SPIKE NOZZLE CONTOUR FOR OPTIMUM THRUST

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Abstract—The contour of a spike nozzle (plug nozzle minus the shroud) to give optimum thrust is considered. The spike length and ambient pressure are held constant and the thrust is maximized with respect to the contour of the spike. The exhaust gases are treated as inviscid and the expansion process is assumed to be isentropic. The variational integral is formulated in terms of flow properties along a suitably chosen control surface. The solution of the variational problem leads to certain flow properties along the control surface and the spike contour to give this flow is constructed by using the method of characteristics. An example is carried out in detail and typical spike contours are given. The results are presented in the form of charts that can be used in selecting parameters such as the expansion ratio and spike length to yield a required thrust coefficient.

1. INTRODUCTION

The exhaust nozzle is an important feature of jet propulsion engines that depend upon the reaction force of the exit jet. In the case of most jet engines, and some rocket applications, it would be necessary to let the exhaust gases expand through a nozzle with a central plug. Where no shroud around the central plug is employed and the expansion is controlled only by the ambient pressure and the plug shape, the nozzle is generally referred to as a spike type nozzle. In nozzles of this type, since all expansion occurs externally, there is no tendency for the gases to over-expand at low altitude operation, with subsequent loss in performance. This behaviour, briefly discussed in Reference (1), is also experimentally indicated in Reference (2). Since the performance of spike-type nozzles appears promising, the problem of computing spike contours is discussed in this paper.

For a given altitude condition, maximum thrust can be obtained by expanding the exhaust gases to the ambient pressure such that an exhaust jet of parallel uniform velocity is obtained. For jet engines and rocket motors designed for high altitudes, such nozzles become excessively long and heavy. The plug can be foreshortened without undue loss in thrust.

The experimental data [3] indicate that at low expansion ratios, a simple conical spike with half cone angles as large as 40° can be used with very little loss in thrust. The thrust performance of such short spike nozzles can be improved by utilizing a plug contour to yield optimum thrust. Variations in the plug contour result in different types of exhaust flows. By considering flow properties along a suitably chosen control surface a variational integral can be formulated for thrust optimization. Formulation of such extremal problem in the case of conventional convergent–divergent nozzles was given elsewhere. [4] A similar approach leads to the design of the plug contour to yield optimum thrust when the length and either the ambient pressure or the expansion ratio are prescribed. The formulation of the variational integral for this problem and its solution are discussed in the present paper.

The exhaust gases are treated as inviscid and the expansion process is assumed to be isentropic. The solution of the variational problem yields certain required flow parameters along a control surface and the method of characteristics is used to construct the spike contour to yield the required flow. Typical plug contours are presented for the case of $\gamma = 1{\cdot}23$. The loss in thrust caused by shortening the length of the plug contour is also discussed. The results are presented in the form of charts that can be used in selecting spike nozzle parameters such as the expansion ratio and the length to yield a required thrust coefficient.

2. FORMULATION OF THE PROBLEM

Let us consider a rocket or a jet engine, where the pressure and temperature in the combustion chamber and the propellant mass flow are given. The phrase, propellant mass flow, is used to include both the fuel and the oxidizer, i.e. the mass flow per unit time of the exhaust products through the nozzle is given. As a result the throat area A^* of the nozzle is prescribed. A plug type nozzle as shown in Fig. 1 can be utilized to expand the exhaust gases from a combustion chamber pressure P_c to an ambient pressure p_a. Maximum thrust can be developed by a spike contour that would yield uniform axial exhaust jet at the ambient pressure. If R_E denotes the radius of the cowl lip of such a nozzle, the ratio $\pi R_E^2/A^*$ is equal to one-dimensional area ratio corresponding to the pressure ratio P_c/p_a. Since A^* and P_c/p_a are prescribed we have R_E also defined depending upon the exhaust gas properties. Such a spike would be called "ideal spike for the pressure ratio" and the contour can easily be computed.

