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Shock reflection and oblique shock waves

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The linear stability of steady attached oblique shock wave and pseudosteady regular shock reflection is studied for the nonviscous full Euler system of equations in aerodynamics. A sufficient and necessary condition is obtained for their linear stability under three-dimensional perturbation. The result confirms the sonic point condition in the study of transition point from regular reflection to Mach reflection, in contrast to the von Neumann condition and detachment condition predicted from mathematical constraint. © *2007 American Institute of Physics.*
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I. INTRODUCTION

As a shock front hits a planar wall with an incident angle α, an oblique reflected shock wave is produced. For small incident angle α, the so-called regular shock reflection happens with the incident and reflected shock fronts intersecting at a point P on the plane surface. Figure 1 shows a regular shock reflection near the intersection point P on an infinite planar wall. In Fig. 1, I is the incident shock wave and R the reflected shock wave, with incident angle α and reflection angle δ.

In wind tunnel experiment, it has been long observed that for fixed shock strength of I, as the incident angle α increases past a critical value α_c, the configuration in Fig. 1 will change into a more complicated Mach reflection,[1,3,10,14,34] with a third shock (Mach stem) connecting the intersection point and the plane surface, as well as the appearance of other features such as vortex sheet.

The relations governing the possible state on two sides of shock front are derived from Rankine-Hugoniot conditions on shock front for the Euler system of equations. Such relations can be graphically represented as a curve called shock polar, see Refs. 10 and 35, also Fig. 3. Any point on the shock polar corresponds to an incident angle α. It is obvious from the shock polar that there is a maximal angle α_d corresponding to the so-called "detachment point" beyond which a regular reflection is simply impossible.

The determination of the exact transition angle α_c from regular reflection to Mach reflection has been one of the focuses of shock wave research since von Neumann. In Refs. 30 and 31, von Neumann introduced the "detachment condition" and "von Neumann condition." The detachment criterion is the above mathematical constraint of detachment point so that for an incident angle $\alpha > \alpha_d$, a regular reflection is impossible. The von Neumann criterion states that there is a von Neumann angle $\alpha_N < \alpha_d$ such that for an incident angle $\alpha < \alpha_N$, a Mach reflection is impossible, see also Refs. 13 and 16.

A sonic point on the shock polar is the angle α_s with $\alpha_s \in (\alpha_N, \alpha_d)$, which corresponds to the sonic speed in downstream flow. The "sonic point criterion" has also been proposed which predicts that the transition from regular reflection to Mach reflection occurs at sonic point.[3] For real gas, the sonic point is very close to detachment point (not larger than 6°, see Refs. 1 and 14, their difference in Fig. 3 is exaggerated) and therefore, it is very difficult to experimentally distinguish the two angles α_s and α_d.

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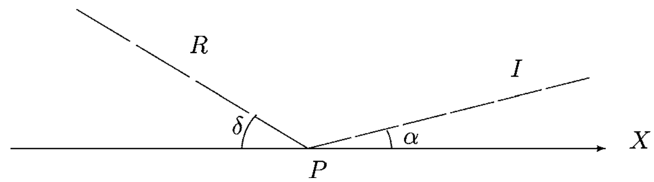

FIG. 1. Regular shock reflection at a planar infinite ramp: I, incident shock front; R, reflected shock front; α, angle between incident shock front and planar ramp; and δ, angle between reflected shock front and planar ramp.

The study on possible criteria for transition from regular to Mach reflection and Mach to regular reflection has been a very active research field for decades. There are massive amount of literatures on such criteria under various conditions, see also the extensive bibliography in Ref. 1. Especially rich is the experimental and numerical work in addition to analytical method.

In particular, it has been shown that the transition point for regular to Mach reflection and for Mach to regular reflection could be different, i.e., there is a hysteresis, see Refs. 4 and 17 and for experimental and numerical, also see Refs. 1, 9, 8, 12, 17, 18, 20, 21, and 36.

It has also been shown, both experimentally and numerically, that the transition point may depend on, among others,

- the Mach number of the incident shock,[17]
- the viscosity,[15,24]
- boundary layer effect,[3] and
- downstream influence.[4]

In a recent mathematical survey on the topic,[34] it is remarked concerning the transition from regular to Mach reflection that "this anticipated transition must be due to some instability, but has not been explained rigorously so far," see Ref. 34 (Sec. 3A).

The purpose of this paper is to address this issue and perform a rigorous three-dimensional stability analysis on the regular shock reflection. The result of the analysis confirms the conjecture in Ref. 34 and provides the mathematical support to the sonic point criterion for the transition from regular shock reflection to Mach reflection.

The shock reflection phenomenon is closely related to the oblique shock waves. An oblique shock wave is produced as an airplane flies supersonically in the air. With other conditions fixed, the shape of such shock waves at the wings of the airplane is determined by the shape of the front edge of the wing. At very small angle θ of a sharp wing edge, the shock front is attached to the wing. But the shock front becomes detached as the angle θ increases past a critical angle θ_c. Figure 2 shows the profile of an attached shock wave S and the flow at a sharp wedge, see Refs. 1, 10, and 34.

Again it is of great interest to know the exact angle θ_c at which an attached shock front transforms into a detached one, since a detached shock front drastically increases resistance to the

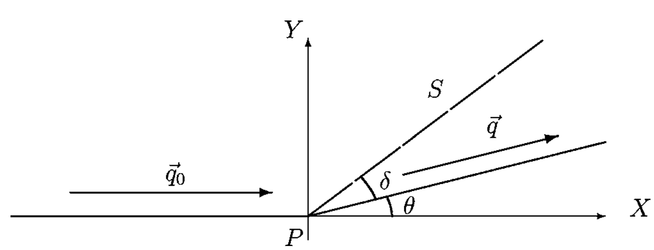

FIG. 2. An attached oblique shock wave in supersonic flight: $\vec{q}_0$, incoming upstream velocity; $\vec{q}$, inflected downstream velocity; S, attached shock front; θ, angle between incoming velocity and solid surface; and δ, shock inflection angle.

flight. Mathematically, it means the determination of the maximal angle θ_c which would guarantee a stable attached oblique shock front and for any angle larger than θ_c, the shock front will become detached.

There are also extensive studies on oblique shock waves using theoretical, numerical, and experimental tools, see the references in Refs. 1 and 34. Rigorous mathematical analysis has been done mostly for various approximate models, such as irrotational potential flow model and others.[5,6,33] Such analysis is only for sufficiently small incident angle α in shock reflection or θ in oblique shock wave, and usually also assumes for very weak incident shock front. Such analysis is limited in dealing with the region far away from the transition point and therefore did not provide any information to the transition criterion of oblique shock from "attached" to "detached," which always happens beyond "small" incident angle.

In Ref. 23, the stability of oblique shock waves is studied for large incident angle for an isentropic Euler system model. Since physical shock waves are always accompanied with entropy change and the shock strength cannot be assumed to be small for the oblique shock wave near the transition from attached to detached, or the shock reflection near transition from regular to Mach reflection, we need to study the stability condition for the full nonisentropic Euler system.

This paper will study oblique shock waves for the general nonviscous gas for arbitrary shock strength and for large incident angle, in particular, for incident angle near transition point. Then the results will be applied to regular shock reflection. The final stability conditions show that the oblique shock wave and regular shock reflection are linearly stable with respect to geometric configuration and upstream perturbation (see also Ref. 19) only up to the sonic angle $\theta_s < \theta_d$. The sonic angle θ_s corresponds to the sonic downstream flow, while angle θ_d is the detachment point (see Theorems 2.1 and 3.1). The theorem provides the analytical support to the sonic point criterion in the transition from regular to Mach reflection[3] and confirms the conjecture in Ref. 34.

The paper is arranged as follows. In Sec. II, we give the mathematical formulation of the problem and state the main theorem (Theorem 2.1) for oblique shock waves. In Sec. III, the main theorem in Sec. II is applied in the analysis of regular shock reflection and obtain Theorem 3.1 regarding its stability and its physical implications. The detailed proof of Theorem 2.1 on the linear stability of oblique shock front is given in Sec. IV.

