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THE MULTIGRID METHOD FOR THE UNSTEADY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS: COMPARATIVE ANALYSIS BETWEEN DIFFERENT CYCLES AND SMOOTHERS

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Abstract. *In this work numerical simulation has been carried out to study the computational cost effective when multigrid method is applied in two-dimensional incompressible and unsteady flows in a driven cavity. A first-order time-accurate fractional-step method for solving Navier–Stokes equations, in an structured staggered mesh, is used. Different structures of multigrid cycle as V–Cycle, W–Cycle and F–Cycle are compared with the iterative solver MSI (Modified Strongly Implicit Procedure) for the pressure correction linear system. Numerical experiments using multigrid V–Cycle algorithms show that one simple acceleration scheme accelerate the convergence rate in the first iterations but do not modify the final computational cost when SOR (Successive Over Relaxation) and SUR (Successive Under Relaxation) are used as smoother.*

keywords: *Multigrid Method, SOR, SUR, Square Driven Cavity, Navier–Stokes*

1. Introduction

In many conventional computers numerical simulation of flows over immersed complex geometries that request the solution of the incompressible Navier-Stokes equations tends to be prohibitive due to the requirements on memory and CPU time, mainly when advanced finite difference methods are used to provide adequate resolution which result in large linear systems.

The multigrid method has been shown to be very effective in solving large linear systems arising from PDE (*Partial Differential Equations*) boundary-value problems (Brandt, 1977) due to their efficiency when compared to conventional iterative solution methods. The essence of multigrid methods is that it do not attempt to solve the linear systems arising from the discretization of PDE directly but guessing an initial solution and correcting it by means of an estimation of the error in the approximation, measured on successively coarser grids (Millar *et al.*, 2002). Although the efficiency of multigrid method is highly dependant on the mechanisms of inter-grid transfer (interpolation and restriction operations) as well as on the smoother type and sweeps number. For various reasons, the Red-Black Gauss–Seidel method has been used extensively as a relaxation in the multigrid methods, but knowing the fact that the restriction is sensitive only to under relaxation, and the interpolation is sensitive only to over relaxation, the Excessive Over-Relaxation Method (SOR/SUR) can be applied (Zhang, 1995).

In addition to solving two-dimensional Poisson equation with accuracy and efficiency some attempts have been made when iterative methods are applied. Gupta *et al*. (1995), suggested a high-order compact difference scheme with multigrid algorithm to solve convection-diffusion equations with constant coefficients which are known to be stable when classical iterative methods are used leading to a dramatic improvement in computed accuracy. Zhang (1996) using the multigrid method to solve linear systems arising from discretized PDE boundary-value problems proposed to optimize the residual injection operator by choosing an optimal residual injection factor, which provides convergence faster than full-weighting. A large number of experiments was made by Ngongang (1998) in order to obtain a better performance. The scaled injection operator as an acceleration technique, the heuristic dynamic injection operator, high-order discretization for 2D and 3D convection-diffusion equations and many classical smoothers were mentioned as options to present high accuracy and efficiency.

The present paper uses multigrid technique to evaluate the effective computational cost in solving unsteady incompressible Navier–Stokes equations. A square two-dimensional driven cavity is used, where a first-order time-accurate fractional-step method leads to a Poisson equation for the pressure correction. Different structures of multigrid cycle as F–Cycle, V–Cycle and W–Cycle provides convergence faster than MSI (*Modified Strongly Implicit Procedure*). The main

advantage occurs at finer grids where the approximate solutions are required to have a high accuracy showing a better performance.

In many problems it has already been shown that the inclusion of SOR in two-dimensional multigrid solvers can increase their efficiency up to 30% (Zhang, 1995), but when unsteady flows are considered, where the solution advance in time in a step-by-step, only the first iterations are affected by relaxation parameter, which do not change the computational time if a large simulation time is required.

2. Mathematical Formulation

Unsteady incompressible flow in a cartesian square driven cavity domain can be modelled by Navier-Stokes equations:

$$\rho\left[\frac{\partial(u_i)}{\partial t}+\frac{\partial(u_i u_j)}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right], \tag{1}$$

$$\frac{\partial u_i}{\partial x_i}=0, \tag{2}$$

where *u* is the velocity vector, *p* is the pressure and ν is the kinematic viscosity.

