above equation decreases and so does the estimated velocity parameter q_{k+1} as the coefficient β increases. Therefore, we can determine the coefficient β with iteration as follows:

$$\beta_{k+1} = \beta_k\left[1 - C_\beta\left(\frac{q}{h_b}\frac{\partial h_b}{\partial q}\right)_k\right], \tag{3.2.12a}$$

$$\beta_0 = \frac{2+\Omega}{3}, \tag{3.2.12b}$$

$$\frac{\partial h_b}{\partial q} = \frac{u_e}{K}\int_{m_o}^{m_b}\left\{\frac{G-q^2}{G+q^2} - \frac{2x}{x^2-1}\left[\ln(x) - \frac{4qu_e(G-G_o)}{(G_o+q^2)(G+q^2)}\right]\right\} \times \frac{x-1}{x+1}\frac{dm}{G+q^2}. \tag{3.2.12c}$$

It is necessary to calculate the integral in Eq. (3.2.12c) by numerical integration. The constant C_β of 1.0 guarantees fast and stable convergence for all cases considered in the present study. In case this coefficient is higher than 2.0, the iterative sequence could be unstable.

Preliminary numerical experiments show that the first order approximation (3.2.11b) converges faster than the third order approximation (3.2.11a) and gives exactly same results. These equations are stable and, thus, converge within several iterations.

3.3. Optimal condition for maximum altitude at stationary state

The rocket in coast phase ascends until stationary state or apogee where the rocket velocity is zero. Hence, the optimal condition for maximizing altitude at stationary state would be different from that at burn-out state. In coast phase, there is no thrust force and thus the governing equation can be expressed as

$$-\frac{dv}{v^2+G_b} = \frac{K}{m_b}\frac{dh}{v}. \tag{3.3.1a}$$

Separating variables yields

$$-\frac{vdv}{v^2+G_b} = \frac{K}{m_b}dh. \tag{3.3.1b}$$

Integrating the governing equation from ground state to stationary state yields

$$\int_o^{v_b}\frac{vdv}{q^2-v^2} - \int_{v_b}^{o}\frac{vdv}{v^2+G_b} = \int_o^{h_b}\frac{K}{m}dh + \int_{h_b}^{h_s}\frac{K}{m_b}dh. \tag{3.3.2a}$$

The second term on the left hand side becomes

$$-\frac{1}{2}\ln\left(v^2+G_b\right)_{v_b}^{o} = \frac{1}{2}\ln\left(\frac{G_b+v_b^2}{G_b}\right). \tag{3.3.2b}$$

Differentiating the above term with respect to the velocity parameter yields

$$\frac{v_b}{G_b+v_b^2}\frac{d}{dq}\left(q\frac{x_b-1}{x_b+1}\right) = \frac{x_b-1}{x_b+1}\frac{q}{G_b+v_b^2}\left[\frac{x_b-1}{x_b+1} + \frac{2q}{(x_b+1)^2}\frac{dx_b}{dq}\right]. \tag{3.3.2c}$$

Differentiating the right hand side of Eq. (3.3.2a) with respect to the velocity parameter leads to

$$\int_o^{h_b}\frac{d}{dq}\frac{K}{m}dh + \frac{K}{m_b}\frac{dh_b}{dq} - \frac{K}{m_o}\frac{dh_o}{dq} + \int_{h_b}^{h_s}\frac{d}{dq}\frac{K}{m_b}dh + \frac{K}{m_b}\frac{dh_s}{dq} - \frac{K}{m_b}\frac{dh_b}{dq}. \tag{3.3.3}$$

The second and the last terms cancel out. The rocket mass after burn-out state is constant and thus its derivative is zero. The derivative of altitude at ground state is zero, and, for the maximum altitude, the derivative of altitude at stationary state should be zero. But, as mentioned in the above section, the first term remains. Hence, the characteristic equation to determine the optimal condition for maximizing altitude at stationary state becomes

$$\frac{1}{2x_b}\frac{x_b-1}{x_b+1}\frac{dx_b}{dq} + \frac{x_b-1}{x_b+1}\frac{q}{G_b+v_b^2}\left[\frac{x_b-1}{x_b+1} + \frac{2q}{(x_b+1)^2}\frac{dx_b}{dq}\right] = \int_o^{h_b}\frac{d}{dq}\frac{K}{m}dh. \tag{3.3.4a}$$