II. FORMULATION AND THEOREM FOR OBLIQUE SHOCK WAVES

The full Euler system for nonviscous flow in aerodynamics is the following:

$$\partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0,$$

$$\partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1,2,3, \tag{2.1}$$

$$\partial_t(\rho E) + \sum_{j=1}^{3} \partial_{x_j}(\rho E v_j + p v_j) = 0.$$

In (2.1), $(\rho, \mathbf{v})$ are the density and the velocity of the gas particles, $E = e + \frac{1}{2}|\mathbf{v}|^2$ is the total energy, and the pressure $p = p(\rho, E)$ is a known function.

In the region where the solution is smooth, the conservation of total energy in (2.1) can be replaced by the conservation of entropy S, see Ref. 10, and system (2.1) can be replaced by the following system:

$$\partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0,$$

$$\partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1,2,3, \tag{2.2}$$

$$\partial_t(\rho S) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j S) = 0,$$

with pressure $p = p(\rho, S)$ satisfying

$$p_\rho > 0, \quad p_{\rho\rho} > 0. \tag{2.3}$$

Shock waves are piecewise smooth solutions for (2.1) which have a jump discontinuity along a hypersurface $\phi(t,x)=0$. On this hypersurface, the solutions for (2.1) must satisfy the following Rankine-Hugoniot conditions, see Ref. 10, 34, and 35:

$$\phi_t \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{bmatrix} + \phi_{x_1} \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ (\rho E + p) v_1 \end{bmatrix} + \phi_{x_2} \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ (\rho E + p) v_2 \end{bmatrix} + \phi_{x_3} \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ (\rho E + p) v_3 \end{bmatrix} = 0. \tag{2.4}$$

Here $[f] = f_1 - f_0$ denotes the jump difference of f across the shock front $\phi(t,x)=0$. In this paper, we will use subscript "0" to denote the state on the upstream side (or, ahead) of the shock front and subscript "1" to denote the state on the downstream side (or, behind).

Rankine-Hugoniot condition (2.4) admits many nonphysical solutions to (2.1). To single out physical solution, we could impose the stability condition, which argues that for observable physical phenomena, the solution to mathematical model should be stable under small perturbation. In the case of one space dimension, this condition is provided by Lax' shock inequality which demands that a shock wave is linearly stable if and only if the flow is supersonic (relative to the shock front) in front of the shock front and is subsonic (relative to the shock front) behind the shock front, see Refs. 10 and 35.

In the case of high space dimension, it is shown that for isentropic polytropic flow, Lax' shock inequality also implies the linear stability of the shock front under multidimensional perturbation. However, an extra condition on shock strength is needed for general nonisentropic flow, see Refs. 27 and 26.

In the study of steady oblique or conical shock waves, the issue is the stability of shock waves with respect to the small perturbation in the incoming supersonic flow or the solid surface. This is the stability independent of time as in Ref. 7, in contrast to the stability studied in Refs. 27 and 38, and is also different from the study of other unsteady flow, see Refs. 5 and 25.

The result on the stability of oblique shock waves for the full Euler system is the following theorem (Theorem 2.1). The corresponding theorem (Theorem 3.1) on regular shock reflection will be stated in Sec. III.

Theorem 2.1: For three-dimensional Euler system of aerodynamics (2.1), a steady oblique shock wave is linearly stable with respect to the three-dimensional perturbation in the incoming supersonic flow and in the sharp solid surface if the following is obtained.

1. The usual entropy condition or its equivalent is satisfied across the shock front. For example, shock is compressive, i.e., the density increases across the shock front:

$$\rho_1 > \rho_0. \tag{2.5}$$

Or equivalently, Lax' shock inequality is satisfied.

2. The flow is supersonic behind the shock front

$$|\mathbf{v}| > a. \tag{2.6}$$

3. The shock strength ρ_1/ρ_0 satisfies

$$\left(\frac{v_n}{|\mathbf{v}|}\right)^2\left(\frac{\rho_1}{\rho_0} - 1\right) < 1. \tag{2.7}$$

In (2.7), v_n denotes the normal component of the downstream flow velocity $\mathbf{v}$.

Conditions (2.5)–(2.7) are also necessary for the linear stability of a planar oblique shock.

Remark 2.1: The necessity part of the theorem follows from the fact that Kreiss' condition[22] is the necessary and sufficient condition for the well posedness of the initial-boundary value problem for hyperbolic systems under consideration.

Remark 2.2: It is interesting to compare condition (2.7) with the following conditions in Ref. 27 [see (1.17) in Ref. 27]:

$$M^2(\rho_1/\rho_0 - 1) < 1, \quad M < 1. \tag{2.8}$$

We notice that (2.7) and (2.8) have very similar forms. The only difference is that the Mach number M in the first relation of (2.8) is replaced here by $v_n/|\mathbf{v}|$ in (2.8). Since Mach number $M<1$ in (2.8) and $|\mathbf{v}|>a$ in (2.6), we have

$$\frac{v_n}{|\mathbf{v}|} < M.$$

Hence condition (2.7) is weaker than conditions (2.8) in Ref. 27.

Despite the similarity, we emphasize that (2.7) and (2.8) deal with two different types of stability. (2.7) is about the time-independent stability with respect to the perturbation of incoming flow and solid surface, while (2.8) is about the transitional stability with respect to the perturbation of initial data, see also Remark 2.3 in the following.

Remark 2.3: In Ref. 38, the linear stability was studied for oblique shock wave and shock reflection and it was shown that all weak (relative to strong, but with large incident angle) oblique shock waves are linearly stable, which obviously differs with the conditions in (2.6) and (2.7) in Theorem 2.1. The difference originates from the fact that different types of stability are considered.

The stability in Ref. 38 is with respect to an initial perturbation and hence is reduced to an initial-boundary problem for a nonstationary linearized system as in Ref. 27. So the result in Ref. 38 only confirms the condition in Ref. 27 and does provide any insight into the effect of geometrical contour on the mechanism of transition from attached shock wave to detached or from regular reflection to Mach reflection.

In this paper, the stability condition in Theorem 2.1 is with respect to a genuine three-dimensional perturbation in the incoming flow and reflection surface for a stationary flow. It is important to notice that the resulting boundary value problem is independent of time. This "global in time" (independent of time) condition is stronger than the ones in Refs. 38 and 27 and produces the criterion which depends purely on the geometrical property of the object. This provides the analytical confirmation to the sonic point criterion to the transition from regular to Mach reflection.

Theorem 2.1 predicts a drastic change in the behavior of oblique shock waves as shock strength increases such that the downstream flow becomes subsonic.

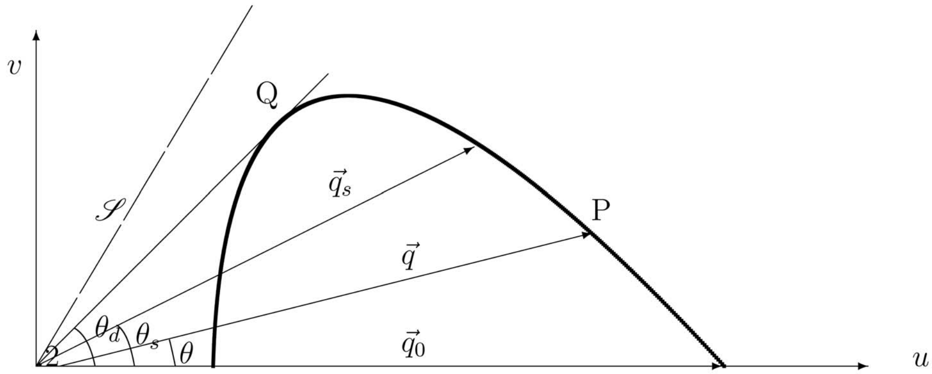

FIG. 3. Shock polar determines the downstream velocity $\vec{q}$: $\vec{q}_0$, incoming upstream velocity; $\vec{q}$, inflected downstream velocity; θ, shock inflection angle; θ_d, the detachment angle, the maximal possible shock inflection angle; $\vec{q}_s$, the velocity with magnitude of sound speed a; θ_s, the sonic angle, the critical angle for shock stability; and $\mathcal{S}$, shock front.