If a driven cavity problem is considered the no-slip condition is imposed at wall, with the velocity components being set to zero except at the upper wall where a velocity different from zero is imposed. Since the velocity is known at the boundaries a Neumann boundary condition is imposed for the pressure correction equation.

The numerical solution of this equation represents a difficult computational challenge. The problem arises from the pressure term in the momentum equations which must be implicitly updated for incompressible flows.

Fractional step methods were conceived to minimize most of these computational challenge, and to allow the system to be solved as a series of individual uncoupled advection-diffusion equations for each of velocity components as a Poisson equation for the pressure correction (Chang *et al.*, 2002). In this method a pseudo-pressure is used to correct the velocity field such that the continuity equation is satisfied at each computational time step. A explicit euler time-advancement scheme is used for both the convective and viscous term in Eq. (1). Knowing that this analysis is independent of any particular spatial discretization, staggered mesh can be applied which eliminates the pressure oscillation. Figure (1) illustrates the nodes for u and v velocities components and the node for pressure.

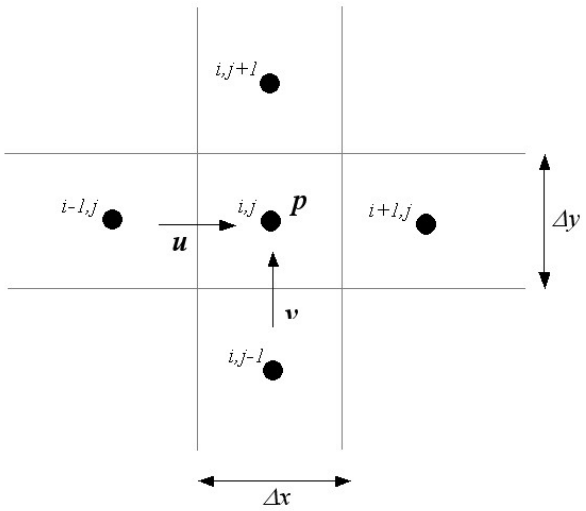

Figure 1: Staggered grid.

Applying the factional-step method to Eqs. (1) and (2), the equations to be solved can be written as follow:

$$\rho\left[\frac{\bar{u}_i^{n+1}-u_i^n}{\Delta t}+\frac{\partial}{\partial x_j}\left(u_i^n u_j^n\right)\right]=-\frac{\partial p^n}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i^n}{\partial x_j}+\frac{\partial u_j^n}{\partial x_i}\right)\right], \tag{3}$$

$$\frac{\partial^2 p'^{n+1}}{\partial x_j \partial x_j}=\frac{\rho}{\Delta t}\frac{\partial \bar{u}_k^{n+1}}{\partial x_k}, \tag{4}$$

$$u_i^{n+1}=\bar{u}_i^{n+1}+\frac{\Delta t}{\rho}\frac{\partial p'^{n+1}}{\partial x_i}, \tag{5}$$

$$p^{n+1} = p'^{n+1} + p^n, \tag{6}$$

where $\bar{u}_i$ is the estimated velocity components, p' is the pressure correction and Δt and n are the computational time step and the sub-step indices, respectively.

The discretization of above equations through second-order finite central difference method in space require especial care in order to control the accumulation of numerical errors. For this reason the pressure correction equation (Eq. 4) or Poisson equation is given by:

$$a_p p_{i,j}^{n+1} + a_w p_{i-1,j}^{n+1} + a_e p_{i+1,j}^{n+1} + a_n p_{i,j+1}^{n+1} + a_s p_{i,j-1}^{n+1} = C_{i,j}, \tag{7}$$

$$\begin{aligned} a_n &= a_s = \beta^2, \\ a_e &= a_w = 1.0, \\ a_p &= -2.0\left(1.0 + \beta^2,\right) \end{aligned} \tag{8}$$

$$C_{i,j} = \Delta x^2 \frac{\rho}{\Delta t}\left(\frac{\bar{u}_{i+1,j}^{n+1} - \bar{u}_{i,j}^{n+1}}{\Delta x} + \frac{\bar{v}_{i,j+1}^{n+1} - \bar{v}_{i,j}^{n+1}}{\Delta y}\right) \tag{9}$$

where, $\beta = \frac{\Delta x}{\Delta y}$.

