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ENHANCEMENTS OF THE SIMPLE METHOD FOR PREDICTING INCOMPRESSIBLE FLUID FLOWS

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Variations of the SIMPLE method of Patankar and Spalding have been widely used over the past decade to obtain numerical solutions to problems involving incompressible flows. The present paper shows several modifications to the method which both simplify its implementation and reduce solution costs. The performances of SIMPLE, SIMPLER, and SIMPLEC (the present method) are compared for two recirculating flow problems.

The paper is addressed to readers who already have experience with SIMPLE or its variants.

INTRODUCTION

Many problems of practical interest require the numerical simulation of two- or three-dimensional, elliptic, incompressible fluid flows, and numerical methods of treating such problems have evolved rapidly over the last two decades. In the early simulation methods vorticity and stream function were usually the calculated variables, but a steady increase in the use of primitive variables has prevailed. The Los Alamos group led the way with development of explicit transient algorithms such as MAC [1] and SMAC [2]. Implicit methods were attractive as a means of avoiding restrictions on the explicit time step, but in 1972 Roache concluded [3] that "no successful implicit formulation has yet been achieved." The SIMPLE algorithm of Patankar and Spalding [4], which appeared in the same year, not only provided a remarkably successful implicit method, but has dominated for a decade the field of numerical simulation of incompressible flows. A clear and detailed description of SIMPLE, together with the improvements that have evolved since 1972, has been provided by Patankar [5, 6].

As with any other algorithm, the application of SIMPLE is not without its difficulties and questions. The problems that have arisen in numerous applications of SIMPLE-type methods over the last several years have forced the present authors to confront some of these issues. These experiences have led to the development of improved methods which a limited circle of other users have found useful. The writing of the present paper was undertaken with the goal of making a wider audience aware of some alternatives that may prove useful to them as well.

The approach adopted in writing this paper assumes that the reader has an intimate

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NOMENCLATURE

a, b, c, d, e, f	coefficients of finite-volume equations	u', v'	velocity corrections
		x, y	Cartesian coordinates
A	area of a control-volume face	α	under-relaxation factor
B, C	coefficients in boundary conditions [Eq. (26)]	α_p	under-relaxation factor for pressure
d	pressure coefficient [Eqs. (15b) and (24b)]	γ	residual reduction factor
		Γ	diffusion coefficient
E	time step multiple	Δt	time step
p	pressure	Δt^*	reference time interval
p^*	best estimate of pressure	ΔV	volume of control volume (equals area for two-dimensional problem)
p'	pressure correction		
r	residual		
S	source term	ρ	density
S_C	constant part of linearized source term	ϕ	general dependent variable
		ϕ°	best estimate of ϕ
S_P	coefficient of ϕ_P in linearized source term [Eq. (2)]		
$\bar{S}$	average value of source term over control volume	Subscripts	
u, v	velocity components in x and y directions	e, n, s, w	control-volume faces (Fig. 1)
		E, N, P, S, W	grid points (Fig. 1)
u^*, v^*	velocities based on p^*	nb	neighbor grid point

familiarity with Patankar's writings [5, 6]. His notation has been used throughout, and only the material that bears directly on developments in this paper has been reproduced. Taken by itself, this paper appears to deal with a somewhat disconnected sequence of topics.

Two points, however, are worthy of mention. First, although this paper is focused on SIMPLE-like methods, some of the ideas advanced may be fruitfully adopted in methods that are quite different from SIMPLE. To avoid confusion, none of these applications is discussed. Second, an undercurrent of concern for problems related to round-off runs through the paper. This stems from the authors' use of single precision on IBM machines. These problems do not arise as frequently for users of computers that carry more significant figures.

It seems likely that SIMPLE and its variants will continue to enjoy widespread use and to evolve. The contributions of this paper are not intended to be the "final word" on SIMPLE, but rather to represent a stage in its continuing development.

FORMULATION OF THE ϕ EQUATION

The General ϕ Equation

The differential equations expressing conservation of momentum, energy, concentration, etc., in two dimensions can be written in Cartesian coordinates in the general form

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right) + S \qquad (1)$$

Under steady conditions the leading term would vanish. Patankar [5, 6] discusses each term in such equations, as well as the application of boundary conditions.

To solve the relevant equations by a finite-volume method, a grid is first generated to cover the domain of interest. A typical control volume within such a grid is denoted by the shaded area in Fig. 1*a*. Equation (1) is integrated over this volume, and each term in the resulting integral balance is approximated in terms of the discrete values of ϕ at the nodal points (i.e., of P, E, N, etc.). The average value of the source term over the volume $\bar{S}$ is approximated by the linear relation

$$\bar{S} = S_C + S_P \phi_P \tag{2}$$

The algebraic approximation of the integral balance for the P control volume in Fig. 1*a* becomes

$$a_P \phi_P = a_E \phi_E + a_W \phi_W + a_N \phi_N + a_S \phi_S + b \tag{3a}$$

or more simply

$$a_P \phi_P = \Sigma\, a_{nb} \phi_{nb} + b \tag{3b}$$

where the summation is over the appropriate neighbor points. The coefficient a_P is given by

$$a_P = a_E + a_W + a_N + a_S - S_P \Delta V \tag{4}$$

A similar equation applies for each control volume in the grid. Details related to the derivation, the application of boundary conditions, the treatment of nonlinearities, etc., have been provided by Patankar [5, 6].

Nonlinearity and Under-Relaxation

The form of Eq. (3) implies that these equations are linear. In fact, the coefficients of Eq. (3) may themselves depend on the solution for any one or more of the dependent variables represented by ϕ.