To better understand the physical implication of the conditions in Theorem 2.1, let us examine the shock polar in Fig. 3, which determines the dependency of downstream velocity $\vec{q}$ upon the angle θ, assuming other parameters unchanged.

In Fig. 3, every incident angle θ corresponds to two theoretically possible oblique shock waves, with the strong ones being well-known unstable. In this paper, we consider only the "weak" ones, even though they may have large incident angle θ, and with relatively big shock strength. The critical velocity $\vec{q}_c=\vec{q}_s$ has magnitude of sound speed and corresponds to a critical angle θ_s on the so-called "sonic point" on shock polar. For all $\theta<\theta_s$, the downstream flow is supersonic ($|\vec{q}|>a$) and the oblique shock wave is linearly stable, and for all $\theta>\theta_c$, the downstream flow is subsonic ($|\vec{q}|<a$) and the linear stability conditions fail. In particular, at the detachment point, the theoretically maximal angle $\theta_d>\theta_s$ (the difference between θ_s and θ_d in Fig. 3 is exaggerated here on purpose), the downstream flow is subsonic. Therefore, for all $\theta \in (\theta_s,\theta_d)$, Theorem 2.1 predicts an unstable weak oblique shock wave. The angle $\theta_s<\theta_d$ provides a prediction of the exact transition angle from an attached shock front to a detached shock front.

III. ANALYSIS OF REGULAR SHOCK REFLECTION AND ITS TRANSITION TO MACH REFLECTION

We consider the planar regular shock reflection along an infinite plane wall, as in Fig. 1. Because the stability result in Theorem 2.1 is with respect to three-dimensional perturbation, our discussion also applies to the case of a curved shock front along an uneven solid surface. In addition it also applies to the local discussion near the intersection point of a regular reflection along a ramp or wedge.

As in Fig. 1, a planar incident shock wave with shock front velocity $\mathbf{v}_0$ is reflected along an infinite wall X and the angle between incident shock front and wall is α, and the angle between reflected shock front and wall is δ.

Because of the Galilean invariance of Euler system of equations, if $(\rho,\mathbf{v},e)$ is a solution, then $(\rho(x+\mathbf{U}t,t),\ \mathbf{v}(x+\mathbf{U}t,t),\ e(x+\mathbf{U}t,t))$ is also a solution for any constant velocity vector $\mathbf{U}$. Therefore, we can choose the coordinates moving with the intersection point P in Fig. 1, which is moving with constant velocity $\mathbf{U}=|\mathbf{v}_0|/\sin\theta$ along the X axis. In this coordinate system, the pseudosteady regular planar reflection at an infinite plane wall becomes steady, with the flow velocity $\vec{q}_0$ in front of incident shock front I, the velocity $\vec{q}_1$ between the incident shock I and the reflected shock R, and the flow velocity $\vec{q}_2$ behind the reflected shock R, as in Fig. 4.

It is obvious that velocity vector $|\vec{q}_0|=|\mathbf{v}_0|/\sin\alpha$. In shock reflection, the state of the flow on two sides of the incident shock I is given, i.e., the state of the flow region of $\vec{q}_0$ and $\vec{q}_1$ is given. The reflected shock front R as well as the flow state in its downstream region need to be determined. It has been known[10,34,38] that the downstream state is uniquely determined by a relation

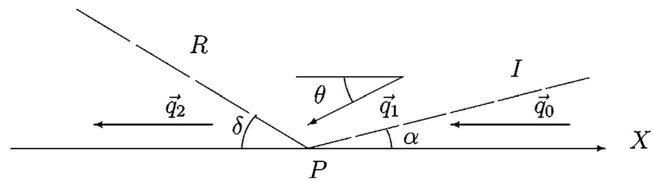

FIG. 4. Steady regular shock reflection at an infinite wall: I, incident shock front; R, reflected shock front; $\vec{q}_0$, upstream velocity in front of incident shock; $\vec{q}_1$, inflected flow velocity between incident and reflected shocks; $\vec{q}_2$, downstream velocity from reflected shock; α, angle between incident shock front and planar ramp; δ, angle between reflected shock front and planar ramp; and θ, inflection angle between $\vec{q}_1$ and planar ramp.

derived from Rankine-Hugoniot conditions on the incident and reflected shock fronts. For a given incident shock I, the incident angle α determines uniquely the downstream flow, in particular, the slope of the vector $\vec{q}_1$ and hence θ.

For the reflected shock front R, the angle θ in Fig. 4 is the same inflection angle θ in the oblique shock wave, as in Figs. 2 and 3. Therefore, the reflected shock R can be looked at as an oblique shock wave generated by an incoming flow $\vec{q}_1$ by a ramp with inflection angle θ. Consequently, we can apply the results in Theorem 2.1 to the regular shock reflection and obtain the following theorem.

Theorem 3.1: For three-dimensional Euler system of gas dynamics (2.1), a steady regular planar shock reflection is linearly stable with respect to the three-dimensional perturbation in the incident shock front I and in the solid surface if the following is obtained.

1. The usual entropy condition or its equivalent is satisfied across the shock front. For example, shock is compressive, or equivalently, Lax' shock inequality is satisfied.
2. The flow is supersonic downstream from the reflected shock front R

$$|\vec{q}_2| > a. \tag{3.1}$$

3. The shock strength ρ_2/ρ_1 satisfies

$$\left(\frac{q_n}{|\vec{q}_2|}\right)^2\left(\frac{\rho_2}{\rho_1}-1\right) < 1. \tag{3.2}$$

Here q_n denotes the component of the flow velocity $\vec{q}_2$ normal to the reflected shock front R.

The above conditions are also necessary for the linear stability of a planar regular shock reflection formed by a uniform incident along an infinite planar wall.

We now turn back to the shock polar in Fig. 3 to see the physical implications of Theorem 3.1, especially in relation to the transition of a regular shock reflection in Fig. 1 to a Mach reflection. Experimental data show that with fixed incident shock I, as incident angle α increases, the reflected angle θ also increases. The regular reflection pattern in Fig. 4 will persist until α reaches a critical value α_c (hence θ reaches a critical value θ_c), beyond which the flow pattern in Fig. 4 will give way to the Mach reflection, with the intersection point P lifted away from the wall and connected to the wall by Mach stem, as well as with the appearance of a slip line or even more complicated features, see Refs. 10, 17, and 34.

The shock relation derived from Rankine-Hugoniot conditions gives the detachment point which corresponds to the maximal possible angle θ_d in Fig. 3. However, there have been neither rigorous analytical proof to pinpoint this point nor accurate experimental data to support this detachment point criterion and exclude the possibility that the transition would actually happen at a smaller $\theta_c < \theta_d$.

In Ref. 14, it has been argued from information criteria that Mach reflection is not possible for supersonic downstream flow, i.e., Mach reflection requires that $\theta > \theta_s$ with θ_s denoting the angle at sonic point corresponding to sonic downstream flow. Otherwise, the results with mathematical rigor are available only for small incident angel α (hence small θ).

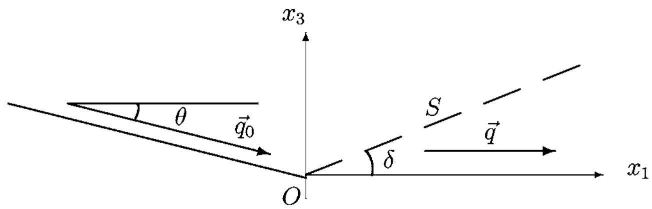

FIG. 5. An attached oblique shock wave in supersonic flight: $\vec{q}_0$, incoming upstream velocity; $\vec{q}$, inflected downstream velocity; S, attached shock front; θ, angle between incoming velocity and solid surface; and δ, shock inflection angle.