Inserting the derivative of x_b with respect to velocity parameter described in Eq. (3.2.5c) yields

$$\frac{G_b+q^2}{2q}\left[\ln(x_b) - \frac{4qu_eG_b(1-\Omega)}{(G_o+q^2)(G_b+q^2)}\right] = q\frac{x_b-1}{x_b+1} - \left(G_b+v_b^2\right)\frac{x_b+1}{x_b-1}\int_o^{h_b}\frac{d}{dq}\frac{K}{m}dh. \tag{3.3.4b}$$

Rearranging this equation with inserting the derivative of the mass with respect to the velocity parameter described in Eq. (3.2.7) yields

$$\ln(x_b) - \frac{4qu_eG_b(1-\Omega)}{(G_o+q^2)(G_b+q^2)} = S_s, \tag{3.3.5a}$$

$$S_s = \frac{2q^2}{G_b+q^2}\frac{x_b-1}{x_b+1} + \frac{G_b+v_b^2}{G_b+q^2}\frac{x_b+1}{x_b-1}\frac{4Kq^2}{G_o+q^2}\int_o^{h_b}\frac{m-\beta m_b}{m^2}dh. \tag{3.3.5b}$$

This characteristic equation can be reduced as the fourth order one. However, with the same reason as mention in the above section, we consider reduced order approximations as follows:

$$q^3\big[\ln(x_b) - S_s\big] - \frac{4u_e G_b(1-\Omega)}{(\frac{G_o}{q^2}+1)(\frac{G_b}{q^2}+1)} = 0, \quad \text{or} \tag{3.3.6a}$$

$$q\big[\ln(x_b) - S_s\big] - \frac{4u_e G_b(1-\Omega)}{(\frac{G_o}{q}+q)(\frac{G_b}{q}+q)} = 0. \tag{3.3.6b}$$

The iterative procedure to solve the above characteristic equations can be represented as

$$q_{k+1} = \sqrt[3]{\frac{4u_e G_b(1-\Omega)}{[\ln(x_{b,k}) - S_{s,k}](\frac{G_o}{q_k^2}+1)(\frac{G_b}{q_k^2}+1)}}, \quad \text{or} \tag{3.3.7a}$$

$$q_{k+1} = \frac{4u_e G_b(1-\Omega)}{[\ln(x_{b,k}) - S_{s,k}](\frac{G_o}{q_k}+q_k)(\frac{G_b}{q_k}+q_k)}. \tag{3.3.7b}$$

The term S_s in the above equation decreases and so does the estimated velocity parameter q_{k+1} as the coefficient β increases. Therefore, we can determine the coefficient β with iteration as follows:

$$\beta_{k+1} = \beta_k\left[1 - C_\beta\left(\frac{q}{h_b}\frac{dh_s}{dq}\right)_k\right] = \beta_k\left[1 - C_\beta\frac{q}{h_b}\left(\frac{dh_b}{dq} + \frac{dh_{bs}}{dq}\right)_k\right], \tag{3.3.8a}$$

$$\beta_0 = \frac{1+\Omega}{2}, \tag{3.3.8b}$$

$$\frac{dh_{bs}}{dq} = \frac{m_b v_b}{G_b + v_b^2}\frac{x_b - 1}{x_b + 1} \times \left\{1 - \frac{2x_b}{x_b^2 - 1}\left[\ln(x_b) - \frac{4qu_e G_b(1-\Omega)}{(G_o+q^2)(G_b+q^2)}\right]\right\}. \tag{3.3.8c}$$

The derivative of altitude at burn-out state with respect to the velocity parameter and the constant C_β are as same as in Eq. (3.2.12).