To account for the resulting inter-equation linkages and nonlinearities, repeated

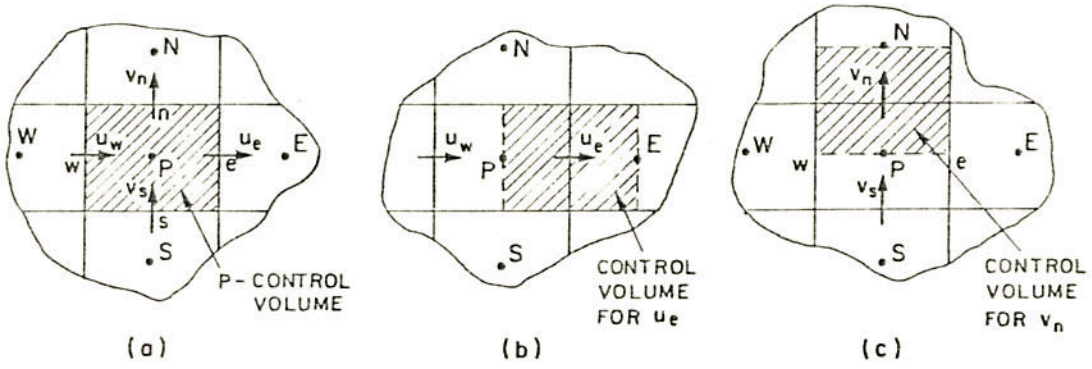

Fig. 1 Section of a Cartesian grid showing placement of control-volume boundaries.

solutions of the nominally linear form of Eq. (3) are required. Each of these solutions is defined herein as a "cycle." At the beginning of each cycle, the coefficients are evaluated using the ϕ values obtained in the previous cycle. With the cycle-by-cycle change in coefficients of Eq. (3), the resulting changes in the ϕ values can be quite large, and this may cause slow convergence or even divergence. To moderate the changes in consecutive solutions for ϕ, and thereby improve convergence, under-relaxation is introduced.

Patankar [5, 6] introduces under-relaxation into Eq. (3) through α as follows:

$$\frac{a_P}{\alpha}\phi_P = \Sigma a_{nb}\phi_{nb} + b + \frac{1-\alpha}{\alpha}a_P\phi_P^0 \tag{5}$$

where ϕ_P^0 is the value of ϕ_P from the previous cycle.

Recommendation 1: The E-factor formulation. The first recommendation, although superficially trivial, is to rewrite Eq. (5) as

$$a_P\left(1+\frac{1}{E}\right)\phi_P = \Sigma a_{nb}\phi_{nb} + b + \frac{a_P}{E}\phi_P^0 \tag{6}$$

The transformation that relates Eqs. (5) and (6) is

$$\alpha = \frac{E}{1+E} \quad \text{or} \quad E = \frac{\alpha}{1-\alpha} \tag{7}$$

Why bother to implement a change that leaves the equations to be solved algebraically equivalent? The primary reason is that E is capable of direct physical interpretation. Equation (6) is precisely the equation that results when the transient term is retained in Eq. (1), so that the solution of Eq. (6) advances ϕ through a time step Δt which is proportional to the reference time interval Δt^*, i.e.,

$$\Delta t = E\,\Delta t^* \quad \text{where } \Delta t^* = \frac{\rho\,\Delta V}{a_P} \tag{8}$$

The reference time interval is related to the times required to respectively diffuse and convect a change of ϕ across the control volume, and is an important scaling parameter in the discrete problem. Values of the time step multiple E of unity or less cause Δt to lie in the range required for stability in an *explicit* solution for ϕ. Because the present implicit method has been adopted in the hope of speeding convergence, values of E well in excess of unity are clearly desirable. Values of E in the range of 4 to 10 are common.

In general, the value of Δt^* will change from one control volume to the next. If a constant E is used, ϕ will be advanced nonuniformly in time across the grid. Furthermore, the value of Δt^* and the E value selected may also be different for different dependent variables, causing them to be advanced at different rates. Such distortions of the transient are often desirable as a means of accelerating convergence [7, 8].

The solution obtained for Eq. (6) can therefore be interpreted as an advancement of ϕ through one time step. Temporal development is so familiar that this interpretation is helpful, even when the time evolution is distorted. As already mentioned, Eq. (5) is algebraically equivalent, but now α bears a nonlinear relation to the time step. The

range of E that is of interest in implicit methods represents a narrow range of α compressed near unity. The extra step required to interpret such α's in terms of the time advancement is undesirable, and can be avoided by using the E-factor formulation.

SOLUTION FOR PRESSURE AND VELOCITY

Treatment of the Pressure-Velocity Coupling

Review of the SIMPLE method. The main purpose of the present paper is to make several recommendations related to the solution for pressure and the velocities. To record the equations that will be needed later, a brief review of the SIMPLE method is necessary. Details that are omitted here may be found in the original references [5, 6].

Using the staggered grid shown in Fig. 1, *b* and *c*, the finite-volume equations for u and v, respectively, have the form of Eq. (6). Rewriting the u-momentum equation for the control volume centered at e to explicitly show the pressure term

$$a_e u_e = \Sigma a_{nb} u_{nb} + b_e + A_e(p_P - p_E) \tag{9}$$

where p is pressure, A_e is the area of the face of the P control volume at e, and

$$a_e = (\Sigma a_{nb} - S_e \Delta V)\left(1 + \frac{1}{E}\right) \tag{10}$$

The S_e in Eq. (10) is the coefficient of u_e in the linearized source term [see Eq. (2)]. For any guessed pressure distribution p^*, the u^* velocities obtained by solving the u-momentum equations [like Eq. (9)] satisfy

$$a_e u_e^* = \Sigma a_{nb} u_{nb}^* + b_e + A_e(p_P^* - p_E^*) \tag{11}$$

The u velocities obtained from Eq. (9) using the correct (but generally unknown) pressure distribution p would satisfy the continuity condition, while the u^* velocities from Eq. (11) in general violate this constraint. Correction of the guessed pressure by $p' = p - p^*$ is therefore necessary to correct the u^* field by $u' = u - u^*$. The relation between p' and u' is obtained by subtraction of Eq. (11) from Eq. (9):

$$a_e u_e' = \underline{\Sigma a_{nb} u_{nb}'} + A_e(p_P' - p_E') \tag{12}$$

The pressure p and velocity u that satisfy both the mass and the momentum constraints are

$$u = u^* + u' \tag{13}$$

$$p = p^* + p' \tag{14}$$

Attention is now turned to the method used to find p'.