Theorem 3.1 tells us that $\theta_c = \theta_s$, i.e., if the downstream flow is supersonic [i.e., (3.1) is satisfied], then the regular reflection pattern is stable with respect to three-dimensional perturbation for moderate shock strength [i.e., (3.2) is automatically satisfied]. This confirms the sonic point transition conclusion in Ref. 14 based on the physical information criteria as well as the stability conjecture in Ref. 34.

Since the condition in Theorem 3.1 is a necessary and sufficient for uniform planar shock and wall, the subsonic downstream flow implies that the onset of instability consequently indicates that the regular reflection pattern in Fig. 1 could not be preserved, unless some extra conditions are imposed in the far fields of downstream flow, see Refs. 2 and 7.

Consequently Theorem 3.1 predicts the transition from regular reflection to Mach reflection exactly at the critical angle $\theta_c = \theta_s$ which is the sonic point corresponding to sonic downstream flow.

IV. PROOF OF THEOREM 2.1

Because of the invariance of Kreiss' conditions for hyperbolic boundary value problems, we need only to consider the linear stability of a uniform oblique shock wave produced by a wedge with plane surface and choose the coordinate system (x_1, x_2, x_3) (see Fig. 5) such that the following is obtained.

- The solid wing surface is the plane $x_3 = 0$.
- The downstream flow behind the oblique shock front is in the positive x_1 direction.
- The angle between the solid wing surface and oblique shock front is δ.
- The angle between the incoming supersonic flow and the solid wing surface is θ.

Consider a small perturbation in the solid surface $x_3 = 0$, as well as in the uniform incoming supersonic steady flow. The perturbed solid surface is $x_3 = b(x_1, x_2)$, with $b(0,0) = b_{x_1}(0,0) = b_{x_2}(0,0) = 0$, the downstream flow after shock front should be close to the direction of positive x_1-axis. The perturbed oblique shock front is described by $x_3 = s(x_1, x_2)$ such that $s(0,0) = s_{x_2}(0,0) = 0$ and $s_{x_1} \sim \lambda = \tan \delta > 0$. Obviously we have $b(x_1, x_2) < s(x_1, x_2)$ for all (x_1, x_2).

In the region $b(x_1, x_2) < x_3 < s(x_1, x_2)$, the steady flow is smooth. Hence Euler system (2.1) can be replaced by (2.2) and we have

$$\sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0,$$

$$\sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, \quad i = 1,2,3, \tag{4.1}$$

$$\sum_{j=1}^{3} \partial_{x_j}(\rho v_j S) = 0.$$

On the shock front $x_3=s(x_1,x_2)$, Rankine-Hugoniot condition (2.4) becomes

$$s_{x_1}\begin{bmatrix}\rho v_1\\ \rho v_1^2+p\\ \rho v_1 v_2\\ \rho v_1 v_3\\ (\rho E+p)v_1\end{bmatrix}+s_{x_2}\begin{bmatrix}\rho v_2\\ \rho v_1 v_2\\ \rho v_2^2+p\\ \rho v_2 v_3\\ (\rho E+p)v_2\end{bmatrix}-\begin{bmatrix}\rho v_3\\ \rho v_1 v_3\\ \rho v_2 v_3\\ \rho v_3^2+p\\ (\rho E+p)v_3\end{bmatrix}=0. \tag{4.2}$$

On the solid surface $x_3=b(x_1,x_2)$, the flow is tangential to the surface and we have the boundary condition

$$v_1\frac{\partial b}{\partial x_1}+v_2\frac{\partial b}{\partial x_2}-v_3=0. \tag{4.3}$$

The study of oblique shock wave consists of investigating the system (4.1) with the boundary conditions (4.2) and (4.3).

System (4.1) can be written as a symmetric system for the unknown vector function $U=(p,v_1,v_2,v_3,S)^T$ in $b(x_1,x_2)<x_3<s(x_1,x_2)$:

$$A_1\partial_{x_1}U+A_2\partial_{x_2}U+A_3\partial_{x_3}U=0, \tag{4.4}$$

where

$$A_1=\begin{pmatrix}v_1/a^2\rho & 1 & 0 & 0 & 0\\ 1 & \rho v_1 & 0 & 0 & 0\\ 0 & 0 & \rho v_1 & 0 & 0\\ 0 & 0 & 0 & \rho v_1 & 0\\ 0 & 0 & 0 & 0 & \rho v_1\end{pmatrix},\quad A_2=\begin{pmatrix}v_2/a^2\rho & 0 & 1 & 0 & 0\\ 0 & \rho v_2 & 0 & 0 & 0\\ 1 & 0 & \rho v_2 & 0 & 0\\ 0 & 0 & 0 & \rho v_2 & 0\\ 0 & 0 & 0 & 0 & \rho v_2\end{pmatrix},$$
$$A_3=\begin{pmatrix}v_3/a^2\rho & 0 & 0 & 1 & 0\\ 0 & \rho v_3 & 0 & 0 & 0\\ 0 & 0 & \rho v_3 & 0 & 0\\ 1 & 0 & 0 & \rho v_3 & 0\\ 0 & 0 & 0 & 0 & \rho v_3\end{pmatrix}. \tag{4.5}$$

When downstream flow is supersonic, we have $v_1^2>a^2$ and it is readily checked that matrix A_1 is positively definite. Therefore (4.4) is a hyperbolic symmetric system[11,32] with x_1 being the timelike direction.

On the fixed boundary $x_3=b(x_1,x_2)$, the boundary matrix

$$A_3-A_1b_{x_1}-A_2b_{x_2}=\begin{pmatrix}0 & -b_{x_1} & -b_{x_2} & 1 & 0\\ -b_{x_1} & 0 & 0 & 0 & 0\\ -b_{x_2} & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}. \tag{4.6}$$

It is readily checked that the boundary condition (4.3) is admissible with respect to system (4.4) in the sense of Friedrichs[11,32] and there is a corresponding energy estimate for the linearized problem. Therefore, we need only to study the linearized problem for (4.1) [or (4.4)] and (4.2) near the shock front.

We perform the coordinate transform to fix the shock front $x_3=s(x_1,x_2)$:

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = x_3 - s(x_1,x_2). \tag{4.7}$$

In the coordinates (x_1',x_2',x_3'), the shock front becomes $x_3'=0$ and the shock front position $x_3 = s(x_1,x_2)$ becomes a new unknown function, coupled with U. To simplify the notation, we will denote the new coordinates in the following again as (x_1,x_2,x_3). The system (4.4) in the new coordinates becomes

$$A_1\partial_{x_1}U + A_2\partial_{x_2}U + \tilde{A}_3\partial_{x_3}U = 0, \tag{4.8}$$

where $\tilde{A}_3 = A_3 - s_{x_1}A_1 - s_{x_2}A_2$. The Rankine-Hugoniot boundary condition (4.2) is now defined on $x_3=0$ and takes the same form

$$s_{x_1}\begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ (\rho E + p)v_1 \end{bmatrix} + s_{x_2}\begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ \rho v_2 v_3 \\ (\rho E + p)v_2 \end{bmatrix} - \begin{bmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ (\rho E + p)v_3 \end{bmatrix} = 0. \tag{4.9}$$

System (4.8) with boundary condition (4.9) is a coupled boundary value problem for unknown variables (U,s) with U defined in $x_3<0$ and s being a function of (x_1,x_2). To examine Kreiss' uniform stability condition, we need to study the linear stability of (4.8)(4.9) near the uniform oblique shock front with downstream flow:

$$U_1 = (p,v_1,0,0,S), \quad s = \lambda x_1, \tag{4.10}$$

where $\lambda=\tan\delta$, with δ being the angle between solid surface and oblique shock front. Under the assumptions in Theorem 2.1, behind the shock front we have

$$v_1 > a, \quad v_n \equiv v_1 \sin\delta < a, \tag{4.11}$$

where v_n is the flow velocity component normal to the shock front.