Preliminary numerical experiments show that the first order approximation (3.3.7b) converges faster than the third order approximation (3.3.7a) and gives exactly same results in most cases. However, in the case with a small mass ratio less than 1.5, the third order approximation shows more stable convergence. The number of iterations for convergence is almost same as that for the burn-out situation.

4. Numerical solutions

If the mass and velocity of the rocket are known at (n) state, then the velocity at $(n+1)$ state can be obtained. The discretized governing equation becomes

$$m_{n+1/2}\frac{v_{n+1} - v_n}{\Delta t} = \dot m_{n+1/2}u_e - K v_{n+1/2}^2 - m_{n+1/2}g, \tag{4.1a}$$

$$\Delta t = t_{n+1} - t_n = \frac{u_e}{g}\ln\left(\frac{m_{n+1}g + Kq^2}{m_n g + Kq^2}\right). \tag{4.1b}$$

The index $n+1/2$ denotes the midpoint between (n) and $(n+1)$ states

$$m_{n+1/2} = \frac{m_n + m_{n+1}}{2}, \tag{4.2a}$$

$$v_{n+1/2} = \frac{v_n + v_{n+1}}{2}, \tag{4.2b}$$

$$\dot m_{n+1/2} = \frac{\dot m_n + \dot m_{n+1}}{2} = \frac{(m_n + m_{n+1})g + 2Kq^2}{2u_e}. \tag{4.2c}$$

The governing equation can be then rewritten as follows:

$$m_{n+1/2}\frac{v_{n+1} - v_n}{\Delta t} = \dot m_{n+1/2}u_e - \frac{K}{4}(v_{n+1}^2 + 2v_n v_{n+1} + v_n^2) - m_{n+1/2}g. \tag{4.3a}$$

This discretized equation becomes a quadratic one as follows:

$$K v_{n+1}^2 + \left(2Kv_n + 4\frac{m_{n+1/2}}{\Delta t}\right)v_{n+1} + Kv_n^2 - 4\left(m_{n+1/2}\frac{v_n}{\Delta t} + \dot m_{n+1/2}u_e - m_{n+1/2}g\right) = 0. \tag{4.3b}$$

The solution of the above equation at $(n+1)$ state becomes

$$v_{n+1} = \frac{-B_n + \sqrt{B_n^2 - KC_n}}{K}, \tag{4.4a}$$

$$B_n = Kv_n + 2\frac{m_{n+1/2}}{\Delta t}, \tag{4.4b}$$

$$C_n = Kv_n^2 - 4\left(m_{n+1/2}\frac{v_n}{\Delta t} + \dot m_{n+1/2}u_e - m_{n+1/2}g\right). \tag{4.4c}$$

The discretized momentum equation for coast phase can be obtained if the thrust term is extracted from the equations and the mass is fixed as the mass at burn-out state.

5. Calculation conditions

The present study considers a rocket dry mass of 750 kg, as well as dry masses of 500 kg and 1000 kg for comparisons. The total mass or the propellant mass is changed according to the mass ratio. In the present study, the mass ratio is varied from 1.5 to 6, which means the rocket total mass is changed from 1025 kg to 4500 kg.

The cross-section diameter of the rocket is 0.4 m. The aerodynamic drag coefficient for a sounding rocket is usually in the range between 0.7 and 1.5. In the present study, the aerodynamic drag coefficient of 1.1 is considered. The effects of the Mach number on the aerodynamic drag coefficient are ignored for simplicity.

The temperature and pressure in the combustor chamber considered in the present study are 2500 K and 100 bar, respectively. The nozzle flow is assumed to be perfectly expanded to the standard atmospheric pressure through the isentropic process. The absolute jet velocity at the nozzle exit is about 1916 m/s.

The numerical solution approaches a more exact result as the number of piecewise intervals increases. The sufficient number of piecewise intervals is determined with a numerical experiment. If the number of intervals is as great as 150, the numerical integration with the trapezoid rule yields almost the same result as that with the Simpson rule. The number of piecewise intervals is fixed as 200 in the present calculations. The mass change during each interval is assumed to be constant.