2.1. The Multigrid Methodology

The multigrid methodology belongs to the fastest and most efficient algorithms for solving linear systems arising from discretization of elliptic differential equations (Zhang, 1995) and offers a convergence rates independent of the size of the problem (Gupta *et al.*, 1995). Thus the basic idea of multigrid technique is to work with a hierarchy of consecutively coarser and coarser grids until convergence is achieved.

There are several possible algorithms for carrying out the basic multigrid idea, each of them with several possible variations. Here three algorithms are presented: V–Cycle, W–Cycle and F–Cycle.

One iteration of a simple multigrid V–Cycle consists of smoothing the error using a relaxation technique (*e.g.* Gauss Seidel and Jacobi which are called smoothers in the multigrid literature) projecting (restricting) the residuals to the coarse grid, solving an approximation to the smoothing error equation on a coarse grid, interpolating (prolongating) the coarse grid error correction back to the fine grid, and finally adding the error correction into the approximation. An important aspect of the multigrid method is that the coarse grid solution can be approximated by recursively using the multigrid idea, that is, on the coarse grid, relaxation is performed to reduce high frequency errors followed by the projection of the residuals on another coarser grid, and so on. Thus the multigrid method requires a series of problems to be solved on a hierarchy of grids with different mesh sizes where a grid reduction factor of 2 is generally employed.

Figures (2), (3) and (4) show the schedule for the grids in the order in which they are visited, where the process goes from the finest grid down to the coarsest grid and back from the coarsest up to the finest, performing η relaxation sweeps on each level.

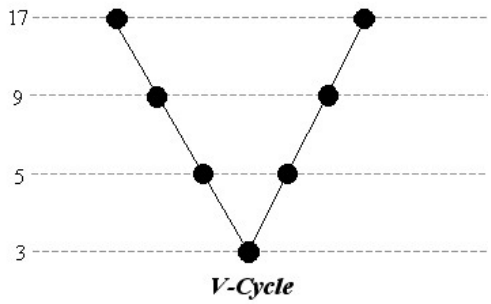

Figure 2: Schedule of grids for V-Cycle.

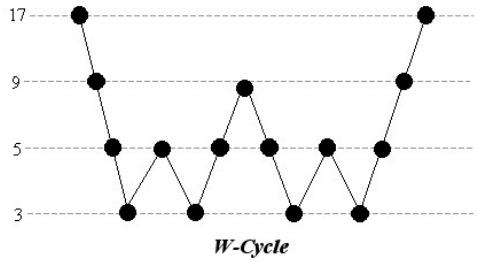

Figure 3: Schedule of grids for W-Cycle.

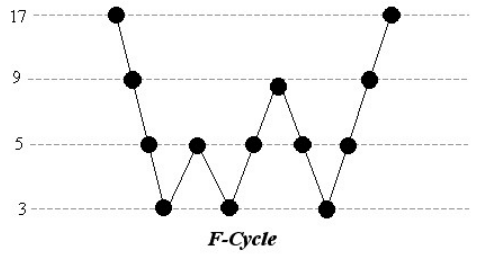

Figure 4: Schedule of grids for F-Cycle.

The simplest projection operators to transfer quantities from fine to coarse grids are the injection operators where the residuals evaluated on the finest grid are injected to the coarse grid space. For RBGS (Red-Black Gauss Siedel), a half-injection projection is commonly used transferring the residuals to the coarse grids points weighted by $1/2$.