Let (V,σ) be the small perturbation of (U,s) with $V=(\dot{p},\dot{v}_1,\dot{v}_2,\dot{v}_3,\dot{S})$. The linearization of (4.8) is the following linear system:

$$A_{10}\partial_{x_1}V + A_{20}\partial_{x_2}V + A_{30}\partial_{x_3}V + C_1\sigma_{x_1} + C_2\sigma_{x_2} + C_3V = f. \tag{4.12}$$

Here $A_{10}=A_1$ in (4.5) and

$$A_{20} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.13}$$

$$A_{30} = \begin{pmatrix} -a^{-2}\rho^{-1}\lambda v_1 & -\lambda & 0 & 1 & 0 \\ -\lambda & -\lambda\rho v_1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda\rho v_1 & 0 & 0 \\ 1 & 0 & 0 & -\lambda\rho v_1 & 0 \\ 0 & 0 & 0 & 0 & -\lambda\rho v_1 \end{pmatrix}. \tag{4.14}$$

The explicit forms of C_1, C_2, and C_3 are of no consequence in the following discussion.

Direct computation shows that A_{30} has one negative triple eigenvalue $-\lambda\rho v_1$ and other two eigenvalues satisfy the quadratic equation

$$y^2 + \lambda v_1\left(\rho + \frac{1}{a^2\rho}\right)y - \frac{1}{a^2}(a^2 + a^2\lambda^2 - \lambda^2 v_1^2) = 0. \tag{4.15}$$

Lax' shock inequality implies that the normal velocity behind the shock front is subsonic, hence $a^2 - v_n^2 > 0$. The quantity $(a^2 + a^2\lambda^2 - \lambda^2 v_1^2)$ in (4.15) will be used often later and will be denoted as

$$\nu^2 = (a^2 + a^2\lambda^2 - \lambda^2 v_1^2) = (1+\lambda^2)(a^2 - v_n^2) > 0. \tag{4.16}$$

Therefore (4.15) has one positive root and one negative root and matrix A_{30} has four negative eigenvalues and one positive eigenvalue.

Denote U_0 and U_1 the upstream and the downstream state of shock front, respectively. To simplify the notation, we drop the subscript 1 when there is no confusion:

$$U_0 = (p_0, v_{10}, 0, v_{30}, S_0), \quad U_1 = (p_1, v_{11}, 0, 0, S_1) \equiv (p, v_1, 0, 0, S).$$

The linearization of boundary condition (4.9) has the form

$$\mathbf{a}_1 \partial_{x_1}\sigma + \mathbf{a}_2 \partial_{x_2}\sigma + BV = g. \tag{4.17}$$

Here $\mathbf{a}_1$ and $\mathbf{a}_2$ are vectors in $\mathbb{R}^5$:

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ 0 \\ a_{14} \\ a_{15} \end{pmatrix} \equiv \begin{pmatrix} \rho v_1 - \rho_0 v_{10} \\ \rho v_1^2 + p - \rho_0 v_{10}^2 - p_0 \\ 0 \\ -\rho_0 v_{10} v_{30} \\ (\rho E + p)v_1 - (\rho_0 E_0 + p_0)v_{10} \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 0 \\ p - p_0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.18}$$

and B is a 5×5 matrix defined by the following differential evaluated at uniform oblique shock front:

$$BdU \equiv \lambda d\begin{pmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ \rho v_1 v_3 \\ (\rho E + p)v_1 \end{pmatrix} - d\begin{pmatrix} \rho v_3 \\ \rho v_1 v_3 \\ \rho v_2 v_3 \\ \rho v_3^2 + p \\ (\rho E + p)v_3 \end{pmatrix}. \tag{4.19}$$

Denote

$$\|u\|_\eta = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-2\eta x_1}|u(x)|^2 dx_3 dx_2 dx_1\right)^{1/2},$$

$$|u|_\eta = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\eta x_1}|u(x_1,x_2,0)|^2 dx_2 dx_1\right)^{1/2},$$

$$|u|_{1,\eta} = \left(\sum_{t_0+t_1+t_2\le 1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta^{2t_0} e^{-2\eta x_1}|\partial_{x_1}^{t_1}\partial_{x_2}^{t_2} u(x_1,x_2,0)|^2 dx_2 dx_1\right)^{1/2}.$$

The boundary value problem (4.12)(4.17) is said to be well posed and the steady oblique shock front is linearly stable if there is an $\eta_0 > 0$ and a constant C_0 such that

$$\eta\|V\|_\eta^2 + |V|_\eta^2 + |\sigma|_{1,\eta}^2 \leqslant C_0\left(\frac{1}{\eta}\|f\|_\eta^2 + |g|_\eta^2\right) \tag{4.20}$$

for all solutions $(V,\sigma) \in C_0^\infty(\mathbb{R}^1 \times \mathbb{R}^2) \times C_1^\infty(\mathbb{R}^2)$ of (4.1)(4.2) and for all $\eta \geqslant \eta_0$.

Denote

$$\tilde{\mathbf{a}}(s,i\omega) = s\mathbf{a}_1 + i\omega\mathbf{a}_2, \tag{4.21}$$

then we have from (4.18),

$$\tilde{\mathbf{a}}(s,i\omega) \neq 0 \quad \text{on } |s|^2 + |\omega|^2 = 1. \tag{4.22}$$

Let Π be the projector in C^5 in the direction of vector $\tilde{\mathbf{a}}(s,i\omega)$, then

$$p(s,i\omega) = (I - \Pi)B \tag{4.23}$$

is a 5×5 matrix of rank 4, with elements being symbols in S^0, i.e., functions of zero-degree homogeneous in $(s,i\omega)$, see Ref. 27. The study of linear stability of oblique shock front under perturbation is reduced to the investigation of Kreiss' condition for the following boundary value problem:

$$\begin{aligned} A_1\partial_{x_1}V + A_{20}\partial_{x_2}V + A_{30}\partial_{x_3}V = f_1 \quad &\text{in } x_3 < 0, \\ PV = g_1 \quad &\text{on } x_3 = 0. \end{aligned} \tag{4.24}$$

Here P is the zero-order pseudodifferential operator[37] with symbol $p(s,i\omega)$ in (4.23).

The stability result of this section is the following theorem about the well posedness of (4.24).

Theorem 4.1: The linear boundary value problem (4.24), describing the linear stability of steady oblique plane shock front, is well posed in the sense of Kreiss[22,29,28] if the following is obtained.

1. $\rho > \rho_0$, i.e., the shock is compressive. This is the usual entropy condition.
2. The downstream flow is supersonic, i.e., $v_1 > a_-$. This guarantees the hyperbolicity of system in (4.24).
3. The following condition on the strength of shock front $\rho/\rho_0 - 1$ is satisfied

$$\left(\frac{v_n}{|\mathbf{v}|}\right)^2\left(\frac{\rho}{\rho_0} - 1\right) < 1. \tag{4.25}$$

The above conditions are also necessary for the problem (4.24) with constant coefficients.