The exact equation for p', derived from Eqs. (12) and (13) and the continuity constraint, is complicated and unsuitable for economic calculations. The SIMPLE procedure derives a more suitable equation by neglecting the underlined term in Eq. (12). Combining the simplified Eq. (12) with Eq. (13) yields

$$u_e = u_e^* + d_e(p'_P - p'_E) \tag{15a}$$

where

$$d_e = \frac{A_e}{a_e} \tag{15b}$$

The continuity equation [5, 6] for the control volume shown in Fig. 1*a* is

$$(\rho uA)_w - (\rho uA)_e + (\rho vA)_s - (\rho vA)_n = 0 \tag{16}$$

Introducing equations like Eq. (15) for u and v into Eq. (16) leads to

$$a_P p'_P = a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S + b \tag{17}$$

where

$$a_E = (\rho Ad)_e \qquad a_W = (\rho Ad)_w \tag{18}$$

$$a_N = (\rho Ad)_n \qquad a_S = (\rho Ad)_s \tag{19}$$

$$a_P = a_E + a_W + a_N + a_S \tag{20}$$

$$b = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n \tag{21}$$

The SIMPLE procedure is implemented by executing the following sequence of steps [5, 6]:

1. Guess a pressure field p^*.
2. Evaluate the coefficients of the momentum equations [such as Eq. (11)] and solve to obtain u^* and v^*.
3. Evaluate the mass source [Eq. (21)] and solve Eq. (17) for p'.
4. Correct the velocity field, using equations like Eq. (15). Correct the pressure field using Eq. (14), with the modification discussed in the following paragraph.
5. Solve other ϕ equations; update properties, coefficients, etc.
6. Using the p found in step 4 as the new p^*, return to step 2. Cycle through this loop until convergence is achieved.

To obtain Eq. (15) from Eq. (12) the term $\Sigma a_{nb} u'_{nb}$ was neglected. This approximation results in p' values that are too large, and this in turn causes slow convergence or divergence of the cycle outlined above. To remedy this, Patankar recommends under-relaxation in the momentum equations by employing $\alpha \approx 0.5$ ($E \approx 1$), and under-relaxation of the pressure correction by replacing Eq. (14) by

$$p = p^* + \alpha_p p' \tag{22}$$

where $\alpha_p \approx 0.8$.

This completes the review of the SIMPLE method. With the equations recorded for easy reference, some recommendations are now made that can dramatically improve the economy of the method and simplify its application.

Recommendation 2: The SIMPLEC approximation. The SIMPLE method just described requires the introduction of the relaxation parameter E in the momentum equations [e.g., Eq. (9)] and of α_p in the pressure correction [Eq. (22)]. The cost of obtaining a solution depends critically on how close to their optimal values E and α_p have

been chosen. The search for the optimal values may, however, be more expensive than simply using nonoptimal values.

The SIMPLEC approximation removes the need for the α_p under-relaxation. The solution cost for any given E with SIMPLEC is roughly equivalent to [or perhaps better when source terms like b_e in Eq. (9) are large] that obtained by using the optimal value of α_p in the SIMPLE method. The SIMPLEC approximation is now described.

Since the main approximation in the SIMPLE method is that the $\Sigma a_{nb}u'_{nb}$ term in Eq. (12) can be neglected, it is appropriate to focus attention on this equation. If the pressure p is changed by p', the velocity at e responds by the change u'_e, while those at the neighbor points respond by u'_{nb}. These velocity changes will all be of the same order. The SIMPLE approximation that $\Sigma a_{nb}u'_{nb}$ can be ignored in Eq. (12), while a term of similar magnitude on the left-hand side is retained [$\Sigma a_{nb}u'_e$ appears on the left-hand side when Eq. (10) for a_e is substituted into Eq. (12)], can therefore be viewed as inconsistent.

To introduce a "consistent" approximation, which still leads to a suitably simple expression for p', the term $\Sigma a_{nb}u'_e$ is subtracted from both sides of Eq. (12). This yields

$$(a_e - \Sigma a_{nb})u'_e = \underline{\Sigma a_{nb}(u'_{nb} - u'_e)} + A_e(p'_P - p'_E) \tag{23}$$

In the SIMPLEC approximation, the underlined term $\Sigma a_{nb}(u'_{nb} - u'_e)$ is neglected; the C is appended to the name SIMPLE as a reminder that, in the sense just discussed, this is a consistent approximation. Replacing u' by $u - u^*$ and adopting the SIMPLEC approximation, Eq. (23) becomes

$$u_e = u^*_e + d_e(p'_P - p'_E) \tag{24a}$$

where now

$$d_e = \frac{A_e}{a_e - \Sigma a_{nb}} \tag{24b}$$

The equations and sequence of steps in SIMPLEC are identical to those in SIMPLE with the following exceptions:

1. The d's are computed from equations such as Eq. (24b) rather than Eq. (15b).
2. These d's replace the previous d's in the p' coefficients [Eqs. (18) and (19)] and in the velocity correction equation [Eq. (24a)].
3. p' should not be under-relaxed so that $\alpha_p = 1$ is used in Eq. (22).

Discussion of SIMPLEC. The changes that are necessary to incorporate SIMPLEC into a SIMPLE code are minor, but the consequences can be great. This will be demonstrated in two problems presented in detail at the end of the paper, but it may be helpful to preview these results by referring to Figs. 6 and 7.

When the S_P portions of the source terms [see Eq. (2)] in the momentum equations are zero, the SIMPLEC method becomes identical to the "consistent time step" (PS3) scheme of Raithby and Schneider [8]. This earlier scheme was derived by constraining the "time step" implied in the pressure correction to be consistent with the

time step implied in the solution of the momentum equations. This requires that,* in Eq. (22),

$$\alpha_p = \frac{1}{1+E} \tag{25}$$

It can be shown that, with this value of α_p, the values of $\alpha_p p'$ in SIMPLE are identical to the corresponding p' values in SIMPLEC, and that the resulting velocity corrections are also identical.