6. Results

6.1. Solution profiles

Fig. 1 compares the velocity profiles between analytic and numerical solutions. The vertical dashed line indicates the burn-out time. The rocket velocities increase gradually with time until burn-out state, but after then, decrease due to the gravity force.

Fig. 1a shows the variation of velocity profiles according to the rocket mass with a fixed rocket mass ratio of 4. Regardless of the rocket masses, each analytic solution is identical to the numerical

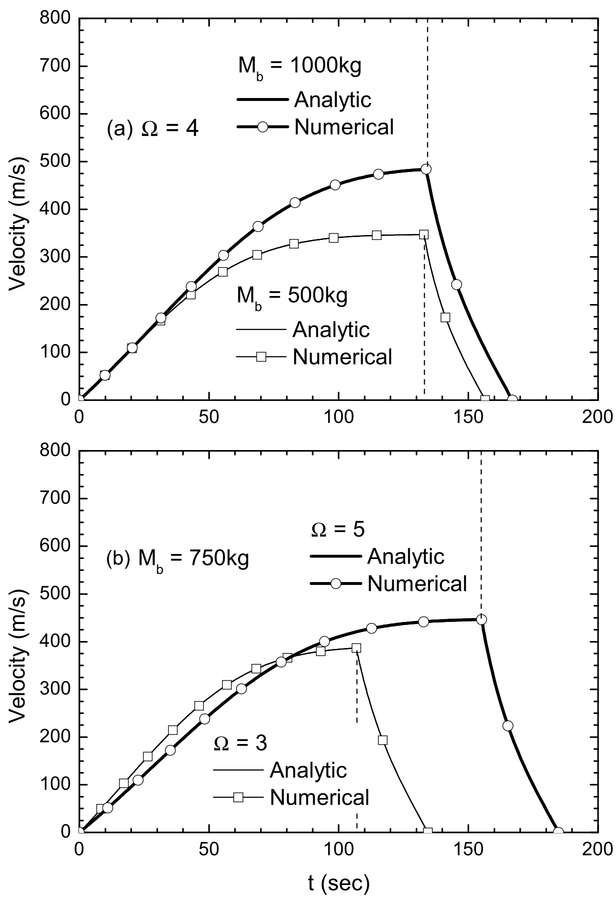

Fig. 1. Velocity profile according to time.

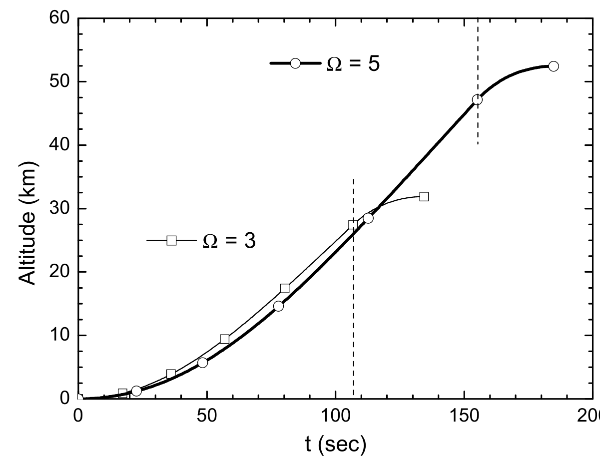

Fig. 2. Changes of altitude according to time.

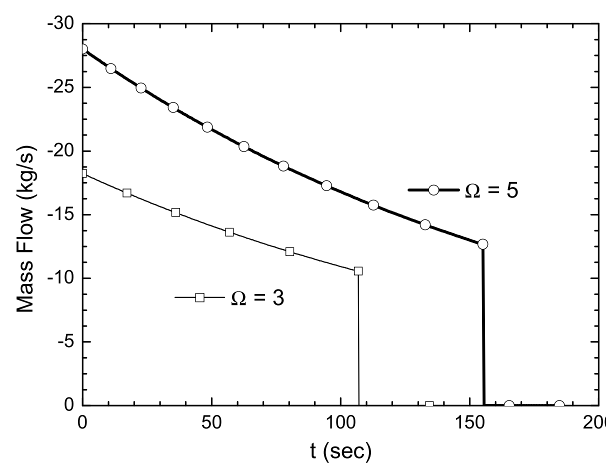

Fig. 3. Changes of mass flow rate according to time.

