# Section 8.4 Quadratic Forms

## DEFINITION OF A QUADRATIC FORM

Expressions of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

occurred in our study of linear equations and linear systems. If $a_1, a_2, \ldots, a_n$ are treated as fixed constants, then this expression is a real-valued function of the $n$ variables $x_1, x_2, \ldots, x_n$ and is called a ***linear form*** on $R^n$. All variables in a linear form occur to the first power and there are no products of variables.

Here we will be concerned with ***quadratic forms*** on $R^n$, which are functions of the form

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 + (\text{all possible terms } a_kx_ix_j \text{ in which } x_i \text{ and } x_j \text{ are distinct})$$

The terms of the form $a_kx_ix_j$ are called ***cross product terms***.

It is common to combine the cross product terms involving $x_i x_j$ with those involving $x_j x_i$ to avoid duplication. Thus, a general quadratic form on $R^2$ would typically be expressed as

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 \tag{1}$$

and a general quadratic form on $R^3$ as

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 \tag{2}$$

If, as usual, we do not distinguish between the number $a$ and the $1 \times 1$ matrix $[a]$, and if we let $\mathbf{x}$ be the column vector of variables, then (1) and (2) can be expressed in matrix form as

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 =$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

Note that the matrix $A$ in these formulas is symmetric and that its diagonal entries are the coefficients of the squared terms and its off-diagonal entries are half the coefficients of the cross product terms.

In general, if $A$ is a symmetric $n \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ column vector of variables, then we call the function

$$Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \tag{3}$$

the ***quadratic form associated with A***. When convenient, (3) can be expressed in dot product notation as

$$Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(A\mathbf{x}) = \mathbf{x} \cdot A\mathbf{x} = A\mathbf{x} \cdot \mathbf{x} \tag{4}$$

In the case where $A$ is a diagonal matrix, the quadratic form $Q_A$ has no cross product terms; for example, if $A$ is the $n \times n$ identity matrix, then

$$Q_A(\mathbf{x}) = \mathbf{x}^T I \mathbf{x} = \mathbf{x}^T \mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

and if $A$ has diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$Q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

## EXAMPLE 1 Expressing Quadratic Forms in Matrix Notation

In each part, express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where $A$ is symmetric.

(a) $2x^2 + 6xy - 5y^2$ (b) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_2$

***Solution*** The diagonal entries of $A$ are the coefficients of the squared terms, and the off-diagonal entries are half the coefficients of the cross product terms, so we obtain

$$2x^2 + 6xy - 5y^2 = [x \quad y]\begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \blacksquare$$

## CHANGE OF VARIABLE IN A QUADRATIC FORM

There are three important kinds of problems that occur in applications of quadratic forms:

1. If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on $R^2$ or $R^3$, what kind of curve or surface is represented by the equation $\mathbf{x}^T A\mathbf{x} = k$?
2. If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on $R^n$, what conditions must $A$ satisfy for $\mathbf{x}^T A\mathbf{x}$ to have positive values for $\mathbf{x} \neq \mathbf{0}$?
3. If $\mathbf{x}^T A\mathbf{x}$ is a quadratic form on $R^n$, what are its maximum and minimum values if $\mathbf{x}$ is constrained to satisfy $\|\mathbf{x}\| = 1$?

We will consider the first two problems in this section and the third problem in the next section.

Many of the techniques for solving these problems are based on simplifying the quadratic form $\mathbf{x}^T A\mathbf{x}$ by making a substitution

$$\mathbf{x} = P\mathbf{y} \tag{5}$$

that expresses the variables $x_1, x_2, \ldots, x_n$ in terms of new variables $y_1, y_2, \ldots, y_n$. If $P$ is invertible, then we call (5) a ***change of variable***, and if $P$ is orthogonal, we call (5) an ***orthogonal change of variable***.

If we make the change of variable $\mathbf{x} = P\mathbf{y}$ in the quadratic form $\mathbf{x}^T A\mathbf{x}$, then we obtain

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T A P\mathbf{y} = \mathbf{y}^T(P^T A P)\mathbf{y} \tag{6}$$

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^T AP)\mathbf{y} \tag{6}$$

The matrix $B = P^T AP$ is symmetric (verify), so the effect of the change of variable is to produce a new quadratic form $\mathbf{y}^T B\mathbf{y}$ in the variables $y_1, y_2, \ldots, y_n$.

In particular, if we choose $P$ to orthogonally diagonalize $A$, then the new quadratic form will be $\mathbf{y}^T D\mathbf{y}$, where $D$ is a diagonal matrix with the eigenvalues of $A$ on the main diagonal; that is,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

Thus, we have the following result, called the ***principal axes theorem***, for reasons that we will explain shortly.