For high pressure ratio conditions such ideal spike nozzles will be excessively long and heavy. Let us consider shortening the spike length and seek a contour that would deliver optimum thrust for the prescribed length. Thus, we have the nozzle throat area A^*, length L and ambient pressure p_a all prescribed. It is required to find a spike nozzle that would yield optimum thrust under the above prescribed conditions. Even though the throat area is prescribed, no restriction need be placed on the diameter of the spike. That is, the radius of the cowl lip can be also varied in addition to the spike contour in considering various spike nozzles to meet the above requirement. Similarly, since only the length is prescribed, it is not necessary that the spike contour terminate in a vertex on the axis. Spike contours with a truncated tip as shown in Fig. 1 can also be considered in meeting the requirements.

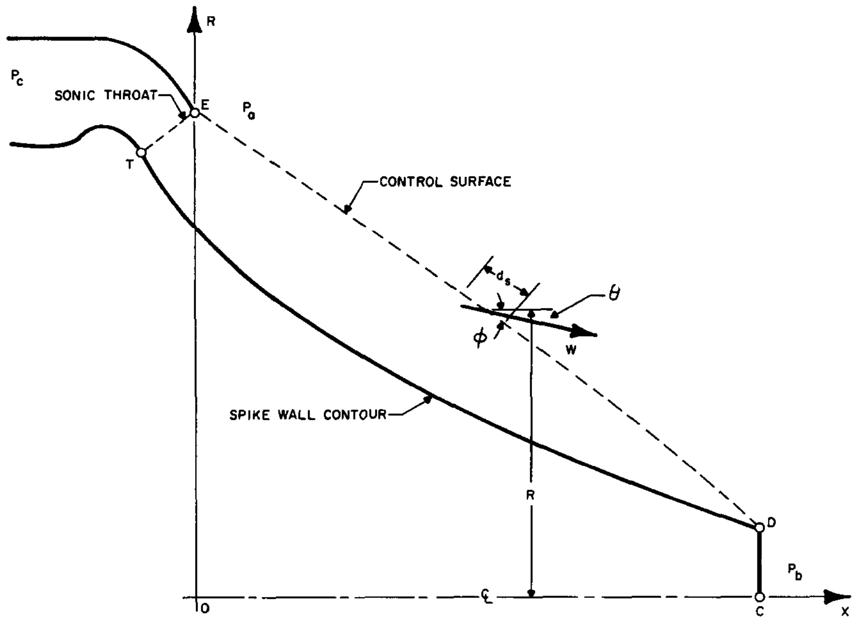

Fig 1. Sketch of a spike nozzle and the control surface.

3. THE VARIATIONAL INTEGRAL AND ITS SOLUTION

Let TD as shown in Fig. 1 represent the meridional cross-section of the spike surface having the prescribed length. Let us consider the control surface, represented by ED, which would encompass all the exit flow of the nozzle. Let p, ρ, w denote the pressure, density and velocity at a point R measured from the axis. Let θ and ϕ (negative values measured from axial) denote respectively the flow angle, and the control surface direction at this point. Let a pressure p_b prevail on truncated portion of the spike represented by CD.

Consider an elemental length ds along ED. The area generated by this element by rotation about the axis is

$$\mathrm{d}A = 2\pi R \cdot \mathrm{d}s$$

The axial distance between the points D and E is held constant in the variation of the plug contour. Hence

$$\int_D^E \cot(-\phi)\, \mathrm{d}R = \text{constant} \qquad (1)$$

The mass flow crossing the control surface is equal to the mass flow through the throat and is a given quantity. Hence, in the variation of the flow parameters, we have to satisfy

$$\int_D^E \rho w \left[\frac{\sin(-\phi+\theta)}{\sin(-\phi)}\right] 2\pi R \mathrm{d}R = \text{constant} \qquad (2)$$

The thrust of the spike nozzle is obtained by integrating impulse function along DE and taking into account the effect of pressures p_a and p_b acting on the respective surfaces. Hence:

Thrust =

$$\int_D^E \left[p + \rho w^2 \frac{\sin(-\phi+\theta)\cos(-\theta)}{\sin(-\phi)}\right] 2\pi R \mathrm{d}R - \\ - \pi R_E^2 p_a + \pi R_D^2 p_b \qquad (3)$$

where R_E and R_D denote the radial coordinates of E and D respectively.

