Section 7.6 The Pivot Theorem and Its Implications

BASIS PROBLEMS REVISITED

Let us reconsider the problem of finding a basis for a subspace W spanned by a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$. There are two variations of this problem:

1. Find any basis for W.
2. Find a basis for W consisting of vectors from S.

We have already seen that the first basis problem can be solved by making the vectors in S into row vectors of a matrix, reducing the matrix to row echelon form, and then extracting the nonzero row vectors (Example 4 of Section 7.3).

One way to solve the second basis problem is to create a matrix A that has the vectors of S as *column vectors*. This makes W into the column space of A and converts the problem into one of finding a basis for the column space of A consisting of column vectors of A.

More generally, suppose that A and B are row equivalent matrices that have been partitioned into column vectors as

$$A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_n] \quad \text{and} \quad B = [\mathbf{c}'_1 \quad \mathbf{c}'_2 \quad \cdots \quad \mathbf{c}'_n]$$

→ row equivalent matrices: matrices related by elementary row operations.

It follows from part (*b*) of Theorem 7.3.7 that the homogeneous linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set and hence the same is true of the vector equations

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{0}$$

and

$$x_1\mathbf{c}'_1 + x_2\mathbf{c}'_2 + \cdots + x_n\mathbf{c}'_n = \mathbf{0}$$

since these are the vector forms of the two homogeneous systems.

It follows from these equations that the column vectors of A are linearly independent if and only if the column vectors of B are linearly independent; and further, if the column vectors of A and B are linearly dependent, then those column vectors have the same dependency relationships.

That is, *elementary row operations do not change linear independence or dependence of column vectors, and in the case of linear dependence they do not change dependency relationships among column vectors.*

It can be proved that these conclusions also apply to any subset of the column vectors, which leads us to the following theorem.

Theorem 7.6.1 *Let A and B be row equivalent matrices.*

(a) *If some subset of column vectors from A is linearly independent, then the corresponding column vectors from B are linearly independent, and conversely.*

(b) *If some subset of column vectors from B is linearly dependent, then the corresponding column vectors from A are linearly dependent, and conversely. Moreover, the column vectors in the two matrices have the same dependency relationships.*

EXAMPLE 1 A Basis for col(A) Consisting of Column Vectors of A

Find a subset of the column vectors of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that forms a basis for the column space of A.

Solution We leave it for you to confirm that reducing A to row echelon form by Gaussian elimination yields

$$U = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since elementary row operations do not alter rank, and since U has three nonzero rows, it follows that A has rank 3 and hence that the column space of A is three-dimensional.

Thus, if we can find three linearly independent column vectors in A, then those vectors will form a basis for the column space of A by Theorem 7.2.6.

For this purpose, focus on the column vectors of U that have the leading 1's (columns 1, 3, and 5):

$$U = \begin{bmatrix} 1 & * & 4 & * & 5 & * \\ 0 & * & 1 & * & -2 & * \\ 0 & * & 0 & * & 1 & * \\ 0 & * & 0 & * & 0 & * \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & * & 4 & * & 5 & * \\ 0 & * & 1 & * & -2 & * \\ 0 & * & 0 & * & 1 & * \\ 0 & * & 0 & * & 0 & * \end{bmatrix}$$

If we progress from left to right through these three column vectors, we see that none of them is a linear combination of predecessors because there is no way to obtain its leading 1 by such a linear combination.

This implies that these column vectors are linearly independent, and hence so are the corresponding column vectors of A by Theorem 7.6.1. Thus, the column vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A. ■

This example shows that if two matrices are assembled, one with columns 1, 3, and 5 of U and the other with the corresponding columns of A, they have the same solution to a homogeneous linear system, which is a trivial solution.

Definition 7.6.2 The column vectors of a matrix A that lie in the column positions where the leading 1's occur in the row echelon forms of A are called the ***pivot columns*** of A.

It is a straightforward matter to convert the method of Example 1 into a proof of the following general result.

Theorem 7.6.3 (***The Pivot Theorem***) *The pivot columns of a nonzero matrix A form a basis for the column space of A.*

REMARK At the end of Section 7.5 we gave a slightly tedious proof of the rank theorem (Theorem 7.5.1). Theorem 7.6.3 now provides us with a simpler way of seeing that result.