**Theorem 8.4.1** (***The Principal Axes Theorem***) *If $A$ is a symmetric $n \times n$ matrix, then there is an orthogonal change of variable that transforms the quadratic form $\mathbf{x}^T A\mathbf{x}$ into a quadratic form $\mathbf{y}^T D\mathbf{y}$ with no cross product terms. Specifically, if $P$ orthogonally diagonalizes $A$, then making the change of variable $\mathbf{x} = P\mathbf{y}$ in the quadratic form $\mathbf{x}^T A\mathbf{x}$ yields the quadratic form*

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

*in which $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$ corresponding to the eigenvectors that form the successive columns of $P$.*

Recall that according to theorem 8.3.4, a matrix is orthogonally diagonalizable iif it is symmetric.

**EXAMPLE 2** An Illustration of the Principal Axes Theorem

Find an orthogonal change of variable that eliminates the cross product terms in the quadratic form $Q = x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$, and express $Q$ in terms of the new variables.

***Solution*** The quadratic form can be expressed in matrix notation as

$$Q = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation of the matrix $A$ is

$$\begin{vmatrix} \lambda - 1 & 2 & 0 \\ 2 & \lambda & -2 \\ 0 & -2 & \lambda + 1 \end{vmatrix} = \lambda^3 - 9\lambda = \lambda(\lambda + 3)(\lambda - 3) = 0$$

so the eigenvalues are $\lambda = 0, -3, 3$.

Distinct eigenvalues.

We leave it for you to show that orthonormal bases for the three eigenspaces are

$$\lambda = 0: \quad \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \qquad \lambda = -3: \quad \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \qquad \lambda = 3: \quad \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

Thus, a substitution $\mathbf{x} = P\mathbf{y}$ that eliminates the cross product terms is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This produces the new quadratic form

$$Q = \mathbf{y}^T(P^TAP)\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -3y_2^2 + 3y_3^2$$

in which there are no cross product terms. ■

**REMARK** If $A$ is a symmetric $n \times n$ matrix, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is a real-valued function whose range is the set of all possible values for $\mathbf{x}^T A\mathbf{x}$ as $\mathbf{x}$ varies over $R^n$.

It can be shown that a change of variable $\mathbf{x} = P\mathbf{y}$ does not alter the range of a quadratic form; that is, the set of all values for $\mathbf{x}^T A\mathbf{x}$ as $\mathbf{x}$ varies over $R^n$ is the same as the set of all values for $\mathbf{y}^T(P^TAP)\mathbf{y}$ as $\mathbf{y}$ varies over $R^n$.

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T A P\mathbf{y} = \mathbf{y}^T(P^TAP)\mathbf{y} \qquad (6)$$

Why principal axes?

'duplo cone circular reto'

# QUADRATIC FORMS IN GEOMETRY

Recall that a ***conic section*** or ***conic*** is a curve that results by cutting a double-napped cone with a plane (Figure 8.4.1). The most important conic sections are ellipses, hyperbolas, and parabolas, which occur when the cutting plane does not pass through the vertex.

Circles are special cases of ellipses that result when the cutting plane is perpendicular to the axis of symmetry of the cone. If the cutting plane passes through the vertex, then the resulting intersection is called a ***degenerate conic***. The possibilities are a point, a pair of intersecting lines, or a single line.