To prove Theorem 4.1 (and hence Theorem 2.1), we construct the matrix $M(s,i\omega)$ as in Refs. 22, 29, and 28

$$M(s,i\omega) = -A_{30}^{-1}(sA_1 + i\omega A_{20}). \tag{4.26}$$

We have

$$sA_1 + i\omega A_{20} = \begin{pmatrix} sv_1/a^2\rho & s & i\omega & 0 & 0 \\ s & s\rho v_1 & 0 & 0 & 0 \\ i\omega & 0 & s\rho v_1 & 0 & 0 \\ 0 & 0 & 0 & s\rho v_1 & 0 \\ 0 & 0 & 0 & 0 & s\rho v_1 \end{pmatrix}$$

and

$$A_{30}^{-1} = \frac{(\lambda\rho v_1)^2}{|D|}\begin{pmatrix} (\lambda\rho v_1)^2 & -\lambda^2\rho v_1 & 0 & \lambda\rho v_1 & 0 \\ -\lambda^2\rho v_1 & (\lambda^2 v_1^2/a^2)-1 & 0 & -\lambda & 0 \\ 0 & 0 & -\nu^2/a^2 & 0 & 0 \\ \lambda\rho v_1 & -\lambda & 0 & \lambda^2((v_1^2/a^2)-1) & 0 \\ 0 & 0 & 0 & 0 & -\nu^2/a^2 \end{pmatrix},$$

where $|D|=(\lambda\rho v_1)^3\nu^2/a^2>0$ is the determinant of A_{30} and

$$\nu^2 = (a^2 + a^2\lambda^2 - \lambda^2 v_1^2) = (1+\lambda^2)(a^2 - v_n^2) > 0.$$

Consider the eigenvalue and eigenvectors of matrix $N(s,i\omega)$:

$$N(s,i\omega) \equiv \frac{|D|}{(\rho\lambda v_1)^2}M(s,i\omega), \tag{4.27}$$

which has the following expression by straightforward computation:

$$N(s,i\omega) = \begin{pmatrix} s\lambda^2\rho v_1(1-(v_1^2/a^2)) & 0 & -i\omega(\lambda\rho v_1)^2 & -s\lambda(\rho v_1)^2 & 0 \\ s & s\rho v_1\nu^2/a^2 & i\omega\lambda^2\rho v_1 & s\lambda\rho v_1 & 0 \\ i\omega\nu^2/a^2 & 0 & s\rho v_1\nu^2/a^2 & 0 & 0 \\ s\lambda(1-(v_1^2/a^2)) & 0 & -i\omega\lambda\rho v_1 & s\lambda^2\rho v_1(1-(v_1^2/a^2)) & 0 \\ 0 & 0 & 0 & 0 & s\rho v_1\nu^2/a^2 \end{pmatrix}.$$

Beside the obvious double eigenvalue $\xi_1 = s\rho v_1\nu^2/a^2$, other eigenvalues are roots of

$$\det\begin{vmatrix} s\lambda^2\rho v_1(1-(v_1^2/a^2))-\xi & -i\omega(\lambda\rho v_1)^2 & -s\lambda(\rho v_1)^2 \\ i\omega\nu^2/a^2 & s\rho v_1\nu^2/a^2-\xi & 0 \\ s\lambda(1-(v_1^2/a^2)) & -i\omega\lambda\rho v_1 & s\lambda^2\rho v_1(1-(v_1^2/a^2))-\xi \end{vmatrix} = 0.$$

Hence the five eigenvalues for $N(s,i\omega)$ are

$$\begin{gathered} \xi_1 = \xi_2 = \xi_3 = s\rho v_1\nu^2/a^2, \\ \xi_{4,5} = s\lambda^2\rho v_1\left(1-\frac{v_1^2}{a^2}\right) \pm \lambda\rho v_1 a^{-1}\sqrt{s^2(v_1^2-a^2)+\omega^2\nu^2}. \end{gathered} \tag{4.28}$$

By $\nu^2 = a^2+\lambda^2 a^2-\lambda^2 v_1^2>0$, we have

$$(\lambda\rho v_1 a^{-1})^2(v_1^2-a^2) > (\lambda^2\rho v_1)^2\left(\frac{v_1^2}{a^2}-1\right)^2.$$

For $\eta=\mathrm{Re}\, s>0$, one of $\xi_{4,5}$ has positive real part and one has negative real part in (4.28). Consequently $N(s,i\omega)$ has four eigenvalues with positive real parts and one with negative real part when $\eta>0$.

For the eigenvalues ξ_1,ξ_2,ξ_3,ξ_4 which have positive real parts when $\eta>0$, we compute the corresponding eigenvectors or generalized eigenvectors for $N(s,i\omega)$.

For the triple eigenvalue $\xi_1=\xi_2=\xi_3$, there are three linearly independent eigenvectors:

$$\alpha_1 = (0,1,0,0,0)^T,$$

$$\alpha_2 = (0,0,s,-i\omega\lambda,0)^T, \tag{4.29}$$

$$\alpha_3 = (0,0,0,0,1)^T.$$

Since

$$s\lambda^2\rho v_1\left(1-\frac{v_1^2}{a^2}\right)-\xi_4=-\lambda\rho v_1 a^{-1}\mu, \quad s\rho v_1\nu^2/a^2-\xi_4=\rho v_1(s-\lambda a^{-1}\mu),$$

the eigenvector α_4 corresponding to the eigenvalue ξ_4 is parallel to

$$\alpha_4 = (-\rho v_1(s-\lambda a^{-1}\mu), s-\lambda a^{-1}\mu, i\omega\nu^2/a^2, a^{-1}\mu - s\lambda(v_1^2/a^2-1), 0)^T, \tag{4.30}$$

where

$$\mu \equiv \sqrt{s^2(v_1^2-a^2)+\omega^2\nu^2}. \tag{4.31}$$

The four eigenvectors α_1, α_2, α_3, and α_4 are linearly independent at $s\neq\lambda a^{-1}\mu$.

At $s=\lambda a^{-1}\mu$, we have $s^2=\lambda^2\omega^2$ and $\xi_1=\xi_2=\xi_3=\xi_4$. α_4 is parallel to $(0,0,i\omega,-a^{-1}\mu,0)^T$ which is parallel to α_2 at $s=\lambda a^{-1}\mu$. A generalized eigenvector needs to be computed.

A. Simplify (4.17)

The Kreiss' condition for (4.24) requires that five vectors $(B\alpha_1, B\alpha_2, B\alpha_3, B\alpha_4)$ and $s\mathbf{a}_1 + i\omega\mathbf{a}_2$ are linearly independent of $|s|^2+|\omega|^2=1$, $\eta\geqslant 0$.

We first simplify the following operator in (4.17) by elementary row operation:

$$\begin{pmatrix} sa_{11} \\ sa_{12} \\ i\omega a_{23} \\ sa_{14} \\ sa_{15} \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ d(\rho v_1^2+p) \\ d(\rho v_1 v_2) \\ d(\rho v_1 v_3) \\ d(\rho v_1 E + p v_1) \end{pmatrix} - \begin{pmatrix} d(\rho v_3) \\ d(\rho v_1 v_3) \\ d(\rho v_2 v_3) \\ d(\rho v_3^2+p) \\ d(\rho v_3 E + p v_3) \end{pmatrix}. \tag{4.32}$$

Noticing that the linearization is at the uniform oblique shock front, (4.32) becomes

$$\begin{pmatrix} sa_{11} \\ sa_{12} \\ i\omega a_{23} \\ sa_{14} \\ s(a_{15}-Ea_{11}) \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ d(\rho v_1^2+p) \\ \rho v_1 dv_2 \\ \rho v_1 dv_3 \\ \rho v_1 dE + d(p v_1) \end{pmatrix} - \begin{pmatrix} \rho dv_3 \\ \rho v_1 dv_3 \\ 0 \\ dp \\ p dv_3 \end{pmatrix}$$

and

$$\begin{pmatrix} sa_{11} \\ s(a_{12}-v_1 a_{11}) \\ i\omega a_{23} \\ sa_{14} \\ s(a_{15}-Ea_{11}) \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ \rho v_1 dv_1 + dp \\ \rho v_1 dv_2 \\ \rho v_1 dv_3 \\ \rho v_1 dE + d(p v_1) \end{pmatrix} - \begin{pmatrix} \rho dv_3 \\ 0 \\ 0 \\ dp \\ p dv_3 \end{pmatrix}.$$