More accurate projection operators are full-weighting (FW) and half-weighting (HW). In those technique the residuals are computed on all of the fine grid (h) and weighted to the coarse grids ($2h$) points by the formula (Gupta, 1995):

$$r^{2h}_{\frac{i}{2},\frac{j}{2}} = \frac{1}{8}\left[4r^h_{i,j} + \left(r^h_{i,j-1} + r^h_{i,j+1} + r^h_{i-1,j} + r^h_{i+1,j}\right)\right], \tag{10}$$

$$r^{2h}_{\frac{i}{2},\frac{j}{2}} = \frac{1}{16}\left[4r^h_{i,j} + 2\left(r^h_{i,j-1} + r^h_{i,j+1} + r^h_{i-1,j} + r^h_{i+1,j}\right) + \left(r^h_{i-1,j-1} + r^h_{i-1,j+1} + r^h_{i+1,j-1} + r^h_{i+1,j+1}\right)\right], \tag{11}$$

for FW and HW respectively. where, $r_{i,j}$ is the residual present on the fine grid and $r_{\frac{i}{2},\frac{j}{2}}$ is the quantities to be transferred to the corresponding coarse grid ($\frac{i}{2}$, $\frac{j}{2}$).

The second class of intergrid transfer functions involves moving vectors from a coarse grid to a fine grid. For two-dimensional problems the bilinear polynomial is commonly employed, which are given by:

$$\begin{aligned}
\varphi^h_{i,j} &= \varphi^{2h}_{\frac{i}{2},\frac{j}{2}}, \\
\varphi^h_{i+1,j} &= \frac{1}{2}\left(\varphi^{2h}_{\frac{i}{2},\frac{j}{2}} + \varphi^{2h}_{\frac{i}{2}+1,\frac{j}{2}}\right), \\
\varphi^h_{i,j+1} &= \frac{1}{2}\left(\varphi^{2h}_{\frac{i}{2},\frac{j}{2}} + \varphi^{2h}_{\frac{i}{2},\frac{j}{2}+1}\right), \\
\varphi^h_{i+1,j+1} &= \frac{1}{4}\left(\varphi^{2h}_{\frac{i}{2},\frac{j}{2}} + \varphi^{2h}_{\frac{i}{2}+1,\frac{j}{2}} + \varphi^{2h}_{\frac{i}{2},\frac{j}{2}+1} + \varphi^{2h}_{\frac{i}{2}+1,\frac{j}{2}+1}\right),
\end{aligned} \tag{12}$$

where φ is the variable to be evaluated. The indices ($\frac{i}{2}$, $\frac{j}{2}$) identify the variable value on the coarse grid and (i, j) on the fine grid.

2.2. The Excessive Over-Relaxation Method

Smoothing methods in multigrid algorithms are usually taken from the class of basic iterative methods. Thus when an iterative method are used, a relaxation parameter may improve the convergency property. But in many situations, using relaxation parameters is not cost-effective when applied to Gauss-Seidel relaxation. However an optimal relaxation parameter (w) in SOR (Successive Over Relaxation, $w > 1.0$) can accelerate the convergence dramatically.

Despite of SOR efficiency, it is not suitable in multigrid methods due to the fact that when SOR is used on the upswing of a multigrid V-Cycle the introduction of low frequency errors on the lower grids levels seriously impairs the smoothing effects of post-relaxation and can result in many extra iterations (Millar et al., 2002). On the other hand under relaxation reduces the low frequencies but excites the high frequencies. The existing believe that SOR is not suitable for using as a smoother in multigrid method was contraried by Zhang (1995), where he has shown that in the V-Cycle, the first (restriction) half cycle is sensitive under relaxation (SUR- Successive Under Relaxation, $w < 1.0$) only and the second (interpolation) half cycle to over relaxation (SOR) only.

In that case using SUR on the second and third finest levels of the down swinging side, where the majority of the low frequency components are removed and SOR where the high frequency components are introduced, that is on the up swinging half cycle, it is possible to accelerate the convergence of the solver. It is important to note that the use of SOR/SUR on the top or bottom levels of the cycles should be avoided. In the finest level the main effort is to smooth the high frequency errors and the employment of a parameter might reduce the effect of smoothing. In case of coarsest grid the residuals are already small and an acceleration helps little (Zhang, 1995). For those cases RBGS ($w = 1.0$) is more suitable. The figure (5) shows both operators (RBGS, SOR and SUR) required at each level to the V-Cycle.