When the S_P in the momentum equations are nonzero, the SIMPLEC formulation should be adopted. SIMPLEC has the property that, when diffusion and advection of momentum are negligible compared to the pressure gradient and source terms, the approximation introduced into Eq. (23) that results in Eqs. (24) becomes exact. These conditions are approached in the analysis of the flow in heat exchangers, porous media, etc., where the pressure gradient is essentially in balance with drag forces.

Both the SIMPLE and SIMPLEC methods are appropriate for the solution of incompressible fluid flow problems for which the predominant factor governing solution cost is the pressure-veelocity coupling [8]. In other situations the interaction between the turbulence model and the momentum equations, or the interplay of the buoyancy forces in the momentum equations with the temperature or concentration, may be the factor mainly responsible for slow convergence. In such cases efficient treatment of the pressure-velocity coupling becomes of less consequence.

The p' Equation where Velocity Boundary Conditions or Pressure is Prescribed

In both the SIMPLE and SIMPLEC procedures, the pressure correction p' is used to both update the pressure, Eq. (14) or (22), and correct the velocities, Eq. (15) or (24). The values of p' are obtained by solving a set of equations, each of which resembles Eq. (17). But before such a solution is attempted, consideration must be given to the treatment of p' at boundaries where velocities boundary conditions are specified and at interior points where pressure is specified. These two topics are addressed in this section.

"Standard" treatment of p' at boundaries where velocities are prescribed. Suppose the boundary condition equation for u_e in Fig. 2*b* is of the form

$$u_e = Bu_w + C \tag{26}$$

where the suitable values of B and C would be selected to impart the desired boundary condition. The momentum equations are solved for u^*, using the corresponding boundary condition

$$u_e^* = Bu_w^* + C \tag{27}$$

where the same B and C are used in Eqs. (26) and (27). With a u^* solution that satisfies Eq. (27), p' is used to correct u^* to u. The boundary conditions on p' should be such that the resulting u field satisfies Eq. (26).

*There was a typographic error [9] in the original paper [8].

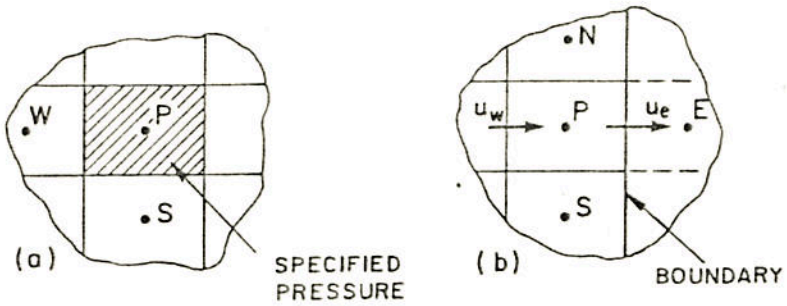

Fig. 2 Control volumes (*a*) for which pressure is specified, and (*b*) for which u_e is related to u_w by a boundary condition.

When the boundary velocity is specified (i.e., $B = 0$ and $C = u_{\text{specified}}$), a zero gradient on p' normal to the boundary gives the desired result [4]. Patankar [5] has since suggested an alternative treatment for this case which requires no explicit application of a p' boundary condition. The question of how p' should be treated for other velocity boundary conditions has not been addressed. Recommendation 3 provides a method that uses the velocity boundary conditions, such as Eq. (26), to implicitly assert the correct conditions on p'.

Recommendation 3: Treatment of p' where velocity boundary conditions are prescribed. Substitution of the expression for u_e given by Eq. (26) into the continuity equation, Eq. (16), for the volume centered at P in Fig. 2*b*, yields

$$(\rho uA)_w - (\rho A)_e(Bu_w + C) + (\rho vA)_s - (\rho vA)_n = 0 \tag{28}$$

The p' equation that results from the substitution for u_w, v_s, and v_n, using equations such as Eq. (24*a*), is again Eq. (17), but with the following coefficients:

$$a_E = 0 \qquad a_W = (\rho Ad)_w - (\rho A)_e d_w B \tag{29}$$

$$a_N = (\rho Ad)_n \qquad a_S = (\rho Ad)_s \tag{30}$$

$$a_P = a_E + a_W + a_N + a_S \tag{31}$$

$$b_P = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n \tag{32}$$

Note that, since a_E is zero, the influence of p'_E on p'_P is reduced to zero; that is, no explicit boundary condition relating p'_E to p'_P is required. A similar procedure can be repeated for each control volume that has one or more faces lying on a boundary where the velocity is prescribed.

This treatment results in coefficients in the p' equations that are dependent on the velocity boundary conditions. Once appropriate values of B and C in equations such as Eq. (26) are specified to convey the proper velocity boundary conditions, the appropriate boundary treatment of p' is implicitly assured.

For the special case of a specified normal velocity at a boundary, the present treatment results in coefficients of the p' equation that are identical to those which arise when Patankar's procedure is followed [5]. A treatment of boundary conditions similar to that described here was proposed independently by Maliska [16] for the solution of the equations of motion in non-orthogonal coordinates.

p' at points where pressure is specified. For incompressible fluid flows, only derivatives of pressure enter the equations of motion. If it desired to solve for a unique pressure

sure distribution, the pressure level at one point must be specified. (Such a pressure specification can itself significantly affect solution cost, but this is discussed in the following section.) In other problems, the flow is driven by a pressure difference that is externally imposed so that pressure specification at two or more points is required. At any location P where the pressure has a specified value p_{SPEC} (see Fig. 2*a*), the guessed pressure p_P^* is assigned this value. The pressure correction should not alter the pressure from this value so that, from Eq. (14),

$$p'_P = 0 \tag{33}$$

Recommendation 4: Modification of p' equation at points where pressure is specified. Equation (33) can be applied by modifying the coefficients in Eq. (17) for the corresponding P node as follows:

$$a_P = 1 \tag{34a}$$

$$a_E = a_W = a_N = a_S = b = 0 \tag{34b}$$

This is obvious and straightforward, but such a practice sometimes results in solution difficulties due to round-off. These arise when the a_P coefficients for the p' equations in the neighbor control volumes are much different from unity in magnitude. The recommended solution to this problem is to set a_P in Eq. (34*a*) to the same magnitude as a_P in the neighbor control volumes.