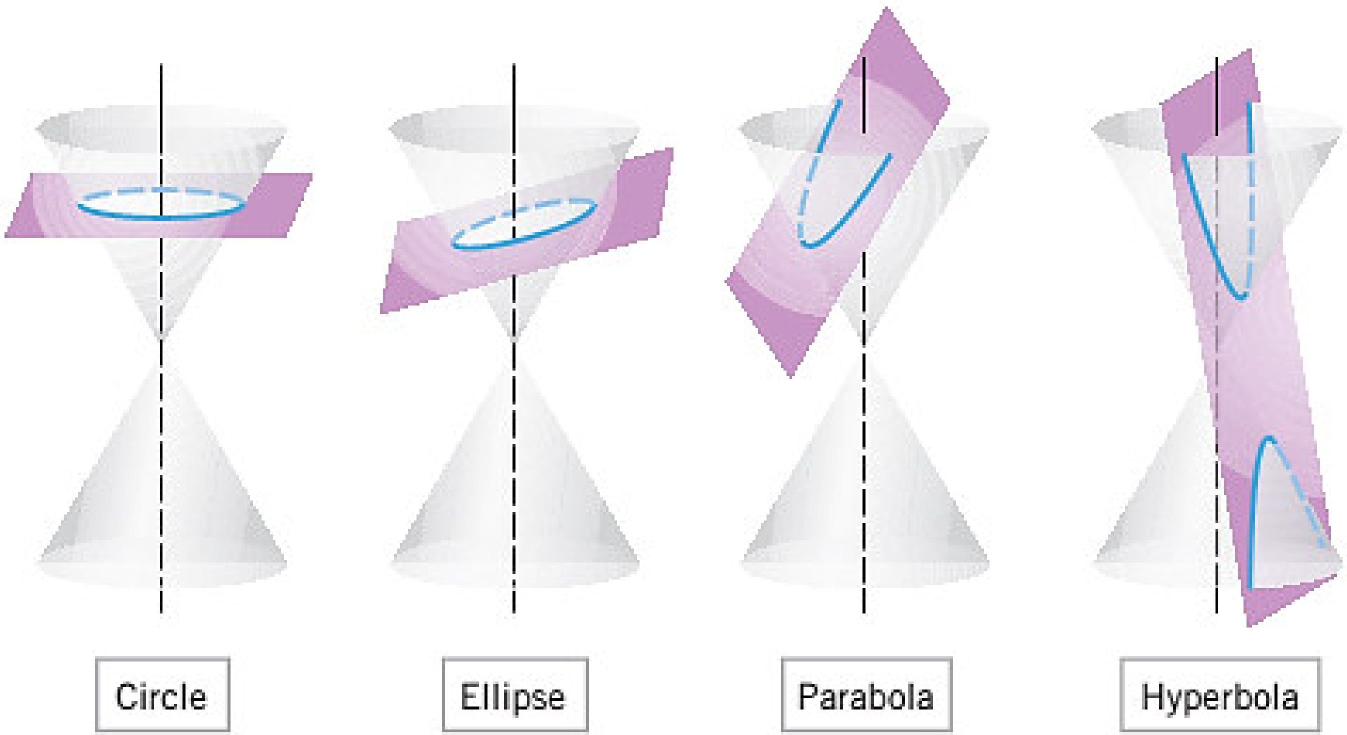



**Figure 8.4.1**

Quadratic forms on $R^2$ arise naturally in the study of conic sections. For example, it is shown in analytic geometry that an equation of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \tag{7}$$

in which $a$, $b$, and $c$ are not all zero, represents a conic section.* If $d = e = 0$ in (7), then there are no linear terms, and the equation becomes

$$ax^2 + 2bxy + cy^2 + f = 0 \tag{8}$$

and is said to represent a ***central conic***. These include circles, ellipses, and hyperbolas, but not parabolas. Furthermore, if $b = 0$ in (8), then there is no cross product term, and the equation

$$ax^2 + cy^2 + f = 0 \tag{9}$$

is said to represent a central conic in ***standard position***.

*We must also allow for the possibility that there are no real values of $x$ and $y$ that satisfy the equation, as with $x^2 + y^2 + 1 = 0$. In such cases we say that the equation has ***no graph*** or has ***an empty graph***.

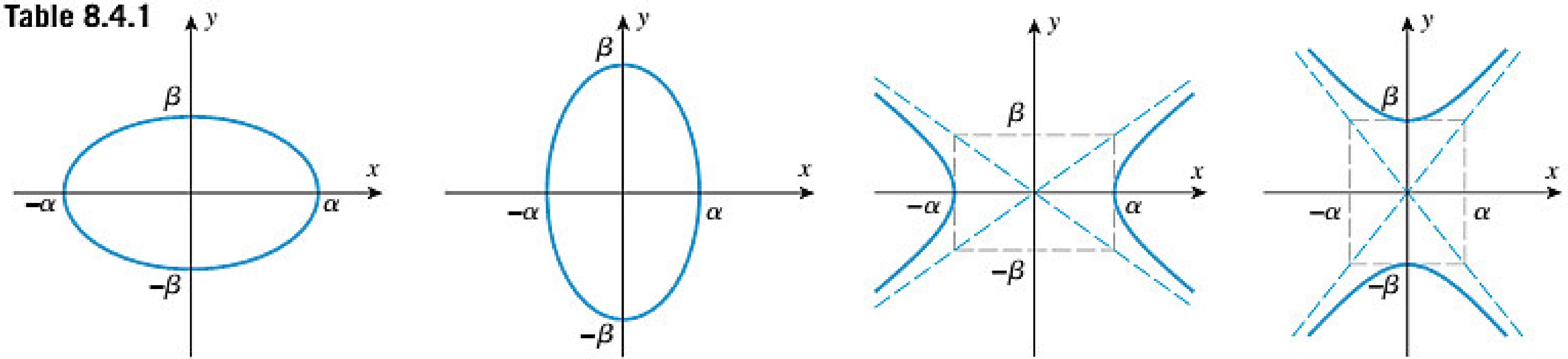
$$ax^2 + cy^2 + f = 0 \tag{9}$$

If $f \neq 0$ in (9), then we can divide through by $-f$ and rewrite this equation in the form

$$a'x^2 + b'y^2 = 1 \tag{10}$$

Furthermore, if the coefficients $a'$ and $b'$ are both positive or if one is positive and one is negative, then this equation represents a nondegenerate conic and can be rewritten in one of the four forms shown in Table 8.4.1 by putting the coefficients in the denominator. These are called the ***standard forms*** of the nondegenerate central conics. In the case where $\alpha = \beta$ the ellipses shown in the table are circles.