We need only observe that the number of pivot columns in a nonzero matrix A is the same as the number of leading 1's in a row echelon form, which is the same as the number of nonzero rows in that row echelon form.

Thus, Theorems 7.6.3 and part (c) of Theorem 7.3.7 imply that the column space and row space have the same dimension.

We now have all of the mathematical machinery required to solve the second basis problem posed at the beginning of this section:

Algorithm 1 If W is the subspace of R^n spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, then the following procedure extracts a basis for W from S and expresses the vectors of S that are not in the basis as linear combinations of the basis vectors.

Step 1. Form the matrix A that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ as successive column vectors.

Step 2. Reduce A to a row echelon form U, and identify the columns with the leading 1's to determine the pivot columns of A.

Step 3. Extract the pivot columns of A to obtain a basis for W. If appropriate, rewrite these basis vectors in comma-delimited form.

Step 4. If it is desired to express the vectors of S that are not in the basis as linear combinations of the basis vectors, then continue reducing U to obtain the reduced row echelon form R of A.

Step 5. By inspection, express each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that contain leading 1's. Replace the column vectors in these linear combinations by the corresponding column vectors of A to obtain equations that express the column vectors of A that are not in the basis as linear combinations of basis vectors.

EXAMPLE 2 Extracting a Basis from a Set of Spanning Vectors

Let W be the subspace of R^4 that is spanned by the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0)$$
$$\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

(a) Find a subset of these vectors that forms a basis for W.

(b) Express those vectors of $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ that are not in the basis as linear combinations of those vectors that are.

Solution ***(a)*** We start by creating a matrix whose column space is W. Such a matrix is

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{array} \end{array}$$

To find the pivot columns, we reduce A to row echelon form U by Gaussian elimination. We leave it for you to confirm that this yields

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1's occur in columns 1, 2, and 4, so the basis vectors for W (expressed in comma-delimited form) are

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_4 = (2, -1, 4, -7)$$

Solution (b) For this problem it will be helpful to take A all the way to reduced row echelon form. We leave it for you to continue the reduction of U and confirm that the reduced row echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}'_1 & \mathbf{v}'_2 & \mathbf{v}'_3 & \mathbf{v}'_4 & \mathbf{v}'_5 \end{array}$$

where we have named the column vectors of R as $\mathbf{v}'_1$, $\mathbf{v}'_2$, $\mathbf{v}'_3$, $\mathbf{v}'_4$, and $\mathbf{v}'_5$. Our goal is to express $\mathbf{v}_3$ and $\mathbf{v}_5$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$.

However, we know that elementary row operations do not alter dependency relationships among column vectors.

Thus, if we can express $\mathbf{v}_3'$ and $\mathbf{v}_5'$ as linear combinations of $\mathbf{v}_1'$, $\mathbf{v}_2'$, and $\mathbf{v}_4'$, then those same linear combinations will apply to the corresponding column vectors of A. By inspection from R,

$$\mathbf{v}_3' = 2\mathbf{v}_1' - \mathbf{v}_2' \quad \text{and} \quad \mathbf{v}_5' = \mathbf{v}_1' + \mathbf{v}_2' + \mathbf{v}_4'$$

$$R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1' & \mathbf{v}_2' & \mathbf{v}_3' & \mathbf{v}_4' & \mathbf{v}_5' \end{array} \end{array}$$

Thus, it follows that

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2 \quad \text{and} \quad \mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$$

As a check, you may want to confirm directly from the components of the vectors that these relationships are correct. ■

BASES FOR THE FUNDAMENTAL SPACES OF A MATRIX

We have already seen how to find bases for three of the four fundamental spaces of a matrix A by reducing the matrix to a row echelon form U or its reduced row echelon form R:

1. The nonzero rows of U form a basis for row(A).
2. The columns of U with leading 1's identify the pivot columns of A, and these form a basis for col(A).
3. The canonical solutions of $A\mathbf{x} = \mathbf{0}$ form a basis for null(A), and these are readily obtained from the system $R\mathbf{x} = \mathbf{0}$.

A basis for null(A^T) can be obtained by using row reduction of A^T to solve $A^T\mathbf{x} = \mathbf{0}$.

fourth fundamental space

However, it would be desirable to have an algorithm for finding a basis for null(A^T) by row reduction of A, since we would then have a common procedure for producing bases for all four fundamental spaces of A. We will now show how to do this.