Solution of the p' Equation

Now that the formulation of the p' equation has been considered in some detail, several points related to the solution of this equation set will now be addressed. The solution of the p' equation can represent as much as 80% of the total cost of solving the fluid flow problem. It is therefore a high priority to solve for p' in an efficient manner. The use of direct solution methods tends to be unattractive because of large storage requirements and computer effort. Extremely fast Poisson equation solvers are available (e.g., [10]), but these are not applicable to the p' equation because of its nonseparable form. Furthermore, the coefficients of p' change on each cycle so that other sparse-matrix solvers that are applicable require a new decomposition each time the coefficients are updated. Iterative methods such as successive over-relaxation (SOR), Stone's [11] strongly implicit procedure (SIP), and the modified strongly implicit (MSI) method of Schneider and Zedan [12] are better suited to this application. Patankar [6] recommends a combination of a block correction to lines, followed by line-by-line iterations based on the tridiagonal matrix algorithm (TDMA). The latter method has the combined advantage of simplicity and low storage requirements. The following section contains a minor variation on Patankar's method which considerably enhances convergence.

The use of iterative solvers also raises the question of when iteration should be terminated. This section also proposes a convergence criterion that is particularly suitable for the p' equation.

Recommendation 5: TDMA solver for p'. Equation (17) for p' can be restated as follows for solution along a line of constant j:

$$a_{ij} p'_{ij} = b_{ij} p'_{i+1j} + c_{ij} p'_{i-1j} + [d_{ij} p'_{ij+1} + e_{ij} p'_{ij-1}] + f_{ij} \tag{35}$$

where

$$a_{ij} = a_P \qquad b_{ij} = a_E \qquad c_{ij} = a_W$$
$$d_{ij} = a_N \qquad e_{ij} = a_S \qquad f_{ij} = \mathrm{b}$$

and where i and j [illegible] locations in the x and y directions. With the dependent [illegible] line [those in brackets in Eq. (35)] temporarily fixed, a direct [illegible] values on the line can be obtained with one TDMA application. [illegible] solution is the basis of an iteration scheme that solves along each j line and [illegible] each i line, and repeats the pattern until convergence is achieved. The rate of convergence of such a scheme depends crucially on the treatment of the off-line dependent variables.

Suppose that a partially converged p' field, denoted by $[p']^0$, has been obtained from one or more TDMA-based iterations. In the current iteration Eq. (35) is to be solved along each j line, sweeping in the direction of increasing j. On the j line the best available estimate of p'_{ij-1} is that obtained from the just-completed solution along the $j-1$ line. This is the off-line value used in Eq. (35). The available estimate of p'_{ij+1} is from the previous iteration, i.e., $[p'_{ij+1}]^0$, and substitution of this value into Eq. (35) can cause slow convergence. To accelerate convergence an approximation, similar to Stone's partial cancellation [11], is introduced. With $[p']_{BE}$ defined as a better estimate of p'

$$[p'_{ij+1}]_{BE} = [p'_{ij+1}]^0 + (\theta - 1)(p'_{ij+1} - [p'_{ij+1}]^0) \tag{36a}$$
$$\approx [p'_{ij+1}]^0 + (\theta - 1)(p'_{ij} - [p'_{ij}]^0) \tag{36b}$$

In these expressions θ is a relaxation parameter such that for $\theta = 1$ $[p'_{ij+1}]_{BE}$ is taken as $[p'_{ij+1}]^0$. The approximation in Eq. (36b) is therefore introduced into Eq. (35) to obtain

$$\{a_{ij} - d_{ij}(\theta - 1)\}p'_{ij} = b_{ij}p'_{i+1j} + c_{ij}p'_{i-1j} + d_{ij}\{[p'_{ij+1}]^0 - (\theta - 1)[p'_{ij}]^0\} + e_{ij}p'_{ij-1} + f_{ij} \tag{37}$$

A similar estimate is made for solutions along i lines.

Figure 3 illustrates how the cost of obtaining a solution to the p' equation, to a given accuracy, depends on the value of θ employed. These results were obtained for the sliding lid problem illustrated in Fig. 4b. Experience to date suggests that, if the aspect ratio of the grid is not too far from unity, the optimal constant value of θ lies in the range 1.85–1.95, while $\theta \geqslant 2$ often results in divergence. A conservative value of $\theta = 1.85$ is therefore recommended.

The same solution method can be used for the transport equations. The dependence of the rate of convergence on θ is similar to that illustrated for p'.

Convergence criteria. If iteration of the p' equation is terminated before sufficient convergence is achieved, the continuity constraint is poorly satisfied by the corrected velocities. These velocities are later used to calculate new coefficients, so that the error is

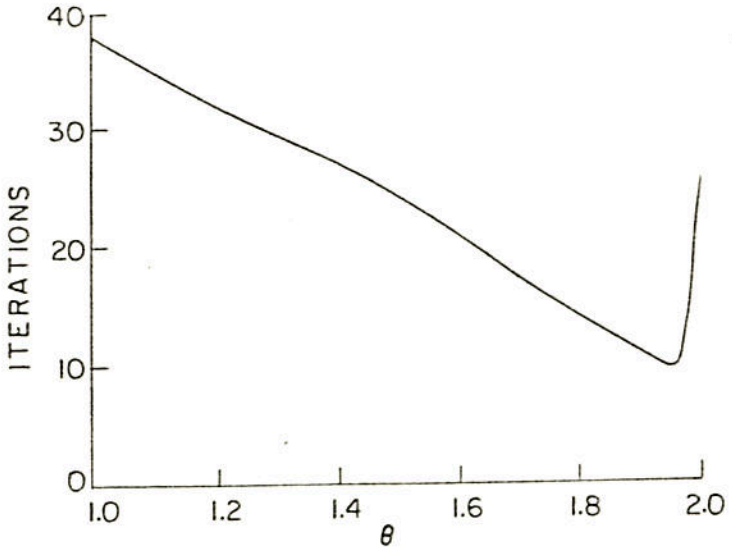

Fig. 3 Dependence on the value of θ of the number of iterations required to reduce the norm of the error in p' to 0.1% of its original value.

propagated with the possible result of divergence or slow convergence. On the other hand, it is uneconomical and wasteful to drive the p' equation to a tight convergence each time it is solved. The performance of the entire solution algorithm depends heavily on the criterion used for terminating iteration on the p' equation; recommendation 6, in the following section, proposes a suitable criterion.