**Table 8.4.1**

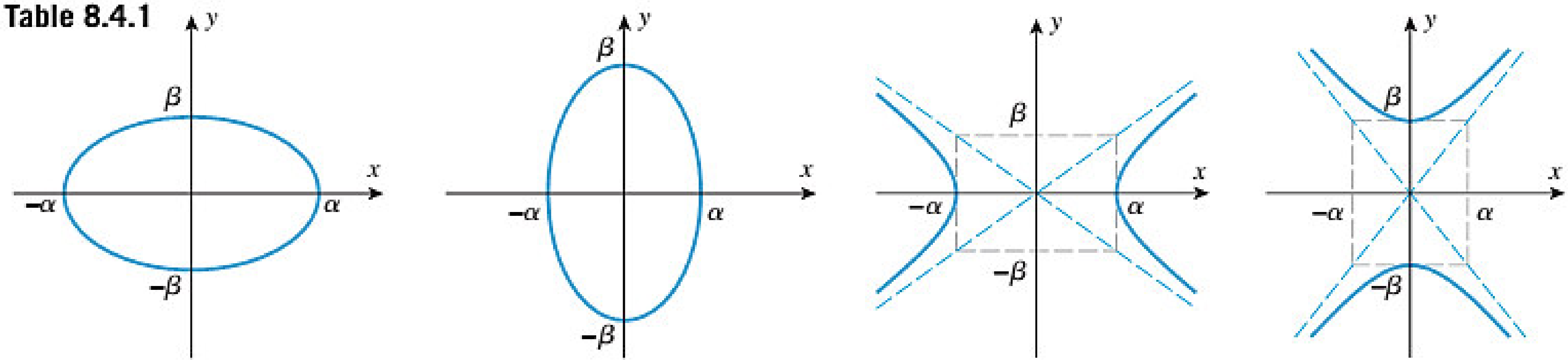



| | | | |
|---|---|---|---|
| $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ $(\alpha \geq \beta > 0)$ | $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$ $(\beta \geq \alpha > 0)$ | $\dfrac{x^2}{\alpha^2} - \dfrac{y^2}{\beta^2} = 1$ $(\alpha > 0, \beta > 0)$ | $\dfrac{y^2}{\beta^2} - \dfrac{x^2}{\alpha^2} = 1$ $(\alpha > 0, \beta > 0)$ |

We assume that you are familiar with the basic properties of conic sections, so we will not discuss such matters in this text. However, you will need to understand the geometric significance of the constants $\alpha$ and $\beta$ that appear in the standard forms of the central conics, so let us review their interpretations.

In the case of an ellipse, $2\alpha$ is its length in the $x$-direction and $2\beta$ its length in the $y$-direction (Table 8.4.1). For a noncircular ellipse, the larger of these numbers is the length of the ***major axis*** and the smaller the length of the ***minor axis***. In the case of a hyperbola, the numbers $2\alpha$ and $2\beta$ are the lengths of the sides of a box whose diagonals are along the asymptotes of the hyperbola (Table 8.4.1).

**Table 8.4.1**




| | | | |
|---|---|---|---|
| $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}=1$ <br> $(\alpha \geq \beta > 0)$ | $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}=1$ <br> $(\beta \geq \alpha > 0)$ | $\dfrac{x^2}{\alpha^2}-\dfrac{y^2}{\beta^2}=1$ <br> $(\alpha > 0, \beta > 0)$ | $\dfrac{y^2}{\beta^2}-\dfrac{x^2}{\alpha^2}=1$ <br> $(\alpha > 0, \beta > 0)$ |

Central conics in standard position are symmetric about both coordinate axes and have no cross product terms. A central conic whose equation has a cross product term results by rotating a conic in standard position about the origin and hence is said to be ***rotated out of standard position*** (Figure 8.4.2).

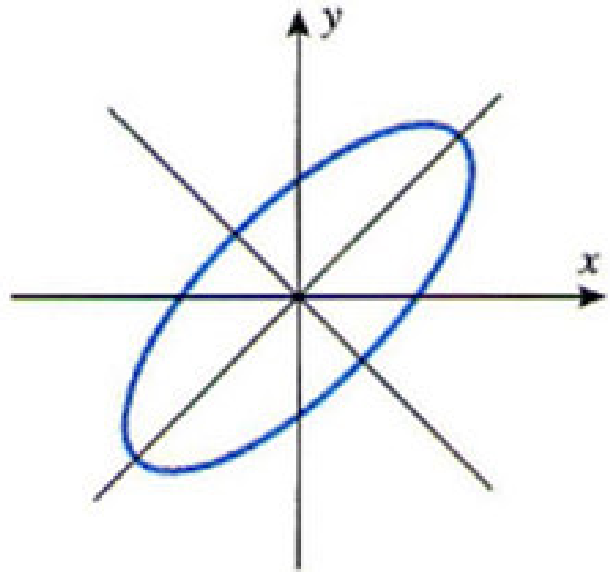



Figure 8.4.2

Quadratic forms on $R^3$ arise in the study of geometric objects called ***quadric surfaces*** (or ***quadrics***). The most important surfaces of this type have equations of the form

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + g = 0$$

in which $a$, $b$, and $c$ are not all zero. These are called ***central quadrics***. A problem involving quadric surfaces appears in the exercises.

