Inversion of the Prandtl-Meyer relation

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SUMMARY

A simple rational fraction is derived which gives the Mach number with an error of less than 0·05 per cent for a given value of the Prandtl-Meyer function.

INTRODUCTION

In the isentropic expansion from sonic conditions through a simple wave in two-dimensional flow the turning angle ν is related to the Mach number M by the Prandtl-Meyer relation.

$$\nu = \frac{1}{\lambda}\tan^{-1}\lambda\beta - \tan^{-1}\beta \quad . \quad . \quad . \quad (1)$$

where

$$\beta = \sqrt{M^2 - 1} \text{ and } \lambda = \sqrt{\frac{\gamma - 1}{\gamma + 1}} \quad . \quad . \quad (2)$$

In many applications it is necessary to obtain the value of M for a given value of ν. One approach is to use the Newton-Raphson method to obtain an iterative formula. Rewriting equation (1) in the form

$$F(\beta) = \frac{1}{\lambda}\tan\{\lambda(\nu + \tan^{-1}\beta)\} - \beta = 0 \quad . \quad (3)$$

and using the standard procedure produces the iterative formula

$$\beta_{n+1} = \beta_n + \frac{(1+\beta_n^2)F(\beta_n)}{(1-\lambda^2)\beta_n^2} \quad . \quad . \quad . \quad (4)$$

Although this gives β to an accuracy of several significant figures after only a few iterations it still requires the repeated calculation of the functions tan and $\tan^{-1}$ whereas a polynomial or rational fraction approximation for M would require much less computing time.

Fraser[1] quotes polynomial approximations which give M to an accuracy of 0·1% over limited ranges for $\gamma = 7/5$ which were obtained by least squares curve fitting using Chebyshev polynomials. A rational fraction approximation is derived below which give a more accurate value of M at any point in the whole range of M from 1 to ∞. The basis of the derivation is to obtain an expression which has the correct asymptotic behaviour at both ends of the range. The numerical values refer to the particular case when $\gamma = 7/5$ but the coefficients are also given as functions of λ so that the formula can be used for other values of γ if required.

ASYMPTOTIC FORMULAE FOR M

For small β equation (1) can be expanded as a series in powers of β. Reversion of this series gives β as a series in powers of $\nu^{2/3}$ which can easily be recast as an equivalent series for M, i.e.:

$$M = 1 + \sum_1^\infty a_n y^n \quad . \quad . \quad . \quad (5)$$

where

$$y = \left[\frac{\nu}{\nu_\infty}\right]^{2/3} \quad . \quad . \quad . \quad (6)$$

and ν_∞, the maximum value of ν, is given by

$$\nu_\infty = \frac{\pi}{2}\left[\frac{1}{\lambda} - 1\right] \quad . \quad . \quad . \quad (7)$$

The coefficients a_1, a_2 and a_3 are given in Table I.

For large β equation (1) can be expanded in powers of β^{-1}. Reversion of this series gives

$$M = \frac{K}{1-y} + \sum_0^\infty b_n(1-y)^n \quad . \quad . \quad . \quad (8)$$

where

$$K = \frac{4}{3\pi}\left(1 + \frac{1}{\lambda}\right) \quad . \quad . \quad . \quad (9)$$

or, for $\gamma = 7/5$,

$$K = 1{\cdot}464009 \quad . \quad . \quad . \quad (10)$$

COMPOSITE EXPANSION FOR M

A composite expansion which has the correct behaviour at the singularity at $y = 1$ and which also has the correct asymptotic behaviour near $y = 0$ is

$$M = \frac{1 + (K-1)y}{1-y} + \sum_1^\infty c_n y^n \quad . \quad . \quad . \quad (11)$$

where

$$c_n = a_n - K \quad . \quad . \quad . \quad (12)$$

Terminating the series after the term $c_3 y^3$ gives M to an accuracy of better than 0·2% over the whole range. The accuracy can be improved, however, by recasting the first three terms of the series as a rational fraction, i.e.:

$$\sum_1^3 c_n y^n = \frac{c_1 y(1 + Ay)}{1 + By} + 0(y^4) \quad . \quad . \quad . \quad (13)$$

where

$$B = -\frac{c_3}{c_2} \text{ and } \qquad A = B + \frac{c_2}{c_1} \quad . \quad . \quad . \quad (14)$$

Inserting the rational fraction into equation (11) and re-writing gives

$$\mathrm{M} = \frac{1 + d_1 y + d_2 y^2 + d_3 y^3}{1 + e_1 y + e_2 y^2} \quad . \quad . \quad . \quad (15)$$

For $\gamma = 1{\cdot}4$

$$\mathrm{M} = \frac{1 + 1{\cdot}3604y + 0{\cdot}0962y^2 - 0{\cdot}5127y^3}{1 - 0{\cdot}6722y - 0{\cdot}3278y^2} \quad . \quad . \quad (16)$$

TABLE I

		$\gamma = 7/5$	
n	a_n	a_n	c_n
1	$\frac{1}{2}\eta_\infty$	2·032646	0·568637
2	$\frac{3 + 8\lambda^2}{40}\eta_\infty^2$	1·790381	0·326372
3	$\frac{-1 + 328\lambda^2 + 104\lambda^4}{2800}\eta_\infty^3$	1·357039	−0·106970

$$\eta_\infty = \left(\frac{3\nu_\infty}{1 - \lambda^2}\right)^{2/3} = \left(\frac{3\pi}{2\lambda(1 + \lambda)}\right)^{2/3}$$

This expression gives M with an error of less than 0·05% over the whole range which corresponds to an uncertainty in ν of less than 0·015 degrees.

If a more accurate value is required, one application of equation (4) will give M correct to eight significant figures.

REFERENCE

1. Fraser, S. M. Calculation of Mach number from given turning angle in supersonic isentropic flow. *The Aeronautical Journal,* Vol 79, No 770, pp 95–96, February 1975.

APPENDIX

For general γ the coefficients d_i and e_i in equation (15) are given below in terms of K, which can be obtained from equation (9), and the a_i's, listed in Table I.

$$d_1 = a_1 - 1 - \frac{a_3 - K}{a_2 - K}$$

$$d_2 = a_2 - a_1 - \frac{(a_1 - 1)(a_3 - K)}{a_2 - K}$$

$$d_3 = \frac{(a_3 - K)(a_1 - K)}{a_2 - K} - a_2 + K$$

$$e_1 = -1 - \frac{a_3 - K}{a_2 - K} = -1 - e_2$$