Since $dE=de+v_1 dv_1$, (4.32) further changes into

$$\begin{pmatrix} sa_{11} \\ s(a_{12}-v_1a_{11}) \\ i\omega a_{23} \\ sa_{14} \\ s(a_{15}-(E-v_1^2)a_{11}-v_1a_{12}) \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ \rho v_1 dv_1 + dp \\ \rho v_1 dv_2 \\ \rho v_1 dv_3 \\ \rho v_1 de + p dv_1 \end{pmatrix} - \begin{pmatrix} \rho dv_3 \\ 0 \\ 0 \\ dp \\ p dv_3 \end{pmatrix}.$$

Multiplying first row by $-p/\rho$ and adding to the fifth row, we obtain

$$\begin{pmatrix} sa_{11} \\ s(a_{12}-v_1a_{11}) \\ i\omega a_{23} \\ sa_{14} \\ s(a_{15}-(E-v_1^2+p/\rho)a_{11}-v_1a_{12}) \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ \rho v_1 dv_1 + dp \\ \rho v_1 dv_2 \\ \rho v_1 dv_3 \\ \rho v_1 (de - p/\rho^2 d\rho) \end{pmatrix} - \begin{pmatrix} \rho dv_3 \\ 0 \\ 0 \\ dp \\ 0 \end{pmatrix}.$$

Because $de = TdS - pd\tau$, $(\tau = 1/\rho)$, (4.32) finally becomes

$$\begin{pmatrix} sa_{11} \\ s(a_{12}-v_1a_{11}) \\ i\omega a_{23} \\ sa_{14} \\ s(a_{15}-(E-v_1^2+p/\rho)a_{11}-v_1a_{12}) \end{pmatrix} + \lambda \begin{pmatrix} d(\rho v_1) \\ \rho v_1 dv_1 + dp \\ \rho v_1 dv_2 \\ \rho v_1 dv_3 \\ \rho v_1 TdS \end{pmatrix} - \begin{pmatrix} \rho dv_3 \\ 0 \\ 0 \\ dp \\ 0 \end{pmatrix}.$$

Therefore, (4.17) is equivalent to

$$\mathbf{b}_1 \partial_{x_1} \sigma + \mathbf{b}_2 \partial_{x_2} \sigma + B_1 V = g, \tag{4.33}$$

with

$$\mathbf{b}_1 = \begin{pmatrix} b_{11} \\ b_{12} \\ 0 \\ b_{14} \\ b_{15} \end{pmatrix} \equiv \begin{pmatrix} \rho v_1 - \rho_0 v_{10} \\ a_{12} - v_1 a_{11} \\ 0 \\ -\rho_0 v_{10} v_{30} \\ a_{15} - (E - v_1^2 + p/\rho)a_{11} - v_1 a_{12} \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \\ p - p_0 \\ 0 \\ 0 \end{pmatrix},$$

and 5×5 matrix B_1 is

$$B_1 \equiv \begin{pmatrix} \lambda v_1/a^2 & \lambda\rho & 0 & -\rho & 0 \\ \lambda & \lambda\rho v_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda\rho v_1 & 0 & 0 \\ -1 & 0 & 0 & \lambda\rho v_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda\rho v_1 T \end{pmatrix}. \tag{4.34}$$

Here in computing B_1, we have made use of the fact that the flow satisfies system (1.2) behind the shock front.

B. Case I: $s \neq \lambda a^{-1}\mu$

Consider the five vectors $(B_1\alpha_1, B_1\alpha_2, B_1\alpha_3, B_1\alpha_4)$ and $s\mathbf{b}_1 + i\omega\mathbf{b}_2$, where B_1 and $\mathbf{b}_j$ are defined as above.

- Vector $B_1\alpha_1 = (\lambda\rho, \lambda\rho v_1, 0, 0, 0)^T$ is parallel to and hence can be replaced by

$$\zeta_1 = (1, v_1, 0, 0, 0)^T.$$

- Vector $B_1\alpha_2=(i\omega\lambda\rho,0,s\lambda\rho v_1,-i\omega\lambda^2\rho v_1,0)^T$ is parallel to

$$\zeta_2=(i\omega,0,sv_1,-i\omega\lambda v_1,0)^T.$$

- Vector $B_1\alpha_3=(0,0,0,0,\lambda\rho v_1 T)^T$ is parallel to

$$\zeta_3=(0,0,0,0,1)^T.$$

- Vector $B_1\alpha_4=(-\rho\mu\nu^2/a^3,0,i\omega\lambda\rho v_1\nu^2/a^2,s\rho v_1\nu^2/a^2,0)^T$ is parallel to

$$\zeta_4=(-a^{-1}\mu,0,i\omega\lambda v_1,sv_1,0)^T.$$

- Vector $s\mathbf{b}_1+i\omega\mathbf{b}_2\equiv\zeta_5$ can be simplified by using Rankine-Hugoniot relations satisfied by the states U_0 and U_1:

$$\begin{aligned}
&\lambda(\rho v_1-\rho_0 v_{10})+\rho_0 v_{30}=0,\\
&\lambda(\rho v_1^2+p-\rho_0 v_{10}^2-p_0)+\rho_0 v_{10}v_{30}=0,\\
&\lambda\rho_0 v_{10}v_{30}+(p-\rho_0 v_{30}^2-p_0)=0,\\
&\lambda((\rho E+p)v_1-(\rho_0E_0+p_0)v_{10})+(\rho_0E_0+p_0)v_{30}=0.
\end{aligned}\tag{4.35}$$

Solving $p-p_0$ from the third equation in (4.35)

$$p-p_0=-\lambda\rho_0 v_{10}v_{30}+\rho_0 v_{30}^2=\rho_0 v_{30}(v_{30}-\lambda v_{10})$$

and substituting it into the second equation in (4.35), we obtain

$$\lambda(\rho v_1^2-\rho_0 v_{10}^2+\rho_0 v_{30}(v_{30}-\lambda v_{10}))+\rho_0 v_{10}v_{30}=0,$$

which simplifies to

$$\lambda\rho v_1^2=\rho_0(v_{10}+\lambda v_{30})(\lambda v_{10}-v_{30}).$$

From the first equation in (4.32), we obtain

$$\lambda\rho v_1=\rho_0(\lambda v_{10}-v_{30}).$$

Combining the two relations above, we obtain

$$v_1=v_{10}+\lambda v_{30}.$$

Therefore, we have

$$\rho_0 v_{10}=\frac{\rho_0+\lambda^2\rho}{1+\lambda^2}v_1,\quad \rho_0 v_{30}=\frac{\lambda(\rho_0-\rho)}{1+\lambda^2}v_1.$$

Consequently we obtain

$$\begin{aligned}
&\rho v_1-\rho_0 v_{10}=\frac{\rho-\rho_0}{1+\lambda^2}v_1,\\
&\rho v_1^2-\rho_0 v_{10}^2+p-p_0=\frac{(\rho-\rho_0)(\rho_0+\lambda^2\rho)}{\rho_0(1+\lambda^2)^2}v_1^2,\\
&p-p_0=\frac{\lambda^2 v_1^2}{1+\lambda^2}\frac{\rho(\rho-\rho_0)}{\rho_0}=\frac{\rho(\rho-\rho_0)}{\rho_0}v_n^2,
\end{aligned}\tag{4.36}$$

$$-\rho_0 v_{10} v_{30} = \frac{\lambda(\rho-\rho_0)(\rho_0+\lambda^2\rho)}{\rho_0(1+\lambda^2)^2} v_1^2.$$