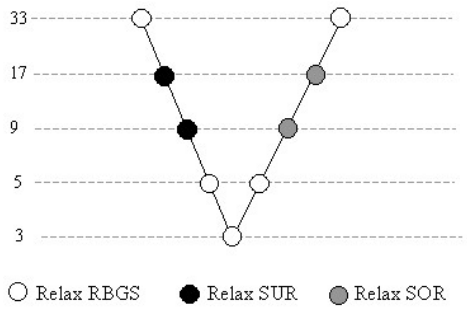

Figure 5: The multigrid V-Cycle showing grid size and operators required at each point in the cycle.

3. Numerical Results

To test the multigrid method and its efficiency the developed code, in the present work, is applied to present a classic benchmark problem in two dimensional unsteady internal flow, in a uniform cartesian grids. In the results presented here a scaled injection which is a injection operator, multiplied by factor equal to 2.0 and a bilinear interpolation are used as projection operators to transfer quantities from fine to coarse grids and vice-versa. It is noted that the use of scaled injection significantly reduces the number of iteration when Gauss-Seidel is used (Ngongang, 1998). The Navier–Stokes equations are dicretized by the usual five-point 2nd-order central difference scheme. All the experiments are done on a PC single processor using FORTRAN 90 programming language in double precision. The program finishes when the residual on the finest grid in L_2 norm is less than 10^{-7} and residual mass is less than 10^{-6}.

3.1. The Driven Cavity Flow

The driven cavity problem has long been used to as a test case for Navier–Stokes solvers, due to simplicity of the geometry and boundary conditions. The two-dimensional square cavity has no slip boundary conditions on all walls. The flow is driven by the top wall sliding over the cavity. The sides and bottom walls are static.

For the current study the problem was solved for a Reynolds number of 100 and 1000 and compared with the results of Ghia *et al.*, (1892) which were generated using a finite difference vorticity-streamfunction solver using a central difference approximation.

The vorticity contours for the flow calculated on a 32×32 mesh using fractional-step method are shown in figure (6). It provides insight into some general features of the flow field as the Reynolds number increases. As the $Re \longrightarrow \infty$ one would expect thin boundary layers to develop along the solid walls, with the central core in almost inviscid motion. This is indeed seen in the figure (6). As Re increases, there is a clearly visible tendency for the core fluid to move as a solid with uniform vorticity (Shankar and Deshpande, 2000).