## IDENTIFYING CONIC SECTIONS

We are now ready to consider the first of the three problems posed earlier, identifying the curve or surface represented by an equation $\mathbf{x}^T A\mathbf{x} = k$ in two or three variables. We will focus on the two-variable case. We noted above that an equation of the form

$$ax^2 + 2bxy + cy^2 + f = 0 \tag{11}$$

represents a central conic. If $b = 0$, then the conic is in standard position, and if $b \neq 0$, it is rotated. It is an easy matter to identify central conics in standard position by matching the equation with one of the standard forms. For example, the equation

$$9x^2 + 16y^2 - 144 = 0$$

can be rewritten as

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

which, by comparison with Table 8.4.1, is the ellipse shown in Figure 8.4.3.

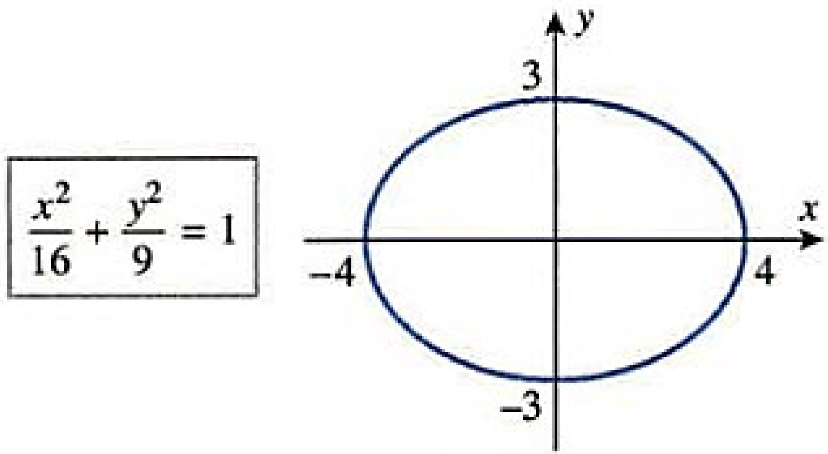



Figure 8.4.3

If a central conic is rotated out of standard position, then it can be identified by first rotating the coordinate axes to put it in standard position and then matching the resulting equation with one of the standard forms in Table 8.4.1.

To find a rotation that eliminates the cross product term in the equation $ax^2 + 2bxy + cy^2 + f = 0$, it will be convenient to take the constant term to the right side and express the equation in the form

$$ax^2 + 2bxy + cy^2 = k$$

or in matrix notation as

$$\mathbf{x}^T A\mathbf{x} = [x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = k \tag{12}$$

To rotate the coordinate axes, we need to make an orthogonal change of variable

$$\mathbf{x} = P\mathbf{x}'$$

See theorems 6.2.3 and 6.2.7.

in which $\det(P) = 1$, and if we want this rotation to eliminate the cross product term, we must choose $P$ to orthogonally diagonalize $A$. If we make a change of variable with these two properties, then in the rotated coordinate system Equation (12) will become

$$\mathbf{x}'^T D\mathbf{x}' = [x' \quad y'] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = k \tag{13}$$

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $A$. The conic can now be identified by writing (13) in the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 = k \tag{14}$$

and performing the necessary algebra to match it with one of the standard forms in Table 8.4.1.

$$\lambda_1 x'^2 + \lambda_2 y'^2 = k \tag{14}$$

For example, if $\lambda_1$, $\lambda_2$, and $k$ are positive, then (14) represents an ellipse with an axis of length $2\sqrt{k/\lambda_1}$ in the $x'$-direction and $2\sqrt{k/\lambda_2}$ in the $y'$-direction. The first column vector of $P$, which is a unit eigenvector corresponding to $\lambda_1$, is along the positive $x'$-axis; and the second column vector of $P$, which is a unit eigenvector corresponding to $\lambda_2$, is a unit vector along the $y'$-axis.

These are called the ***principal axes*** of the ellipse, which explains why Theorem 8.4.1 is called "the principal axes theorem." Also, since $P$ is the transition matrix from $x'y'$-coordinates to $xy$-coordinates, it follows from Formula (29) of Section 7.11 that the matrix $P$ can be expressed in terms of the rotation angle $\theta$ as

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{15}$$

(Figure 8.4.4).