Therefore we obtain

$$\zeta_5 = \begin{pmatrix} s\frac{\rho-\rho_0}{1+\lambda^2}v_1 \\ s\frac{(\rho-\rho_0)^2 v_1^2\lambda^2}{\rho_0(1+\lambda^2)^2} \\ i\omega\frac{\lambda^2 v_1^2}{1+\lambda^2}\frac{\rho(\rho-\rho_0)}{\rho_0} = \frac{\rho(\rho-\rho_0)}{\rho_0}v_n^2 \\ s\frac{\lambda(\rho-\rho_0)(\rho_0+\lambda^2\rho)}{\rho_0(1+\lambda^2)^2}v_1^2 \\ sb_{15} \end{pmatrix}, \tag{4.37}$$

where b_{15} can be computed from Rankine-Hugoniot condition

$$b_{15} = -\frac{(\rho-\rho_0)^2 v_1^3 \lambda^2}{\rho_0(1+\lambda^2)^2}.$$

It will be obvious in the following that the explicit form of b_{15} is of no importance. Hence ζ_5=is parallel to

$$(s(1+\lambda^2)\rho_0, s(\rho-\rho_0)\lambda^2 v_1, i\omega(1+\lambda^2)\lambda^2\rho v_1, s\lambda(\rho_0+\lambda^2\rho)v_1, -s(\rho-\rho_0)\lambda^2 v_1^2)^T.$$

Kreiss' condition states that the oblique steady shock front is linearly stable if five vectors $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5$ are linearly independent, or the following matrix with these five vectors as column vectors is uniformly nondegenerate on $|s|^2+|\omega|^2=1$, $\eta>0$:

$$\begin{pmatrix} 1 & i\omega & 0 & -a^{-1}\mu & s(1+\lambda^2)\rho_0 \\ v_1 & 0 & 0 & 0 & s(\rho-\rho_0)\lambda^2 v_1 \\ 0 & sv_1 & 0 & i\omega\lambda v_1 & i\omega(1+\lambda^2)\lambda^2\rho v_1 \\ 0 & -i\omega\lambda v_1 & 0 & sv_1 & s\lambda(\rho_0+\lambda^2\rho)v_1 \\ 0 & 0 & 1 & 0 & -s(\rho-\rho_0)\lambda^2 v_1^2 \end{pmatrix}. \tag{4.38}$$

Obviously, it is nondegenerate if and only if the following 4×4 matrix J is nondegenerate:

$$J = \begin{pmatrix} 1 & i\omega & -\mu & s(1+\lambda^2)\rho_0 \\ 1 & 0 & 0 & s(\rho-\rho_0)\lambda^2 \\ 0 & s & i\omega\lambda a & i\omega(1+\lambda^2)\lambda^2\rho \\ 0 & -i\omega\lambda & sa & s\lambda(\rho_0+\lambda^2\rho) \end{pmatrix}. \tag{4.39}$$

Compute the determinant of J,

$$\det J = s^3 a(\lambda^2\rho-\rho_0-2\lambda^2\rho_0) - sa\omega^2\lambda^2(1+\lambda^2)(\rho-2\rho_0) - \lambda\mu[s^2(\rho_0+\lambda^2\rho) - \omega^2(1+\lambda^2)\lambda^2\rho]. \tag{4.40}$$

We have the following lemma.

Lemma 4.1: Under condition (4.25), there exists an $\epsilon>0$ such that for all (s,ω), with $s \neq \lambda a^{-1}\mu$,

$$|\det J| \geq \epsilon, \quad \forall |s|^2 + |\omega|^2 = 1, \quad \eta = \mathrm{Re}\, s > 0. \tag{4.41}$$

Proof: Noticing that (4.40) is the same as (4.20) in Ref. 23, we can prove similarly as in Ref. 23. Hence the details are omitted here.

C. Case II: $s=\lambda a^{-1}\mu$

In the case $s=\lambda a^{-1}\mu$, we have $s=\lambda\omega>0$ and $\mu=\omega a>0$. Since

$$s\lambda^2\rho v_1\left(1-\frac{v_1^2}{a^2}\right)-\xi_1=-s\rho v_1,$$

$$N(s,i\omega)-\xi_1 I=\begin{pmatrix} -s\rho v_1 & 0 & -i\omega(\lambda\rho v_1)^2 & -s\lambda(\rho v_1)^2 & 0 \\ s & 0 & i\omega\lambda^2\rho v_1 & s\lambda\rho v_1 & 0 \\ i\omega\nu^2/a^2 & 0 & 0 & 0 & 0 \\ s\lambda(1-(v_1^2/a^2)) & 0 & -i\omega\lambda\rho v_1 & -s\rho v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.42}$$

At the point $s=\lambda a^{-1}\mu$, the vectors α_2 in (4.29) and α_4 in (4.30) are parallel, and there are only three linearly independent eigenvectors corresponding to the eigenvalue ξ_1:

$$\begin{aligned} \alpha_1 &= (0,1,0,0,0)^T, \\ \alpha_2 &= (0,0,1,-i,0)^T, \\ \alpha_3 &= (0,0,0,0,1)^T. \end{aligned} \tag{4.43}$$

A generalized eigenvector α_4' corresponding to ξ_1 can be found by solving the equation $(N(s,i\omega)-\xi_1 I)\alpha_4'=\alpha_2$, i.e.,

$$\begin{aligned} a^2\eta_1+\lambda\rho v_1(i\eta_3+\eta_4) &= 0, \\ a^2\eta_1+\lambda\rho v_1(i\eta_3+\eta_4) &= 0, \\ i\nu^2\eta_1 &= 1, \\ \lambda^2(a^2-v_1^2)\eta_1-\lambda\rho v_1(i\eta_3+\eta_4) &= -i. \end{aligned} \tag{4.44}$$

System (4.44) has a solution of generalized eigenvector

$$\alpha_4'=(-ia^2\omega^{-1}\nu^{-2},0,a^2(\lambda\omega\rho v_1)^{-1}\nu^{-2},0,0)^T,$$

which is parallel to

$$(\lambda\rho v_1,0,i,0,0)^T.$$

Computing $B_1\alpha_1$, $B_1\alpha_2$, $B_1\alpha_3$, $B_1\alpha_4'$, and $s\mathbf{b}_1+i\omega\mathbf{b}_2$ at $s=\lambda a^{-1}\mu$, we obtain the matrix corresponding to (4.38) as follows

$$\begin{pmatrix} 1 & i\omega & 0 & \lambda^2\rho v_1^2 a^{-2} & s(1+\lambda^2)\rho_0 \\ v_1 & 0 & 0 & \lambda^2\rho v_1 & s(\rho-\rho_0)\lambda^2 v_1 \\ 0 & s v_1 & 0 & i\lambda\rho v_1 & i\omega(1+\lambda^2)\lambda^2\rho v_1 \\ 0 & -i\omega\lambda v_1 & 0 & -\lambda\rho v_1 & s\lambda(\rho_0+\lambda^2\rho)v_1 \\ 0 & 0 & 1 & 0 & -s(\rho-\rho_0)\lambda^2 v_1^2 \end{pmatrix}, \tag{4.45}$$

which is nondegenerate if and only if

$$\det\begin{pmatrix} 1 & i\omega & \lambda^2\rho v_1^2 a^{-2} & s(1+\lambda^2)\rho_0 \\ v_1 & 0 & \lambda^2\rho v_1 & s(\rho-\rho_0)\lambda^2 v_1 \\ 0 & s v_1 & i\lambda\rho v_1 & i\omega(1+\lambda^2)\lambda^2\rho v_1 \\ 0 & -i\omega\lambda v_1 & -\lambda\rho v_1 & s\lambda(\rho_0+\lambda^2\rho)v_1 \end{pmatrix} \neq 0,$$

i.e.,

$$\det J' \equiv \det\begin{pmatrix} 1 & -1 & \lambda^2 v_1^2 a^{-2} & (1+\lambda^2)\rho_0 \\ 1 & 0 & \lambda^2 & (\rho-\rho_0)\lambda^2 \\ 0 & 1 & 1 & (1+\lambda^2)\rho \\ 0 & 1 & -1 & (\rho_0+\lambda^2\rho) \end{pmatrix} \neq 0. \tag{4.46}$$

It is readily checked that

$$\det J' = (\rho-\rho_0)\nu^2/a^2 + 2(\rho+\rho_0) + \lambda^2(3\rho+\rho_0) > 0.$$

This completes the proof for the case $s=\lambda a^{-1}\mu$. The proof of Theorem 4.1 and hence Theorem 2.1 is complete.

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