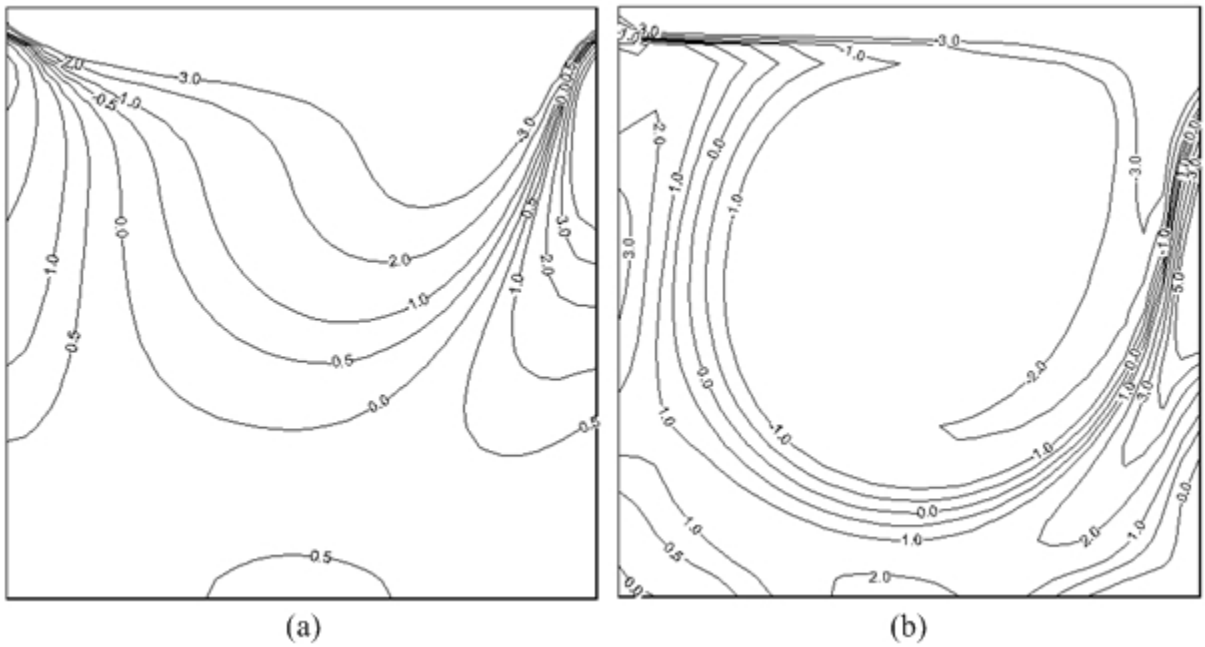

Figure 6: Vorticitty contours in a square driven cavity at (a) $Re = 100$ and (b) $Re = 1000$.

Plots of x and y components of velocity at $Re = 100$ and $Re = 1000$ using F-Cycle are given in the figures (7) and (8) in order to examine the accuracy of the numerical method. For Re=1000 the values were either calculated on mesh 64×64 which show better convergence when compared with Ghia *et al.*, (1892).

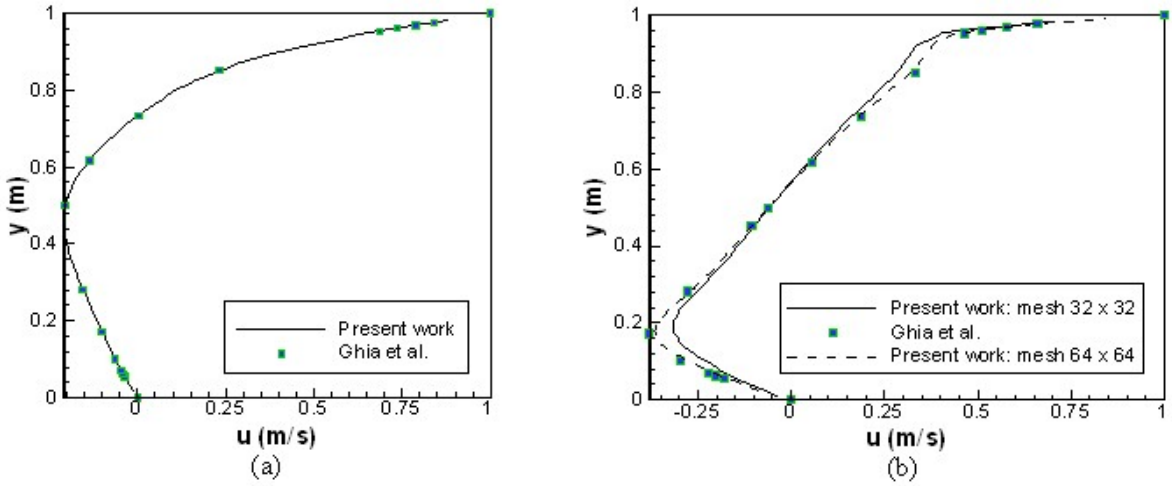

Figure 7: Profiles of velocity u along the vertical centreline for (a) $Re = 100$ and (b) $Re = 1000$.

