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with $\mu \neq 0$. A solution for G is

$$G(t) = G_0 e^{-\mu^2 t/a} \tag{5.17}$$

To construct a solution for F, we must also construct $k(x, y)$. After a bit of trial and error, we found the following solution for $\mu = 1$:

$$\begin{aligned} F(x,y) &= e^x \cos y \\ k(x,y) &= e^x \sin y - x \end{aligned} \tag{5.18}$$

This solution can be scaled to ensure it satisfies the requirements outlined in Section 5.2.1.

5.2.5.2 *Steady incompressible flow with no source term*

Manufactured solutions can also be constructed for nonlinear systems of homogeneous equations. We illustrate using the equations for two-dimensional, steady, incompressible, laminar flow:

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\frac{\partial}{\partial x}\left(u^2 + h\right) + \frac{\partial}{\partial y}(uv) = \upsilon \nabla^2 u \\ &\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}\left(v^2 + h\right) = \upsilon \nabla^2 v \end{aligned} \tag{5.18}$$

with $h = P/\rho$ and υ a constant. To satisfy the continuity equation, let $\phi = \phi(x, y)$ and set

$$\begin{aligned} u &= -\frac{\partial \phi}{\partial y} \\ v &= +\frac{\partial \phi}{\partial x} \end{aligned} \tag{5.19}$$

The momentum equations then become

$$\begin{aligned} \frac{\partial h}{\partial x} &= R \\ \frac{\partial h}{\partial y} &= Q \end{aligned} \tag{5.20}$$

where

$$R = \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} - \upsilon \frac{\partial}{\partial y}\left(\nabla^2 \phi\right)$$
$$Q = \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial y} - \upsilon \frac{\partial}{\partial x}\left(\nabla^2 \phi\right) \tag{5.21}$$

In order for h to exist, we must have

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial x} \tag{5.22}$$

This means that ϕ must satisfy

$$\upsilon \nabla^4 \phi - \frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2 \phi\right) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2 \phi\right) = 0 \tag{5.23}$$

To construct a manufactured solution, choose ϕ so that $\nabla^2\phi = \mu$ (a constant). Then the last equation is automatically satisfied. The velocity components u and v are then computed from the derivatives of ϕ. By computing R and Q from ϕ, one can find h by integration. For example, let

$$\phi(x, y) = e^x \cos y - e^y \sin x \tag{5.24}$$

Then $\mu = 0$ because $\nabla^2\phi = 0$. From the derivatives of ϕ, we find

$$u(x, y) = e^x \sin y + e^y \sin x$$
$$v(x, y) = e^x \cos y - e^y \cos x \tag{5.25}$$

The functions R and Q are then

$$R(x, y) = -e^{2x} - e^{x+y}\left[\sin(x+y) - \cos(x+y)\right]$$
$$Q(x, y) = -e^{2y} - e^{x+y}\left[\sin(x+y) - \cos(x+y)\right] \tag{5.26}$$

and finally,

$$h(x, y) = -\frac{1}{2}e^{2x} - \frac{1}{2}e^{2y} + e^{x+y}\cos(x+y) \tag{5.27}$$