Maximizing thrust under the two above constraints of equations (1) and (2) leads to the following variational integral

$$I = \int_D^E (f_1 + \lambda_2 f_2 + \lambda_3 f_3)\mathrm{d}R - \pi R_E^2 p_a + \pi R_D^2 p_b \quad (4)$$

where

$$f_1 = \left[p + \rho w^2 \frac{\sin(-\phi+\theta)\cos(-\theta)}{\sin(-\phi)}\right] 2\pi R$$

$$f_2 = \rho w \frac{\sin(-\phi+\theta)}{\sin(-\phi)} 2\pi R$$

$$f_3 = \cot(-\phi),$$

λ_2 and λ_3 are the Lagrangian multiplier constants.

In maximizing the above integral, we can vary w, θ and ϕ between D and E. Also, we can vary R_E and R_D, since we required only the length to be constant.

The Euler equations of the variational problem would reduce to

$$\phi = \theta - \alpha \qquad (5)$$

$$w \frac{\cos(-\theta-\alpha)}{\cos\alpha} = -\lambda_2 \qquad (6)$$

and

$$R \rho w^2 \sin^2\theta \tan\alpha = -\lambda_3 \qquad (7)$$

along DE, where α is the Mach angle corresponding to velocity w. Also,

$$[(p - p_b)/\tfrac{1}{2}\rho w^2] \cot\alpha = \sin(-2\theta) \quad \text{at the point } E \quad (8)$$

and

$$[(p - p_a)/\tfrac{1}{2}\rho w^2] \cot\alpha = \sin(-2\theta) \quad \text{at the point } D \quad (9)$$

From equations (6) and (7) one can obtain that the flow parameters along ED satisfy the following relation

$$\mathrm{d}\theta + \frac{\mathrm{d}w}{w}\cot\alpha - \frac{\sin\alpha\sin\theta}{\sin(\theta-\alpha)} \cdot \frac{\mathrm{d}R}{R} = 0 \quad (10)$$

This relation is the compatibility condition between flow parameters along a right characteristic direction in supersonic axisymmetric flow. Compliance of the above condition is necessary since the control surface is a right characteristic according to equation (5). Caution is recommended regarding the use of equation (9). Only if the base pressure p_b is assumed independent of spike contour and the height

CD, this equation (9) can be derived. In general such is not the case and p_b depends upon the geometry of spike vertex and flow conditions at D. One may yet obtain suitable answers if p_b were held constant.

4. METHOD OF COMPUTING OPTIMUM SPIKE NOZZLE PARAMETERS

In the above two sections, the problem has been formulated in terms of a prescribed length and ambient pressure and the solution is given in terms of a control surface and flow parameters along the control surface. In carrying out the computations it is convenient to choose Mach number M_E, and flow direction θ_E on the control surface ED at the cowl-lip E. These values of M_E and θ_E would then serve as parameters to define the ratio $\pi R_E^2/A^*$ and the length of the nozzle. The ambient pressure for which the nozzle is optimized can be obtained, a posteriori, by substituting these parameters into equation (8).

For a typical example, $M_E = 2{\cdot}4$ and $\theta_E = -8{\cdot}25°$ are chosen. These parameters, substituted into equation (8), indicate that the optimum thrust contour thus computed corresponds to an ambient pressure condition $p_a = 0{\cdot}0355\ P_c$. The values of M and θ along the control surface are computed by solving above equations (6) and (7) using $\gamma = 1{\cdot}23$. In Fig. 2 are shown the values thus obtained plotted as functions of nondimensionalized ordinate R/R_E. The solution is to be carried out until the point D is reached compatible with equation (9) and the base pressure P_b assumed. For the computations given in this paper P_b is assumed zero. A higher value of assumed base pressure would terminate the control surface at a larger value of R/R_E. With the flow parameters along the control surface known, the area ratio $\pi R_E^2/A^*$ of the nozzle can be computed from

$$\pi R_E^2/A^* = \pi/\int_E^D (\rho w/\rho^* w^*)[\sin\alpha/\sin(\theta - \alpha)] \times \times 2\pi(R/R_E)\cdot d(R/R_E) \quad (11)$$