Suppose that A is an $m \times n$ matrix with rank k, and we are interested in finding a basis for null(A^T) using elementary row operations on A. Recall that the dimension of null(A^T) is $m - k$ (number of rows − rank), so if $k = m$, then null(A^T) is the zero subspace of R^m, which has no basis.

This being the case, we will assume that $k < m$. With this assumption, we are guaranteed that every row echelon form of A has at least one zero row (why?). Here is the procedure (which is justified by a proof at the end of this section):

Algorithm 2 If A is an $m \times n$ matrix with rank k, and if $k < m$, then the following procedure produces a basis for null(A^T) by elementary row operations on A.

Step 1. Adjoin the $m \times m$ identity matrix I_m to the right side of A to create the partitioned matrix $[A \mid I_m]$.

Step 2. Apply elementary row operations to $[A \mid I_m]$ until A is reduced to a row echelon form U, and let the resulting partitioned matrix be $[U \mid E]$.

Step 3. Repartition $[U \mid E]$ by adding a horizontal rule to split off the zero rows of U. This yields a matrix of the form

$$\begin{array}{cc} \left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right] & \begin{array}{c} k \\ m-k \end{array} \\ \begin{array}{cc} n & \quad m \end{array} & \end{array}$$

where the margin entries indicate sizes.

Step 4. The row vectors of E_2 form a basis for null(A^T).

EXAMPLE 3 A Basis for null(A^T) by Row Reduction of A

In Example 1 we found a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Apply Algorithm 2 to find a basis for null(A^T) by row reduction.

Solution In Example 1 we found that A has rank 3, so we know without performing any computations that null(A^T) has dimension 1 (number of rows − rank). Following the steps in the algorithm we obtain

$$[A \mid I_4] = \left[\begin{array}{cccccc|cccc} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 & 1 & 0 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 & 0 & 0 & 1 \end{array}\right] \quad \textbf{Step 1}$$

$$[U \mid E] = \left[\begin{array}{rrrrrr|rrrr} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \textbf{Step 2}$$

$$\left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right] = \left[\begin{array}{rrrrrr|rrrr} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \textbf{Step 3}$$

As anticipated, the matrix E_2 has only one row vector, namely

$$\mathbf{w} = [1 \quad 0 \quad 0 \quad 1]$$

This vector is a basis for $\text{null}(A^T)$.

row(A^T)

Since we know that null(A^T) and col(A) are orthogonal complements, the vector $\mathbf{w}$ should be orthogonal to the basis vectors for col(A) obtained in Example 1. We leave it for you confirm that this is so by showing that $\mathbf{w} \cdot \mathbf{c}_1 = 0$, $\mathbf{w} \cdot \mathbf{c}_3 = 0$, and $\mathbf{w} \cdot \mathbf{c}_5 = 0$. ■

$$\mathbf{w} = [1 \quad 0 \quad 0 \quad 1] \qquad \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

OPTIONAL *Proof of Algorithm 2*

Assume that A is an $m \times n$ matrix with rank k, where $k < m$, and apply elementary row operations to $[A \mid I_m]$ until A is reduced to a row echelon form U. The row operations that reduce A to U can be performed by multiplying A on the left by an appropriate product E of elementary matrices. Thus,

$$EA = U \tag{5}$$

where E is invertible, since it is a product of elementary matrices, each of which is invertible. Multiplying $[A \mid I_m]$ on the left by E yields

$$E[A \mid I_m] = [EA \mid EI_m] = [U \mid E]$$

An elementary matrix is a matrix that results from applying a single elementary row operation to an identity matrix.
Elementary matrices are always square and invertible (see section 3.3 of Anton&Busby).

Now partition the matrix $[U \mid E]$ as

$$[U \mid E] = \left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right] \tag{6}$$

where V is the $k \times n$ matrix of nonzero rows in U. It now follows from (5) and (6) that

$$\left[\begin{array}{c} V \\ \hline 0 \end{array}\right] = U = EA = \left[\begin{array}{c} E_1 \\ \hline E_2 \end{array}\right] A = \left[\begin{array}{c} E_1 A \\ \hline E_2 A \end{array}\right]$$

from which we see that $E_2 A = 0$.

We see that $E_2A = 0$.