The convergence criteria used to terminate the iterative solution of the momentum equations as well as other scalar equations are usually not so critical. The relaxation factor [E in Eq. (9)] built into the equation specifies the "time" through which the equation advances the solution. If the equation is not driven to convergence, the solution is roughly equivalent to that for a smaller time advance than is specified. The effect of this is usually harmless.

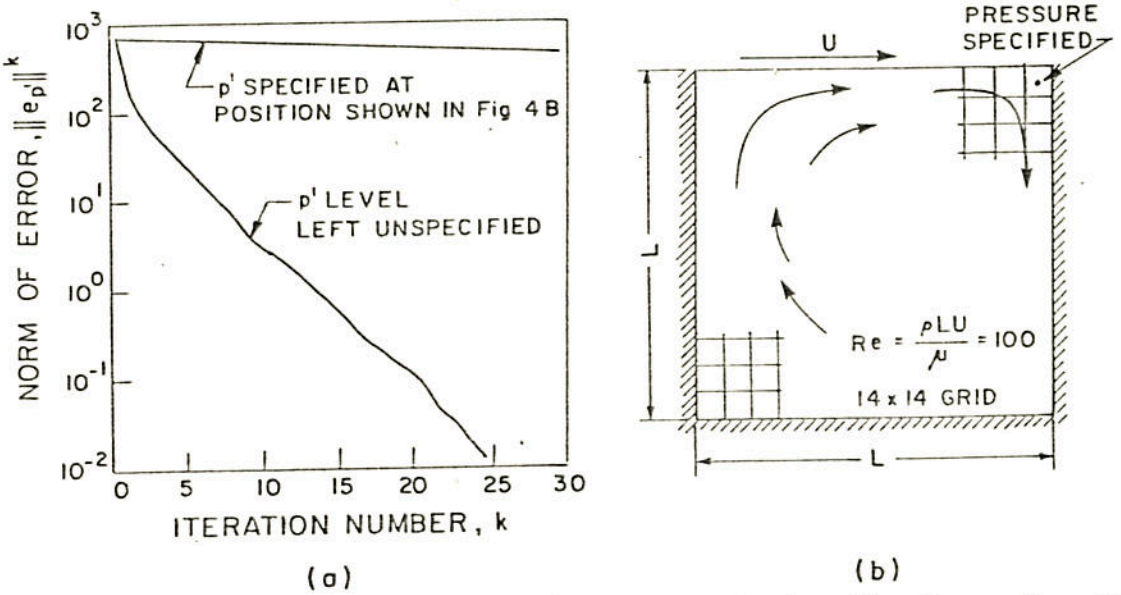

Fig. 4 (*a*) Reduction of the residual of the p' equation as a function of iteration number with and without p' specification. The problem is sketched in (*b*).

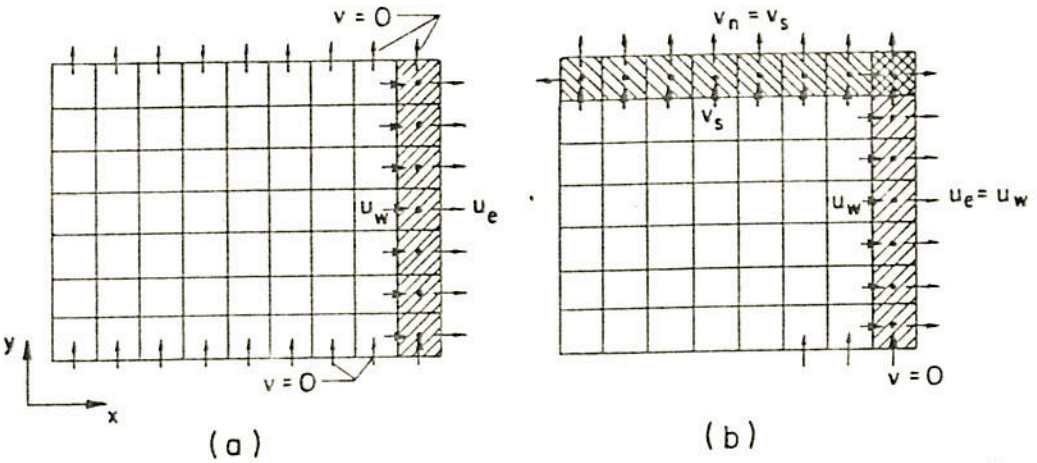

Fig. 5 Boundary conditions that lead to a redundant pressure equation (*a*) along a line or (*b*) at a point.

Recommendation 6: Convergence criteria for the p' equation. If $\overset{\circ}{p}{}'$ is the initial (guessed and normally assigned to be zero) p' distribution, the initial value of the Euclidean norm of the residuals $\|r_p\|^0$ is given by

$$\|r_p\|^0 = [\Sigma\,(a_E\overset{\circ}{p}{}'_E + a_W\overset{\circ}{p}{}'_W + a_N\overset{\circ}{p}{}'_N + a_S\overset{\circ}{p}{}'_S + b_P - a_P\overset{\circ}{p}{}'_P)^2\,]^{1/2} \tag{38a}$$

where the summation is over all interior volumes. After k iterations, the corresponding norm $\|r_p\|^k$ is

$$\|r_p\|^k = [\Sigma\,(a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S + b_P - a_P p'_P)^2\,]^{1/2} \tag{38b}$$

Iteration is continued until the following criterion is satisfied:

$$\|r_p\|^k \leqslant \gamma_p \|r_p\|^0 \tag{39}$$

Equation (39) guarantees that iteration has reduced the residual to at least the fraction γ_p of its initial value. Optimal values of the residual reduction factor γ_p typically range from 0.25 to 0.05.