one. The case with a higher mass yields a higher velocity but a similar running time to the others. Fig. 1b shows the variations of velocity profile according to the rocket mass ratio with a fixed rocket mass of 750 kg. Regardless of the rocket mass ratios, each analytic solution is identical to the numerical one. The case with a higher mass ratio yields a higher rocket velocity and a longer running time. The profile in coast phase is concave because the drag is proportional to the square of velocity.

Fig. 2 shows the changes of altitude with time. The rocket mass at burn-out state is 750 kg. The vertical dashed line indicates the burn-out time. In boost phase, the rocket altitude increases through concave curves until burn-out state, since the velocity increases. In coast phase, the rocket altitude changes through a convex curve, since the rocket is decelerated by gravity force. The increase of rocket mass ratio results in a proportional increase of the maximum altitude.

Fig. 3 shows the changes of the mass flow rate with time. The rocket mass at burn-out state is 750 kg. The mass flow rate changes according to Eq. (3.1.1a). Regardless of the rocket mass ratios, the mass flow rate decreases gradually almost in a linear mode. The abrupt drop of the curve indicates the burn-out time. The increase of mass ratio results in the proportional increase of the mass flow rate.

6.2. Optimal conditions at burn-out state

To estimate the characteristic changes of the altitude at burn-out state according to the velocity parameter, the following normalized parameters are introduced.

$$\phi_b = \frac{q - q_{opt,b}}{q_{opt,b}}. \quad (6.1a)$$

$$\eta_b = \frac{h_b}{h_b(q_{opt,b})}. \quad (6.1b)$$

Fig. 4 shows variations of the normalized altitude at burn-out state according to the normalized velocity parameter. The vertical dashed line indicates the optimal velocity parameter calculated by the characteristic equation. The reduced order approximations of the characteristic equation (3.2.11) give the exact predictions of the optimal velocity parameter regardless of the rocket mass ratios or of the rocket masses. The change of the normalized altitude on the left side where the velocity parameter is lower than the optimal one is more sensitive to the change of the velocity parameter than the other side.

Fig. 4a represents the effects of the mass ratio on the variations of the normalized altitude. The case with a lower mass ratio results in a little more sensitive variations of the normalized altitude according to the change of the normalized velocity parameter. Fig. 4b represents the effects of the rocket mass on the variations of the normalized altitude. The normalized curves with different rocket masses nearly coincide even though the changes of the rocket mass are remarkably large, which suggests that the variation of the normalized altitude is almost irrelevant to the rocket mass.

Fig. 5 shows variations of the optimal velocity parameters with the rocket mass or the rocket mass ratio at burn-out state. For a given rocket mass ratio, the optimal velocity parameter grows with the rocket mass but the growth rate slightly decreases as the rocket mass increases. For a given rocket mass, the optimal velocity parameter increases with the rocket mass ratio in a linear mode.