Since ϕ, the slope of the control surface, is known, the length of the spike nozzle can be computed from

$$x/R_E = \int_D^E \cot(-\theta - \alpha)\, d\,(R/R_E)$$

Similarly, the thrust coefficient of the spike nozzle is computed from

$$C_F = \frac{\text{Thrust}}{P_c A^*} = \frac{p^*}{P_c}\cdot\frac{\pi R_E^2}{A^*} \times \int_D^E \frac{p}{p^*}\left[1 + \frac{\rho w^2}{p}\frac{\sin\alpha\cos(-\theta)}{\sin(-\theta-\alpha)}\right] 2\frac{R}{R_E} d(R/R_E) \quad (12)$$

In the present example, where $M_E = 2{\cdot}4$, $\theta_E = -8{\cdot}25°$ and $\gamma = 1{\cdot}23$ are chosen, one obtains according to above equations

$X_D/R_E = 1{\cdot}164$; $R_D/R_E = 0{\cdot}137$;
$\pi R_E{}^2/A^* = 3{\cdot}81$ and $C_F = 1{\cdot}58$.

An ideal spike nozzle, i.e. a nozzle yielding uniform exit flow parallel to the nozzle axis can be designed for this area ratio 3·81. Such a nozzle will have a length ratio $X/R_E = 2{\cdot}428$; and a thrust coefficient $C_F = 1{\cdot}591$. Hence, the nozzle in this particular example has a thrust coefficient as high as 0·993 times the ideal thrust coefficient for the area ratio, even though the length is only 47·9 per cent of the length of the ideal spike nozzle.

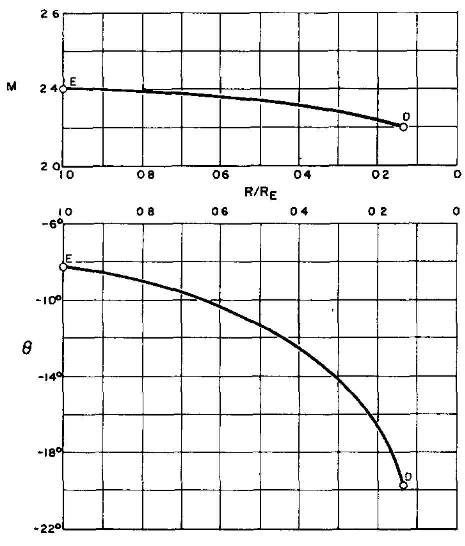

Fig. 2. Variation of M and θ along the control surface.