If the solution is driven into a very tight convergence, the values of $\|r_p\|^0$ may already be near the round-off limits of the machine and Eq. (39) will demand convergence to better than round-off. This should be avoided by terminating iteration before $\|r_p\|^k$ reaches the round-off limit.

There are two advantages of the proposed criterion. First, the same value of γ_p can be used for most problems. This avoids costly trials to determine a suitable value of the convergence criterion. The second is that the number of iterations on the p' equation for each cycle is roughly the same [13]. Some other criteria, such as

$$\|r_p\|^k \leqslant \epsilon_p \tag{40}$$

where ϵ_p is a prescribed tolerance, require too many iterations in the early cycles, and too few as the converged solution to the fluid flow problem is approached. The result is excessive computer time requirements. Like many others, the criterion given by Eq. (39) suffers the disadvantage that computational effort is required to calculate the residuals $\|r_p\|$.

A similar convergence criterion can also be used for the other equations.

Fixing the level of p'. It was mentioned in the previous section that fixing the pressure at a point could significantly affect the solution cost. This and other consequences of pressure specifications are now addressed.

Suppose that in a two-dimensional, constant-density problem all normal velocities were specified at the boundaries; these, of course, must be specified in such a way that mass is conserved globally. Now, if mass conservation is enforced at all interior control volumes except for one, then global mass conservation guarantees satisfaction also for the volume at which it was not locally enforced. In fact, the local enforcement of mass conservation for this volume results in a redundancy in the p' equation set, which, in turn, causes a singular matrix to be encountered when a direct solver is used. This problem arises because the p' equation and its boundary conditions only establish p' to within an additive constant; i.e., the level of p' is not fixed. Specification of a value of p' at any interior point in place of the continuity constraint establishes the level and removes the problem. For further information on this, the reader is referred to a lucid discussion by Patankar [5].

Iterative solvers, as opposed to direct solvers, may often be successfully used without specifying the level of p'. In fact, by allowing p' to "seek its own level" rather than enforcing the level, the convergence rate can be increased [5, 14]. Just how important this effect can be is not well appreciated. To illustrate this, the norm of the error in p' (the square root of the sum of the squares of the error, $\|e'_p\|$) is plotted against the number of solver iterations for the sliding lid problem [15] illustrated in Fig. 4; the solver described in the section 'Solution of the p' Equation' was used. In the first case, $p' = 0$ was specified at the point shown in Fig. 4*b*, while in the second, no specification was made. The dramatic reduction in the rate of convergence that results from the p' specification is illustrated in Fig. 4*a*.

It is usual to specify the gradient of velocity, rather than velocity itself, at boundaries where the fluid leaves the calculation domain, and this gives rise to quite different pressure specification requirements. Suppose in Fig. 5*a* that $u_e = u_w$ is specified on the outflow boundary, while $v = 0$ is prescribed along the two boundaries of constant y. For the bottom control volume at the outflow boundary, the boundary conditions, together with the continuity constraint, dictate (for constant density) that v at the upper surface of this control volume is zero. A similar application to each but the uppermost volume requires that all the v's be zero so that the continuity constraint for the top volume is redundant. Just as in the previous case, in which the equations established the pressure in the full two-dimensional region only to within an additive constant, now the pressure *along this line* can only be determined to within an additive constant. It also follows, as before, that a direct solution would fail, but now a line solver applied to this line is a direct solver. To apply the TDMA solver along this line would require that one of the continuity equations on the line be replaced by a specification of p'. It is interesting to note that a point-iterative solver would not encounter difficulties.

In some cases, gradients in the normal velocity are specified as zero along two boundaries that intersect. In this case, the boundary conditions make the continuity constraint redundant for the doubly shaded control volume in Fig. 5*b*, which lies at the intersection of the boundaries. Unless this redundancy is removed, the TDMA solver would fail along either line and a point-iterative solver would fail at the corner volume. Replacement of the continuity constraint by a specification of p' at the intersection volume is required to remove these solution difficulties.

Extension of recommendations to the application of SIMPLER. The SIMPLER method of Patankar [5, 6] improves upon the method used by SIMPLE to estimate the pressure p by solving for a separate equation for pressure. The equation for pressure has the same coefficients as the p' equation, with the source term alone being different. The above recommendations related to the application of boundary conditions and to the solution of the p' equation are readily extended to the p equation. Although it is possible to extend the SIMPLEC approximation to the SIMPLER procedure, it is not necessary since, in SIMPLER, separate pressure-like equations are solved to correct the velocity and to estimate the new pressure field.

Pressure, unlike p', does not vanish as the solution to the equation set is approached. For some problems, the authors have experienced convergence difficulties, when solving the pressure equation of SIMPLER, which were caused by computer round-off. This problem can be overcome by rewriting the p equation as a pressure correction equation. Not only does this reduce the effect of round-off, but the similarity between this equation and the original p' equation permits the subroutines developed for p' to be directly used for the solution of this pressure correction.

COMPARISON OF SOLUTION METHODS

For the purpose of demonstrating the applicability of the proposals set forth and of evaluating the relative performance of SIMPLE, SIMPLER, and SIMPLEC, two fluid flow problems involving, respectively, an internal confined flow and an external flow are considered. The laminar recirculating flow of water in a 4:1 rectangular tank on a 20 X 20 grid (see Fig. 6*a*) was used as the internal flow test problem. As an external flow problem, the laminar flow of air over a rearward-facing step, as shown in Fig. 6*b*, was computed on a 25 X 25 grid. All calculations were carried out on an IBM 4341-Type 1