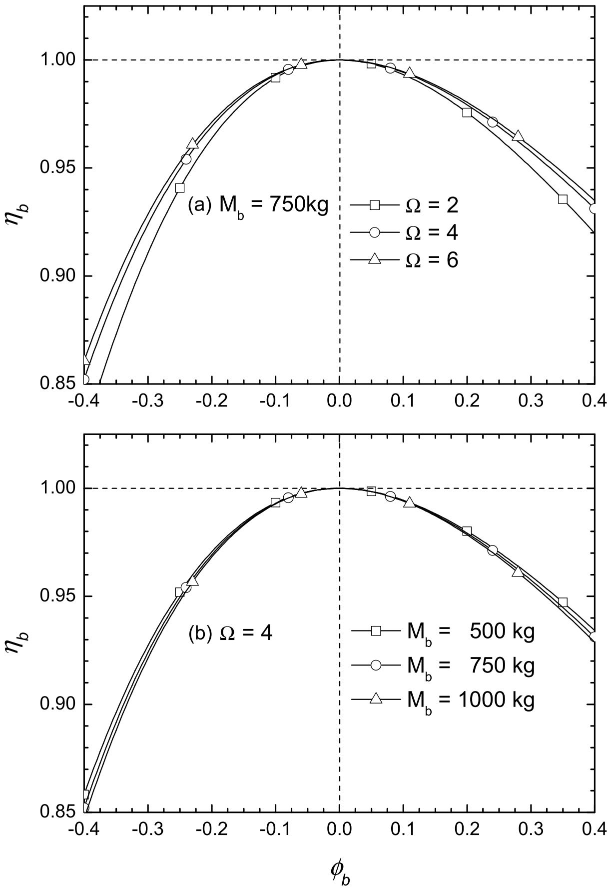

Fig. 4. Variation of altitude at burn-out state with velocity parameter.

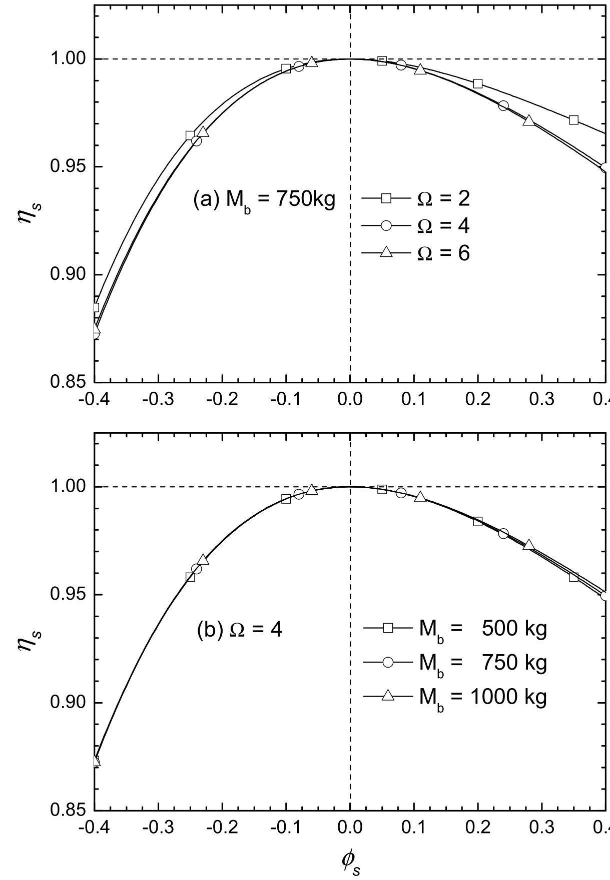

Fig. 6. Variation of altitude at stationary state with velocity parameter.

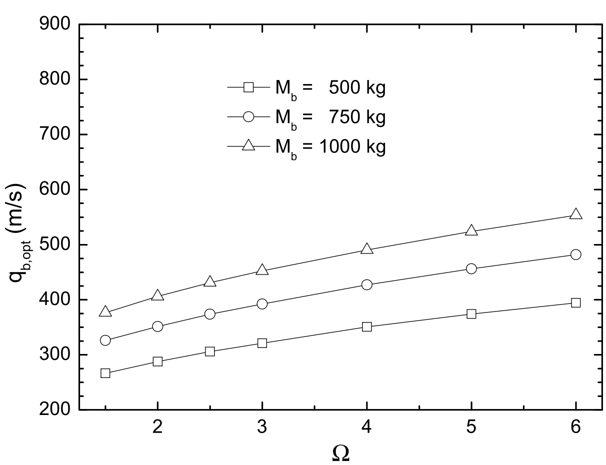

Fig. 5. Variation of optimal velocity parameter at burn-out state.