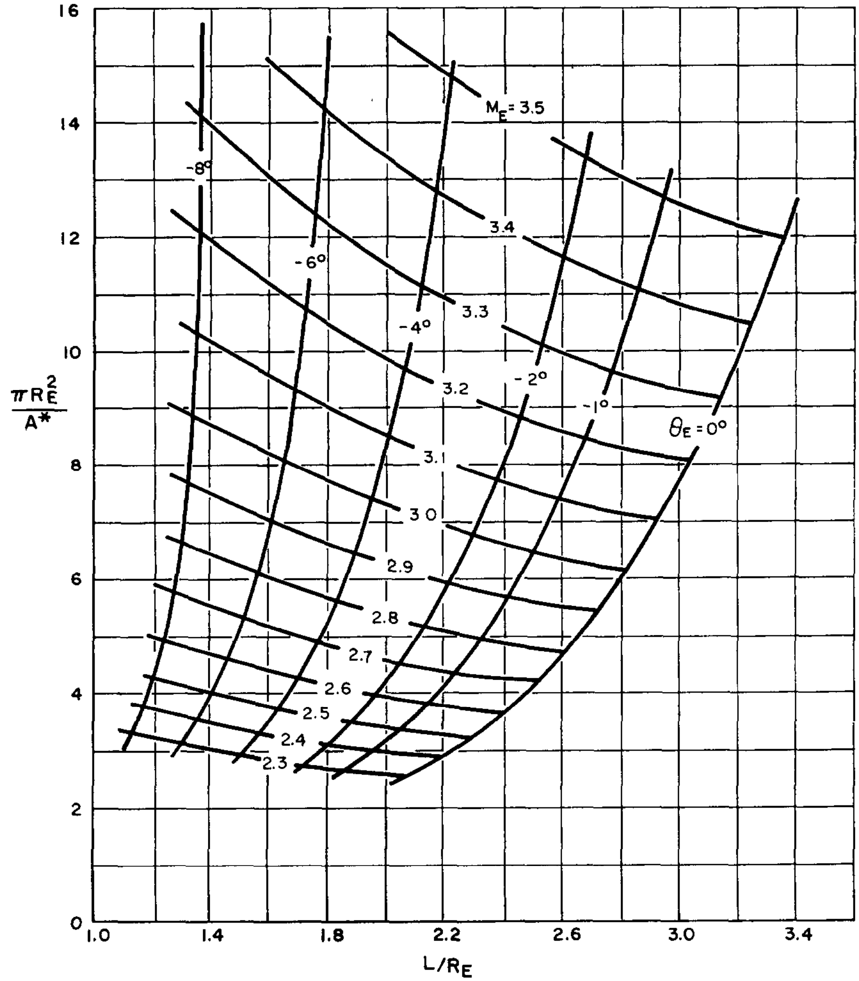

Fig. 3. Area ratio R_E^2/A^* and length ratio L/R_E as functions of parameters M_E and θ_E.

With the same value of M_E, several values of θ_E can be chosen and one obtains various values of area ratio $\pi R_E^2/A^*$, length ratio X/R_E and thrust coefficient C_F. These values are shown in Figs. 3 and 4 in a parametric form. In Fig. 3, the points obtained with the same value M_E are joined and the value of M_E is shown against each such line. Along each constant M_E line points are obtained, by interpolation, for drawing constant thrust coefficient lines in Fig. 4. The thrust coefficient as computed by equation (12) represents the thrust of the spike nozzle when discharging into vacuum. Division by the vacuum thrust-coefficient of an ideal one-dimensional nozzle for the area ratio, as given by equation (11), would represent the performance as a percentage. These percentages are shown in Fig. 4 against each thrust line.

Let us examine the reduction in thrust coefficient caused by considering shorter lengths. This can be readily seen from Fig. 4 by drawing a horizontal line at $\pi R_E^2/A^* = \text{const}$. In Fig. 5 are shown results of such computations for area ratios 3·81 and 10·7. The reduction in thrust performance due to shortening the length seems to be very small.

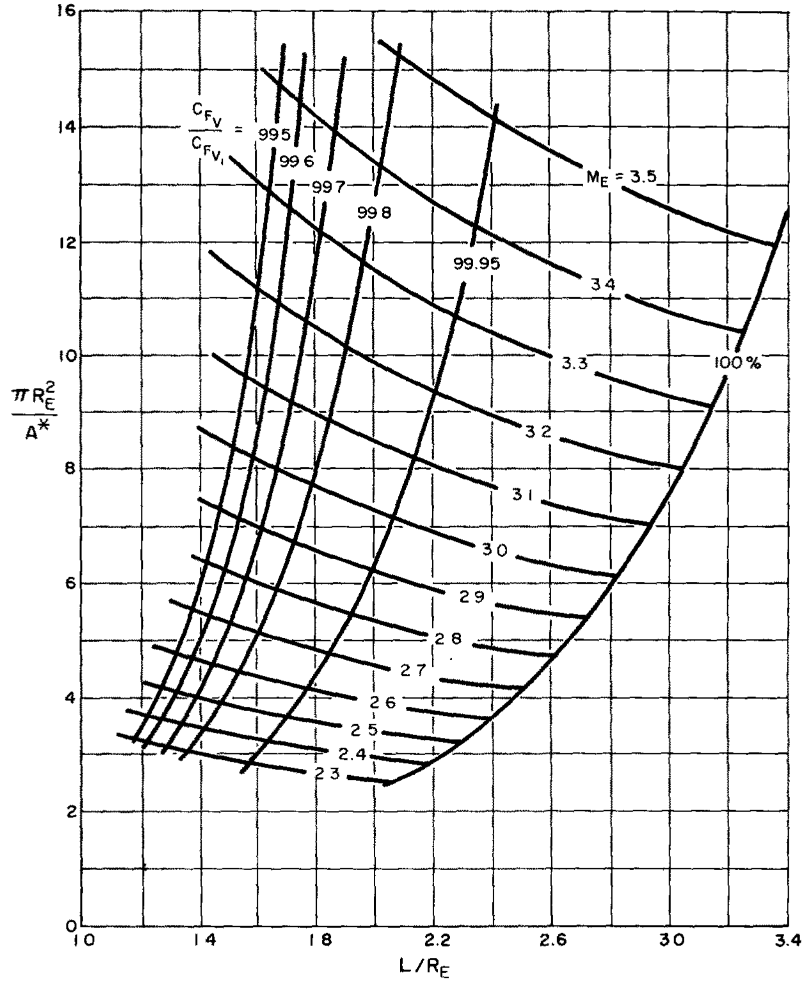

Fig. 4. Thrust performance vs. nozzle parameters.