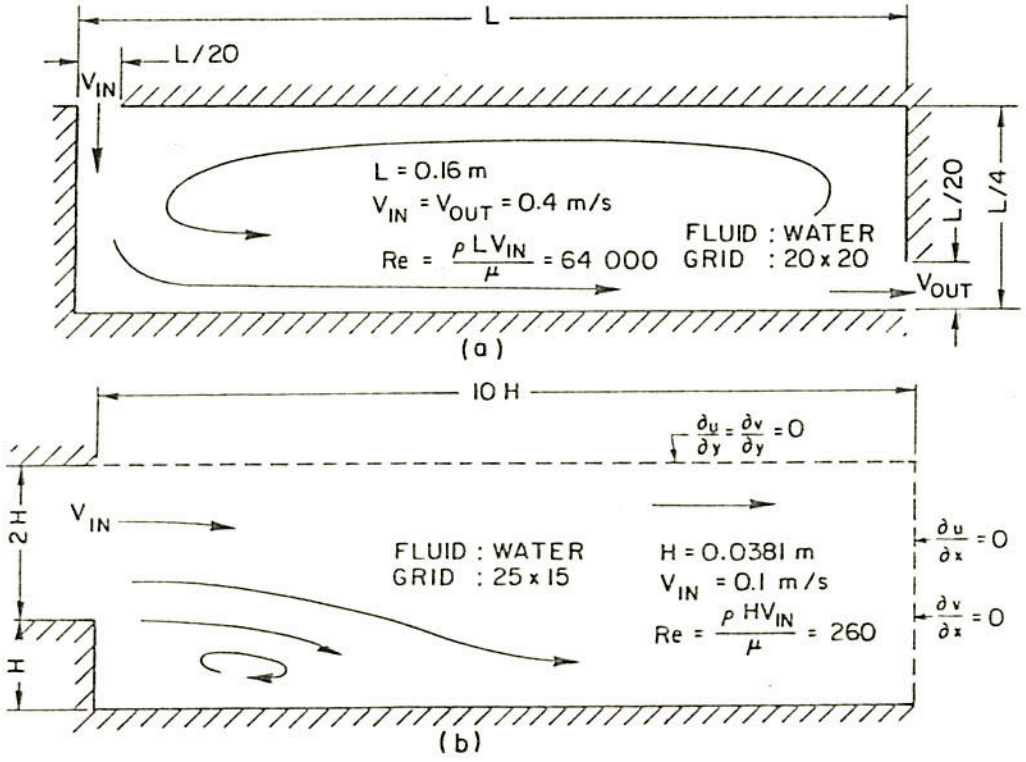

Fig. 6 Geometries for the two comparison problems.

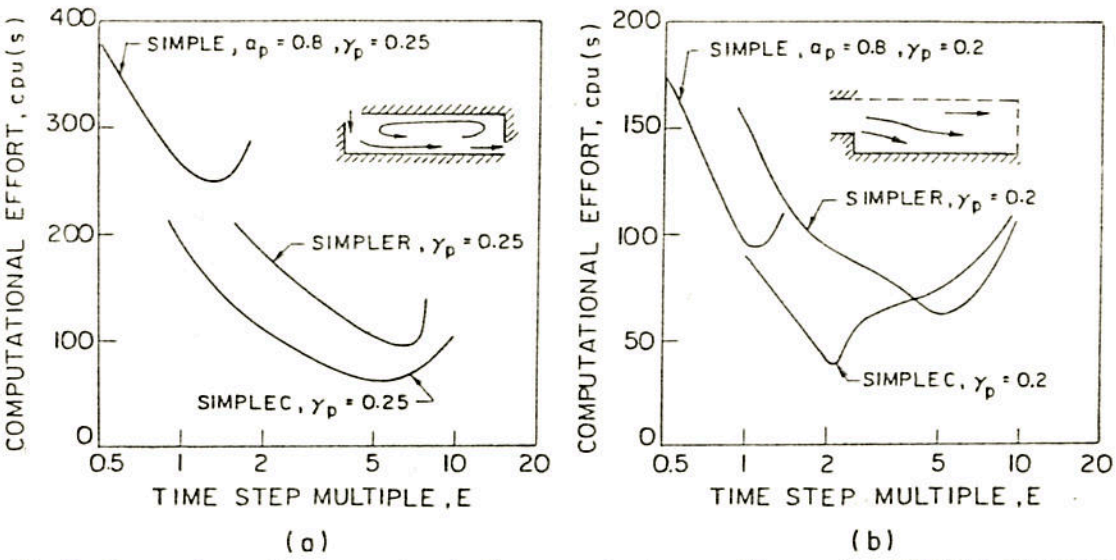

Fig. 7 Comparison of computational effort to solve two problems using SIMPLE, SIMPLER, and SIMPLEC.

computer, using single-precision FORTRAN H-Extended. All the numerical results were obtained with the same basic code, so that the TDMA solver, the criteria for terminating iteration, the application of boundary conditions, etc., were as recommended and only the method of treating the pressure-velocity coupling changed. Comparisons are presented of the computational effort required to achieve pressures that agree with "exact" values to within 0.5% of the range of pressure in the problem. The "exact" solution was established, using the same grid and differencing scheme, by driving the solution to a very tight convergence.

The effort required for each of the methods to satisfy the above requirement is dependent on a number of parameters, including the distorted time step multiple E and the residual reduction factor γ_p. Figure 7, *a* and *b*, illustrate the sensitivity of computational effort to E for each method; the residual reduction factors were chosen so that the computational effort was minimized at its optimal E value. The values of γ_p are indicated in Fig. 7.

The results shown in Fig. 7 report effort for optimal residual reduction factors. It has been determined through numerical experiments that the sensitivity of each of the methods to the residual reduction factors (i.e., γ_p) is very similar. In addition, to ensure that only the nature of the pressure-velocity coupling is being observed, numerical experiments similar to those outlined above were performed with the coefficients of the momentum conservation equations held fixed at their "exact" values. The results of these experiments indicate trends very similar to those illustrated in Fig. 7.

CONCLUDING REMARKS

A number of modifications to the application of SIMPLE and its variants have been recommended in this paper, including

- The E-factor formulation
- The SIMPLEC approximation
- Treatment of p' where velocity boundary conditions are specified and where pressure is specified

- TDMA solver for p'
- Convergence criterion for the p' equation

These modifications result in either simplification of the application of SIMPLE-like methods or improved economy. The applicability of the proposals set forth was illustrated in two problems and the computational costs of SIMPLE, SIMPLER, and SIMPLEC were compared. These results illustrate that both SIMPLER and SIMPLEC are substantially more economic than SIMPLE, and that SIMPLEC is usually less expensive than SIMPLER for the problems solved.

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