6.3. Optimal conditions at stationary state

To determine the characteristic changes of the altitude at stationary state or apogee according to the velocity parameter, the following normalized parameters are introduced.

$$\phi_s = \frac{q - q_{opt,s}}{q_{opt,s}}. \tag{6.2a}$$

$$\eta_s = \frac{h_s}{h_s(q_{opt,s})}. \tag{6.2b}$$

Fig. 6 shows variations of the normalized altitude at stationary state according to the velocity parameter. The vertical dashed line indicates the optimal velocity parameter at stationary state. The reduced order approximations of the characteristic equation (3.3.7) give the exact predictions of the optimal velocity parameter regardless of the rocket mass ratios or the rocket masses. The change of the normalized altitude on the left side where the velocity parameter is lower than the optimal one is more sensitive than the other side.

Fig. 6a represents the effects of the mass ratio on the variations the normalized altitude. On the contrary to the situation at burn-out state, the case with a lower mass ratio shows a less sensitive variation of the normalized altitude to the change of the normalized velocity parameter. Fig. 6b represents the effects of the rocket mass on the variations the normalized altitude. Like the situation at burn-out state, the normalized curves with different rocket masses almost coincide even though the difference of the rocket mass is remarkably large, which suggests that the variation of the normalized altitude is almost irrelevant to the rocket mass.

Fig. 7 shows variations of the optimal velocity parameters with the rocket mass or the rocket mass ratio at stationary or apogee state. For a given rocket mass ratio, the optimal velocity parameter grows with rocket mass. While, on the contrary to the situation at burn-out stare, for a given rocket mass, the optimal velocity parameter decreases steeply with the mass ratio until the minimum value and, after then, bounce back and grows gradually with the rocket mass ratio in a linear mode.

7. Conclusions

The one-dimensional rocket momentum equation including thrust, gravitational force, and aerodynamic drag is examined to determine analytically the optimal condition for maximizing altitude of a sounding rocket at burn-out state or at stationary state.

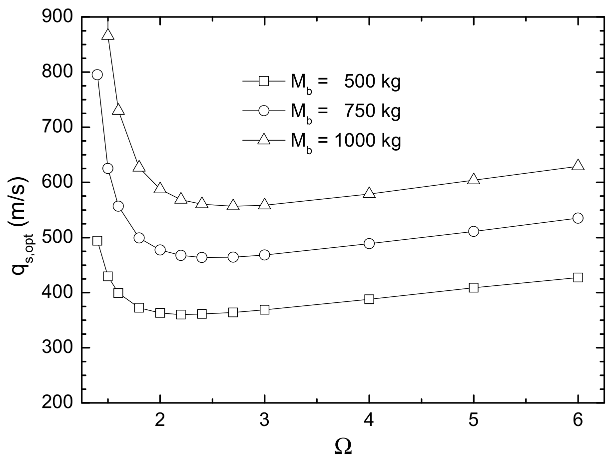

Fig. 7. Variation of optimal velocity parameter at stationary state.

The rocket flights in a constant atmosphere where the air density is constant are considered.

The rocket velocity for a given velocity parameter can be obtained analytically that matches the numerical one. Characteristic equations are derived from the analytic solutions to determine the optimal velocity parameter for maximizing altitudes at burn-out state and at stationary state. The characteristic equations provide accurate predictions of the optimal conditions at burn-out state and at stationary state.

The velocity parameter for maximizing altitude at burn-out state exists. The increase of the rocket mass at a given mass ratio results in the increases of the optimal velocity parameter but the increasing rate is reduced as the rocket mass increases. And, the optimal velocity parameter at a given rocket mass grows with the rocket mass ratio in a linear mode. Also, the velocity parameter for maximizing altitude at stationary state exists and is higher than that at the burn-out situation. Like the situation at burn-out state, the optimal velocity parameter grows with rocket mass but, on the contrary to the burn-out situation, there is a mass ratio where the optimal velocity parameter is the minimum at a given rocket mass.

As a beginning study, the present works are restricted to the rocket fights in a constant atmosphere where the air density is constant regardless of the altitude. In the future, further studies will be conducted to extend the present analytic approach to the flight in a real atmosphere.

Conflict of interest statement

None declared.

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