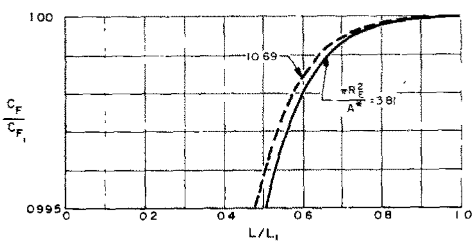

Fig. 5. Thrust performance vs. nozzle length.

5. CONTOUR OF THE OPTIMUM PLUG NOZZLE

The results presented in Fig. 3 would indicate the values of the parameters M_E and θ_E that have to be chosen to design a nozzle for any particular area ratio and length. After the parameters M_E and θ_E are chosen, the control surface itself and the flow parameters all along the control surface can be computed as shown in the above section. Next step is to compute the plug wall contour along which the exhaust gases would expand and yield the flow parameters required along the control surface.

The control surface *ED* along which the Mach number and flow direction are given in Fig. 2 is shown in Fig. 6. Let us assume that the nozzle throat occurs at *E* such that all the expansion is external and occurs across a fan of simple waves centered at the point *E*. Starting with values of *M* and θ prescribed along *ED*, and the simple expansion fan located at E, the characteristics net can be completed in the region upstream of ED and above the axis. Such a network of characteristic lines can be computed sufficiently close to the throat location. With the flow field in this region known, the stream line passing through D can be drawn. This stream line as shown in Fig. 6, then forms the plug wall contour. The throat section can be located by the following simple procedure. From M_E and θ_E the sonic line can be easily found from the Prandtl–Meyer relation between flow angle and Mach number in the simple expansion fan. Thus, θ^*, the direction of flow at the throat section can be obtained. Let R_T denote the radial coordinate of the plug wall at the throat section. From the relation of conservation of mass flow, we obtain

$$\pi(R_E^2 - R_T^2)/\cos\theta^* = \pi R_E^2/\epsilon \qquad (12)$$

where ϵ is the area ratio $\pi R_E^2/A^*$ as computed from equation (11). With the known value of θ^*, the above equation (12) can be used to locate the throat section for each plug nozzle under consideration.

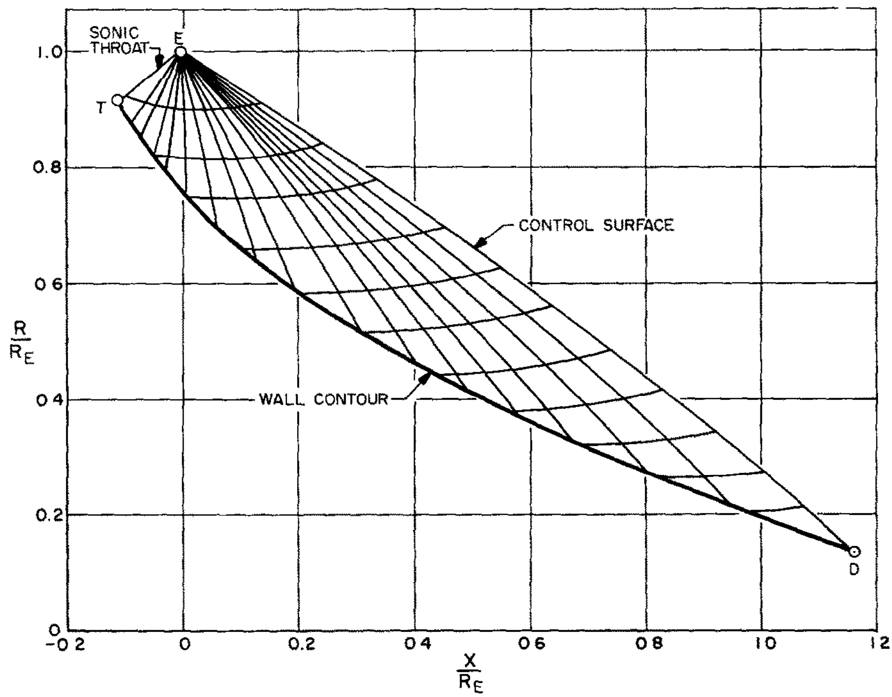

Fig. 6. Construction of spike wall contour.