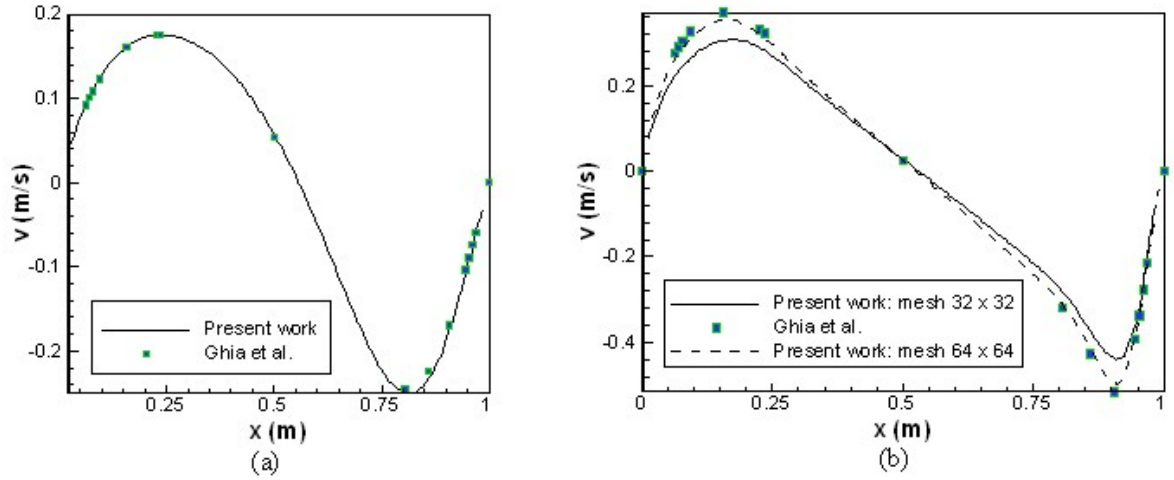

Figure 8: Profiles of velocity v along the vertical centreline for (a) $Re = 100$ and (b) $Re = 1000$.

3.2. Computational Cost

For a comparison of computational cost between multigrid method using RBGS as smoother which performs 4 relaxation sweep on each level and other iterative solver, the MSI (Modified Strongly Implicit Procedure) is applied considering a simulated time of $0.1s$ on a mesh of 32×32 and $Re = 100$.

More sophisticated adaptive cycling strategies besides of V-Cycle, such as W- and F-Cycle are either evaluated but did not offer advantage at the Reynolds of interest. The table (1) shows the CPU time evaluated at each case quoted previously.

Table 1: Performance comparison on IBM PC among different structures of multigrid cycles and MSI solver.

	CPU (s)			
Grid size	**F-Cycle**	**V-Cycle**	**W-Cycle**	**MSI**
16×16	3.645	2.503	3.064	2.103
32×32	13.039	9.494	10.005	11.086
64×64	59.647	48.782	51.525	322.117
128×128	336.290	338.850	336.763	1734.492

The values above can be plotted to understand the exponential solvers behavior (Fig. 9). It is found that in the coarsest grids there is no advantage between the solvers but in the followings grids a very lower computational time is verified at V-Cycle when compared with the MSI solver.

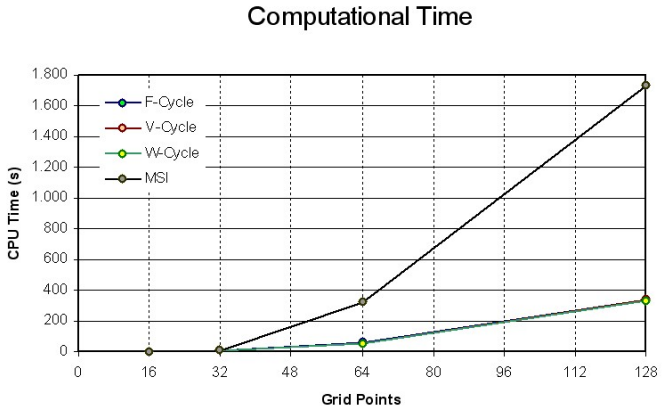

Figure 9: Performance comparison among different structures of multigrid cycles and MSI solver.

3.3. Computational Cost at Excessive Over Relaxation

Based on foregoing analysis, results for two acceleration scheme are presented here. The V-Cycle algorithm with SUR and SOR as smoothers which embed under-relaxation ($w_1 < 1.0$) and over-relaxation ($w_2 > 1.0$) on their second and third finest level is applied in a driven cavity flow considering a simulated time of $0.1s$, Re=100 and different mesh sizes.

To evaluate the effect of excessive over relaxation, some parameters are defined at the first iteration or at the first time-step where the convergence is more difficult to be attained. Due to this difficult the number of multigrid V-Cycles, denoted by CN required to satisfy a given error tolerance (10^{-7}) is an important factor to test the computational efficiency and their high values occur at the first iterations. Other parameter, the contraction number, k, or the average residual reduction factor is defined by the equation (13):

$$k = \left(\frac{\|r_n\|_2}{\|r_o\|_2} \right)^{\frac{1}{CN}}, \tag{13}$$

where the $\|r_o\|_2$ is the initial discrete error in L_2 norm and $\|r_n\|_2$ is the final discrete error.