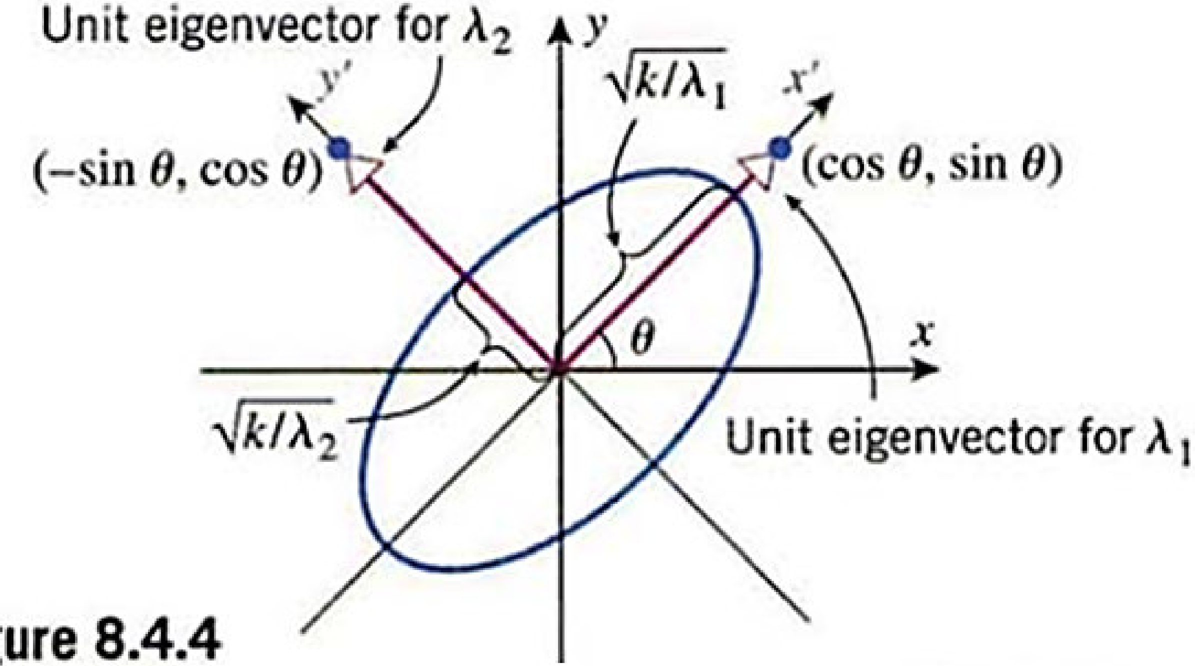



**Figure 8.4.4**

**EXAMPLE 3** Identifying a Conic by Eliminating the Cross Product Term

(a) Identify the conic whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ by rotating the $xy$-axes to put the conic in standard position.

(b) Find the angle $\theta$ through which you rotated the $xy$-axes in part (a).

***Solution (a)*** The given equation can be written in the matrix form

$$\mathbf{x}^T A\mathbf{x} = 36$$

where

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic polynomial of $A$ is

$$\begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{vmatrix} = (\lambda - 4)(\lambda - 9)$$

so the eigenvalues are $\lambda = 4$ and $\lambda = 9$.

We leave it for you to show that orthonormal bases for the eigenspaces are

$$\lambda = 4: \quad \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \quad \lambda = 9: \quad \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Thus, $A$ is orthogonally diagonalized by

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \tag{16}$$

Moreover, it happens by chance that $\det(P) = 1$, so we are assured that the substitution $\mathbf{x} = P\mathbf{x}'$ performs a rotation of axes. Had it been the case that $\det(P) = -1$, then we would have interchanged the columns to reverse the sign.

It follows from (13) that the equation of the conic in the $x'y'$-coordinate system is

$$[x' \quad y'] \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 36$$

which we can write as

$$4x'^2 + 9y'^2 = 36 \quad \text{or} \quad \frac{x'^2}{9} + \frac{y'^2}{4} = 1$$

We can now see from Table 8.4.1 that the conic is an ellipse whose axis has length $2\alpha = 6$ in the $x'$-direction and length $2\beta = 4$ in the $y'$-direction.

***Solution (b)*** It follows from (15) that

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which implies that

$$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{2}$$

Thus, $\theta = \tan^{-1}\frac{1}{2} \approx 26.6°$ (Figure 8.4.5).

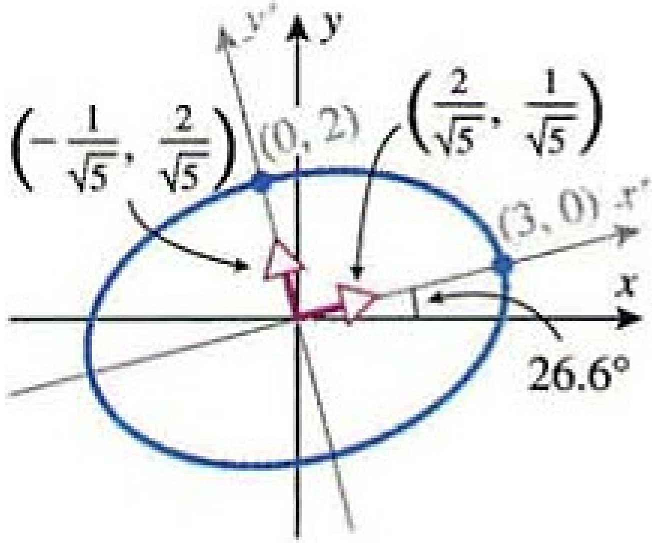



**Figure 8.4.5**

**REMARK** In the exercises we will ask you to show that if $b \neq 0$, then the cross product term in the equation

$$ax^2 + 2bxy + cy^2 = k$$

can be eliminated by a rotation through an angle $\theta$ that satisfies

$$\cot 2\theta = \frac{a-c}{2b} \tag{17}$$

We leave it for you to confirm that this is consistent with part (b) of the last example.