The plug wall contour thus computed in the present example is shown in Fig. 7 and the contour coordinates are given in Table 1. In Fig. 7 is also shown an ideal plug nozzle for this area ratio 3·81. Following similar procedure, an optimum plug nozzle is designed for an area ratio 10·69 and length ratio $L/R_E = 1{\cdot}731$, by choosing $M_E = 3{\cdot}2$ and $\theta_E = -6°$. The contour thus obtained is shown in Fig. 8 compared with an ideal plug nozzle for the area ratio, and the wall contour coordinates are given in Table 2.

The ideal plug contour can be terminated at the same length as the optimum nozzle and the thrust can be estimated by subtracting from the ideal nozzle thrust the pressure forces acting on the deleted portion. The contour parameters and the thrust performance of such truncated nozzles and the optimum nozzles are compared in Table 3. As can be expected from theory, the optimum plug nozzle yields higher thrust coefficient than the plug contour obtained by

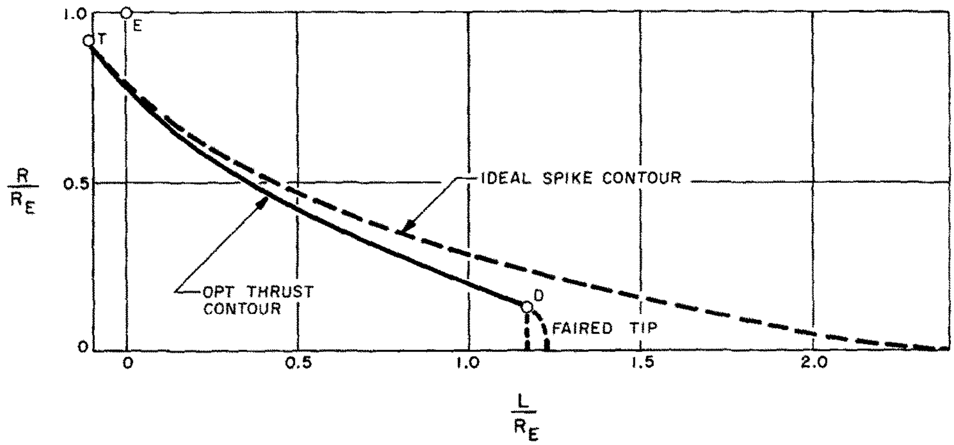

Fig. 7. Optimum contour of spike for $\pi R_E^2/A^* = 3{\cdot}81$; $L/R_E = 1{\cdot}164$; and $\gamma = 1{\cdot}23$.

Table 1. Coordinates of a Spike Nozzle Contour Designed for $\gamma = 1{\cdot}23$, $\pi R_E^2/A^* = 3{\cdot}81$ and $L/R_E = 1{\cdot}164$

X/R_E	R/R_E	$\theta°$
0	1·000	(Cowl-lip location)
−0·109	0·9165	−51·92 (throat location on spike contour)
−0·016	0·782	−50·10
0·034	0·728	−45·18
0·087	0·678	−41·21
0·132	0·640	−38·44
0·192	0·596	−35·59
0·267	0·545	−32·73
0·314	0·516	−31·21
0·367	0·484	−29·72
0·429	0·450	−28·23
0·502	0·413	−26·75
0·588	0·371	−25·32
0·690	0·324	−23·92
0·815	0·271	−22·59
0·969	0·209	−21·47
1·164	0·137	−19·73 (end point of the computed contour)