Those parameters are dependent of the mesh sizes and of the sweep numbers. In tables 2, 3 and 4, the parameters k and the computational time given in seconds (CPU) are shown for sweep numbers going from 1 to 4, and for three mesh sizes - 16×16, 32×32 and 64×64, respectively. It is important to note that the under (w_1) and over (w_2) relaxation coefficients are the optimum values found, for this problem and either dependent of the mesh size, through experiments analysis.

Table 2: Performance comparison between RBGS and SOR/SUR on a 16×16 mesh size.

	1 sweep		2 sweeps		3 sweeps		4 sweeps	
w_1	1.0	0.9	1.0	0.9	1.0	0.9	1.0	0.9
w_2	1.0	1.5	1.0	1.5	1.0	1.5	1.0	1.5
CN	21	15	12	9	9	7	7	6
k	0.606	0.475	0.424	0.289	0.335	0.222	0.287	0.187
$CPU(s)$	1.983	1.873	2.103	2.083	2.314	2.253	2.583	2.523

Table 3: Performance comparison between RBGS and SOR/SUR on a 32×32 mesh size.

	1 sweep		2 sweeps		3 sweeps		4 sweeps	
w_1	1.0	0.9	1.0	0.9	1.0	0.9	1.0	0.9
w_2	1.0	1.2	1.0	1.2	1.0	1.2	1.0	1.2
CN	26	22	15	12	11	9	9	8
k	0.645	0.580	0.470	0.388	0.383	0.299	0.331	0.244
$CPU(s)$	7.300	6.970	7.582	7.470	8.523	8.392	9.414	9.304

Table 4: Performance comparison between RBGS and SOR/SUR on a 64×64 mesh size.

	1 sweep		2 sweeps		3 sweeps		4 sweeps	
w_1	1.0	0.9	1.0	0.9	1.0	0.9	1.0	0.9
w_2	1.0	1.1	1.0	1.1	1.0	1.1	1.0	1.1
CN	32	28	18	15	13	12	11	10
k	0.675	0.641	0.507	0.460	0.421	0.357	0.366	0.307
$CPU(s)$	107.580	106.315	57.654	46.718	51.224	48.062	49.662	48.392

Regarding the values shown above, the use of SOR/SUR in multigrid is actually dependent of the sweep number. The major efficiency occur at one relaxation sweep, where the number of multigrid V-Cycles decreases drastically, but do not show computational time advantages when one more relaxation sweep is considered for coarser grids. If a fine grid is considered more than one sweep relaxation leads to a better performance. However the average residual reduction factor tends to decrease if a sweep number increases. In theory it could affect the mass conservation, but it was observed that

this aspect do not change the final computational time neither the residual mass in each iteration of the step-fractional method.

Other aspect demonstrated in the results is that only two relaxation sweep is sufficient when RBGS or SOR/SUR are considered which supplies the major percentual gain in CPU time.

4. Conclusions and Remarks

In this work a effective computational cost analysis has been performed to the Navier-Stokes equations for unsteady flow. The step-fractional method with a central difference scheme in primitive variables and the Neumann boundary condition for the pressure on a staggered grid were used to solve these equations.

The numerical results were carried out for multigrid method and its different structure cycles as F-Cycle, V-Cycle and W-Cycle. Other iterative solver (MSI) was used to compare the efficiency of multigrid method on a range of grids. The results performed on a driven cavity flow were compared with the existing solutions showing a very good agreement.

Both cycles have been tested and showed similar CPU time reduction rates with excellent rates, when compared to the MSI. It is important to note that when refined grids are applied, the reduction rates behavior in CPU time are exponential which confirm the high multigrid method efficiency. However, the excessive over relaxation in which under and over relaxation is adopted on the smoother do not show advantage over RBGS if two sweep relaxation is used. The results implies that only two relaxation sweeps are needs for a given level of accuracy leading to faster solution times.

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