## POSITIVE DEFINITE QUADRATIC FORMS

We will now consider the second of the two problems posed earlier, determining conditions under which $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero values of $\mathbf{x}$. We will explain why this is important shortly, but first we introduce some terminology.

**Definition 8.4.2** A quadratic form $\mathbf{x}^T A\mathbf{x}$ is said to be

***positive definite*** if $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$
***negative definite*** if $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$
***indefinite*** if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values

The terminology in this definition is also applied to the matrix $A$; that is, we say that *a symmetric matrix $A$ is positive definite, negative definite, or indefinite in accordance with whether the associated quadratic form $\mathbf{x}^T A\mathbf{x}$ has that property.*

The following theorem provides a way of using eigenvalues to determine whether a matrix $A$ and its associated quadratic form $\mathbf{x}^T A\mathbf{x}$ are positive definite, negative definite, or indefinite.

**Theorem 8.4.3** *If $A$ is a symmetric matrix, then:*

(*a*) $\mathbf{x}^T A\mathbf{x}$ *is positive definite if and only if all eigenvalues of $A$ are positive.*

(*b*) $\mathbf{x}^T A\mathbf{x}$ *is negative definite if and only if all eigenvalues of $A$ are negative.*

(*c*) $\mathbf{x}^T A\mathbf{x}$ *is indefinite if and only if $A$ has at least one positive eigenvalue and at least one negative eigenvalue.*

***Proofs (a) and (b)*** It follows from the principal axes theorem (Theorem 8.4.1) that there is an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ for which

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{18}$$

Moreover, it follows from the invertibility of $P$ that $\mathbf{y} \neq \mathbf{0}$ if and only if $\mathbf{x} \neq \mathbf{0}$, so the values of $\mathbf{x}^T A\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$ are the same as the values of $\mathbf{y}^T D\mathbf{y}$ for $\mathbf{y} \neq \mathbf{0}$. Thus, it follows from (18) that $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$ if and only if all of the $\lambda$'s in that equation (which are the eigenvalues of $A$) are positive, and that $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$ if and only if all of the eigenvalues are negative. This proves parts (*a*) and (*b*).

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{18}$$

***Proof* (c)** Assume that $A$ has at least one positive eigenvalue and at least one negative eigenvalue, and to be specific, suppose that $\lambda_1 > 0$ and $\lambda_2 < 0$ in (18). Then

$$\mathbf{x}^T A\mathbf{x} > 0 \quad \text{if} \quad y_1 = 1 \text{ and all other } y\text{'s are } 0$$

and

$$\mathbf{x}^T A\mathbf{x} < 0 \quad \text{if} \quad y_2 = 1 \text{ and all other } y\text{'s are } 0$$

which proves that $\mathbf{x}^T A\mathbf{x}$ is indefinite.

Conversely, if $\mathbf{x}^T A\mathbf{x} > 0$ for some $\mathbf{x}$, then $\mathbf{y}^T D\mathbf{y} > 0$ for some $\mathbf{y}$, so at least one of the $\lambda$'s in (18) must be positive. Similarly, if $\mathbf{x}^T A\mathbf{x} < 0$ for some $\mathbf{x}$, then $\mathbf{y}^T D\mathbf{y} < 0$ for some $\mathbf{y}$, so at least one of the $\lambda$'s in (18) must be negative, which completes the proof. ■

Also see parts (a) and (b).

**Definition 8.4.2** A quadratic form $\mathbf{x}^T A\mathbf{x}$ is said to be

***positive definite*** if $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$

***negative definite*** if $\mathbf{x}^T A\mathbf{x} < 0$ for $\mathbf{x} \neq \mathbf{0}$

***indefinite*** if $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values

**REMARK** The three classifications in Definition 8.4.2 do not exhaust all of the possibilities. For example, a quadratic form for which $\mathbf{x}^T A\mathbf{x} \geq 0$ if $\mathbf{x} \neq \mathbf{0}$ is called ***positive semidefinite***, and one for which $\mathbf{x}^T A\mathbf{x} \leq 0$ if $\mathbf{x} \neq \mathbf{0}$ is called ***negative semidefinite***.

Every positive definite form is positive semidefinite, but not conversely, and every negative definite form is negative semidefinite, but not conversely (why?).

By adjusting the proof of Theorem 8.4.3 appropriately, one can prove that $\mathbf{x}^T A\mathbf{x}$ is positive semidefinite if and only if all eigenvalues of $A$ are nonnegative and is negative semidefinite if and only if all eigenvalues of $A$ are nonpositive.

## EXAMPLE 4 Positive Definite Quadratic Forms

One cannot usually tell from the signs of the entries in a symmetric matrix $A$ whether that matrix is positive definite, negative definite, or indefinite. For example, the entries of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

are nonnegative, but the matrix is indefinite, since its eigenvalues are $\lambda = 1, 4, -2$ (verify). To see this another way, let us write out the quadratic form as

$$Q_A(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$$

We can now see, for example, that $Q_A = 4$ for $x_1 = 0$, $x_2 = 1$, $x_3 = 1$ and $Q_A = -4$ for $x_1 = 0$, $x_2 = 1$, $x_3 = -1$. ■

## IDENTIFYING POSITIVE DEFINITE MATRICES

Positive definite matrices are the most important symmetric matrices in applications, so it will be useful to learn a little more about them.