If we view the entries in E_2A as dot products of row vectors from E_2 with column vectors from A, then the equation $E_2A = 0$ implies that each row vector of E_2 is orthogonal to each column vector of A. This places the row vectors of E_2 in the orthogonal complement of $\text{col}(A)$, which is $\text{null}(A^T)$.

Thus, it only remains to show that the row vectors of E_2 form a basis for $\text{null}(A^T)$.

Let us show first that the row vectors of E_2 are linearly independent. Since E is invertible, its row vectors are linearly independent by Theorem 7.4.4, and this implies that the row vectors of E_2 are linearly independent, since they are a subset of the row vectors of E.

Moreover, there are $m - k$ row vectors in E_2, and the dimension of $\text{null}(A^T)$ is also $m - k$, so the row vectors of E_2 must be a basis for $\text{null}(A^T)$ by Theorem 7.2.6.

■

Theorem 7.6.4 (**Column-Row Factorization**) *If A is a nonzero $m \times n$ matrix of rank k, then A can be factored as*

$$A = CR \tag{1}$$

where C is the $m \times k$ matrix whose column vectors are the pivot columns of A and R is the $k \times n$ matrix whose row vectors are the nonzero rows in the reduced row echelon form of A.

Proof As in Algorithm 2, adjoin the $m \times m$ identity matrix I_m to the right side of A, and apply elementary row operations to $[A \mid I_m]$ until A is in its reduced row echelon form R_0. If the resulting partitioned matrix is $[R_0 \mid E]$, then E is the product of the elementary matrices that perform the row operations, so

$$EA = R_0 \tag{2}$$

Partition R_0 and E^{-1} as

$$R_0 = \left[\begin{array}{c} R \\ \hline 0 \end{array}\right] \quad \text{and} \quad E^{-1} = [C \mid D]$$

where the matrix R consists of the nonzero row vectors of A, the matrix C consists of the first k column vectors of E, and the matrix D consists of the last $m - k$ columns of E.

E^{-1} E^{-1}

Thus, we can rewrite (2) as

$$A = E^{-1}R_0 = [C \mid D]\left[\frac{R}{0}\right] = CR + D0 = CR \tag{3}$$

It now remains to show that the successive columns of C are the successive pivot columns of A. For this purpose suppose that the pivot columns of A (and hence R_0) have column numbers

$$c_1, c_2, \ldots, c_k$$

A moment's reflection should make it evident that the column vectors of R in those positions are the standard unit vectors

(→ see, for instance, solution (b) of Example 2.)

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$$

in R^k. Thus, (3) implies that the jth pivot column of A is $C\mathbf{e}_j$, which is the jth column of C. ■

To fully understand the final statement above, it should be noticed in (3) that:

(a) Elementary row operations do not change linear independence of column vectors, which applies to A and R_0 (Theorem 7.6.1);

(b) R consists of nonzero row vectors of the reduced row echelon form R_0;

(c) The pivot columns of R_0 correspond to columns of R composed of standard unit vectors.

EXAMPLE 4

Column-Row Factorization

The reduced row echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 8 \\ -1 & -1 & -5 \\ 2 & 5 & 19 \end{bmatrix}$$

is

$$R_0 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(verify), so A has the column-row factorization

$$\underset{A}{\begin{bmatrix} 1 & 2 & 8 \\ -1 & -1 & -5 \\ 2 & 5 & 19 \end{bmatrix}} = \underset{C}{\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 2 & 5 \end{bmatrix}} \underset{R}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}}$$

■

Theorem 7.6.5 (**Column-Row Expansion**) *If A is a nonzero matrix of rank k, then A can be expressed as*

$$A = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_k\mathbf{r}_k \tag{4}$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are the successive pivot columns of A and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ are the successive nonzero row vectors in the reduced row echelon form of A.

EXAMPLE 5

Column-Row Expansion

From the column-row factorization obtained for the matrix A in Example 4, the corresponding column-row expansion of A is

$$\begin{bmatrix} 1 & 2 & 8 \\ -1 & -1 & -5 \\ 2 & 5 & 19 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [1 \quad 0 \quad 2] + \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} [0 \quad 1 \quad 3] = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & -2 \\ 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 6 \\ 0 & -1 & -3 \\ 0 & 5 & 15 \end{bmatrix}$$

■