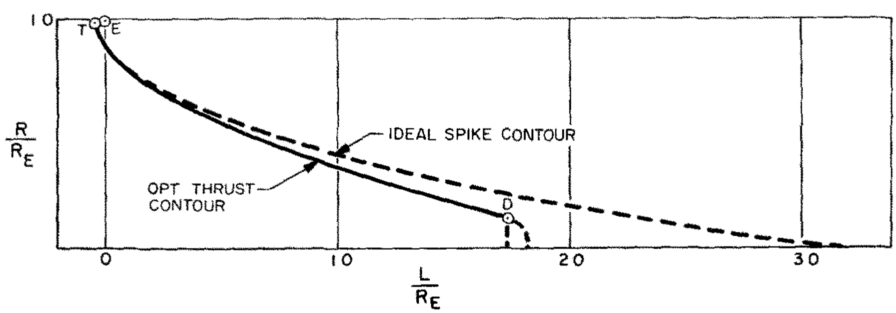

Fig. 8. Optimum contour of spike for $\pi R_E^2/A^* = 10{\cdot}69$; $L/R_E = 1{\cdot}731$; and $\gamma = 1{\cdot}23$.

Table 2. Coordinates of a Spike Nozzle Contour Designed for $\gamma = 1{\cdot}23$, $\pi R_E^2/A^* = 10{\cdot}69$; $L/R_E = 1{\cdot}731$

X/R_E	R/R_E	$\theta°$
0	1·000	(Cowl-lip location)
−0·045	0·9856	−71·43 (throat location on spike contour)
−0·021	0·917	−63·06
−0·002	0·886	−56·23
0·024	0·852	−50·27
0·054	0·818	−45·73
0·101	0·775	−40·56
0·173	0·718	−35·52
0·245	0·670	−32·04
0·347	0·611	−28·45
0·413	0·577	−26·66
0·494	0·537	−24·90
0·594	0·493	−23·12
0·717	0·442	−21·32
0·874	0·384	−19·56
1·076	0·316	−17·85
1·347	0·233	−16·34
1·731	0·126	−14·94 (end point of the computed contour)

Table 3. Comparison of Thrust Coefficients

Nozzle description	L/R_E	R_D/R_E	C_F	L/L_i	C_F/C_{F_i}
Opt. thrust contour for $\epsilon = 10{\cdot}69$	1·731	0·126	1·7269	0·529	0·9967
Ideal spike contour for $\epsilon = 10{\cdot}69$	3·271	0	1·7326	1·00	1·00
Above contour truncated	1·731	2·3	1·7252	0·529	0·9957
Opt. thrust contour for $\epsilon = 3{\cdot}81$	1·164	0·137	1·5804	0·479	0·9934
Ideal spike contour for $\epsilon = 3{\cdot}81$	2·433	0	1·5909	1·00	1·00
Above contour truncated	1·164	0·245	1·5783	0·479	0·9921

shortening the ideal contour. Furthermore, the spike contours computed for optimum thrust have less surface area than the truncated ideal contours, as can be deduced from Figs. 7 and 8.

The endpoint D of the computed wall contour of an optimum plug nozzle does not lie on the nozzle axis. For short lengths the distance of the point D from the nozzle axis can be appreciable. The vertex portion D to C can be of any suitable shape. Remembering that the plug nozzle is designed assuming zero pressure in this region, one would find that changes in this portion of the contour would not alter the thrust performance of the plug nozzle.

6. CONCLUDING REMARKS

The method of determining the spike nozzle contour that yields optimum thrust is derived through the application of calculus of variations. The nozzle expansion ratio, and the spike length are the governing parameters and the spike contour for optimum thrust can be uniquely determined. Illustrative numerical examples are computed using $\gamma = 1{\cdot}23$. The results presented in Figs. 3 and 4 can be helpful in the choice of design parameters for a spike nozzle to deliver a required thrust performance. Two representative optimum thrust contours are shown in Figs. 7 and 8.

Since the method presented here determines the contour uniquely, the problem of thrust improvement by contour changes is answered directly.

The exhaust gases are treated as frictionless in the theory presented here and the examples computed. One can evaluate the displacement thickness of the boundary layer along the spike wall and apply the correction to the contours presented in Figs. 7 and 8. Decreasing the radial coordinates of the spike wall contour to accommodate the boundary layer would ensure the exit flow for which the nozzle is designed.

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