We already know that a symmetric matrix is positive definite if and only if its eigenvalues are all positive; now we will give a criterion that can be used to determine whether a symmetric matrix is positive definite without finding the eigenvalues.

For this purpose we define the ***k*th principal submatrix** of an $n \times n$ matrix $A$ to be the $k \times k$ submatrix consisting of the first $k$ rows and columns of $A$. For example, here are the principal submatrices of a general $4 \times 4$ matrix:

$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \quad \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right]$$

First principal submatrix | Second principal submatrix | Third principal submatrix | Fourth principal submatrix $= A$

**Theorem 8.4.5** *A symmetric matrix $A$ is positive definite if and only if the determinant of every principal submatrix is positive.*

No proof given.

## EXAMPLE 5 Working with Principal Submatrices

The matrix

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$$

is positive definite since the determinants

$$|2| = 2, \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, \quad \begin{vmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{vmatrix} = 1$$

are all positive. Thus, we are guaranteed that all eigenvalues of $A$ are positive and $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$. ■

**Theorem 8.4.6** *If $A$ is a symmetric matrix, then the following statements are equivalent.*

(a) *$A$ is positive definite.*

(b) *There is a symmetric positive definite matrix $B$ such that $A = B^2$.*

(c) *There is an invertible matrix $C$ such that $A = C^TC$.*

***Proof* (a) ⇒ (b)** Since $A$ is symmetric, it is orthogonally diagonalizable. This means that there is an orthogonal matrix $P$ such that $P^TAP = D$, where $D$ is a diagonal matrix whose entries are the eigenvalues of $A$. Moreover, since $A$ is positive definite, its eigenvalues are positive, so we can write $D$ as $D = D_1^2$, where $D_1$ is the diagonal matrix whose entries are the square roots of the eigenvalues of $A$. Thus, we have $P^TAP = D_1^2$, which we can rewrite as

$$A = PD_1^2P^T = PD_1D_1P^T = PD_1P^TPD_1P^T = (PD_1P^T)(PD_1P^T) = B^2 \tag{22}$$

where $B = PD_1P^T$. We leave it for you to confirm that $B$ is symmetric.

We will show that $B$ is positive definite by proving that it has positive eigenvalues. The eigenvalues of $B$ are the same as the eigenvalues of $D_1$, since eigenvalues are a similarity invariant and $B$ is similar to $D_1$. Thus, the eigenvalues of $B$ are positive, since they are the square roots of the eigenvalues of $A$.

***Proof*** $(b) \Rightarrow (c)$ Assume that $A = B^2$, where $B$ is symmetric and positive definite. Then $A = B^2 = BB = B^T B$, so take $C = B$.

For the invertibility of B, recall that B = $PD_1P^T$.

***Proof*** $(c) \Rightarrow (a)$ Assume that $A = C^T C$, where $C$ is invertible. We will show that $A$ is positive definite by showing that $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$. To do this we use Formula (26) of Section 3.1 and part $(e)$ of Theorem 3.2.10 to write

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T C^T C\mathbf{x} = (C\mathbf{x})^T (C\mathbf{x}) = C\mathbf{x} \cdot C\mathbf{x} = \|C\mathbf{x}\|^2 \geq 0$$

But the invertibility of $C$ implies that $C\mathbf{x} \neq \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$, so $\mathbf{x}^T A\mathbf{x} > 0$ for $\mathbf{x} \neq \mathbf{0}$. ■

$$\boxed{\mathbf{u}^T\mathbf{v} = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T\mathbf{u} \qquad (26)}$$

**EXAMPLE 6** The Factorization $A = B^2$

In Example 1 of Section 8.3 we showed that the matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

has eigenvalues $\lambda = 2$ and $\lambda = 8$ and that

$$P^TAP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = D$$

Since the eigenvalues are positive, Theorem 8.4.6 implies that $A$ can be factored as $A = B^2$ for some symmetric positive definite matrix $B$. One way to obtain such a $B$ is to use Formula (22) and take $B = PD_1P^T$, where $D_1$ is the diagonal matrix whose diagonal entries are the square roots of the diagonal entries of $D$.

This yields

$$B = PD_1P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} \end{bmatrix}$$

We leave it for you to confirm that $A = B^2$. ■

## CHOLESKY FACTORIZATION

We know from Theorem 8.4.6 that if $A$ is a symmetric positive definite matrix, then it can be factored as $A = C^T C$, where $C$ is invertible.

More specifically, one can prove that if $A$ is symmetric and positive definite, then it can be factored as $A = R^T R$, where $R$ is upper triangular and has positive entries on the main diagonal. This is called a ***Cholesky factorization*** of $A$.

Cholesky factorizations are important in many kinds of numerical algorithms, and programs such as MATLAB, *Maple*, and *Mathematica* have built-in commands for computing them.