Matrices, Vectors

A **matrix** is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the **entries** (or sometimes the *elements*) of the matrix. For example,

$$
(1)\qquad
\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \qquad
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},
$$

$$
\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \qquad
\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \qquad
\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}
$$

are matrices.

source: Advanced Engineering Mathematics, Erwin Kreyszig

We shall denote matrices by capital boldface letters **A, B, C**, $\cdots$, or by writing the general entry in brackets; thus $\mathbf{A} = [a_{jk}]$, and so on. By an $m \times n$ **matrix** (read *m by n matrix*) we mean a matrix with m rows and n columns—rows come always first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\text{(2)} \qquad \mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Each entry in (2) has two subscripts. The first is the *row number* and the second is the *column number*. Thus a_{21} is the entry in Row 2 and Column 1.

If $m = n$, we call **A** an $n \times n$ **square matrix.** Then its diagonal containing the entries $a_{11}, a_{22}, \cdots, a_{nn}$ is called the **main diagonal** of **A.**

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a, b,** $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots, \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal,** written $\mathbf{A} = \mathbf{B}$**,** if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different.** Thus, matrices of different sizes are always different.

Equality of Matrices

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

The following matrices are all different. Explain!

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix} \blacksquare$$

Addition

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of **A** and **B.** Matrices of different sizes cannot be added.

As a special case, the **sum** $\mathbf{a} + \mathbf{b}$ of two row vectors or two column vectors, which must have the same number of components, is obtained by adding the corresponding components.

Addition of Matrices and Vectors

If $\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$. then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$.

Scalar Multiplication

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Here $(-1)\mathbf{A}$ is simply written $-\mathbf{A}$ and is called the **negative** of **A.** Similarly, $(-k)\mathbf{A}$ is written $-k\mathbf{A}$. Also, $\mathbf{A} + (-\mathbf{B})$ is written $\mathbf{A} - \mathbf{B}$ and is called the **difference** of **A** and **B** (which must have the same size!).

Scalar Multiplication

If $\mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}$, then $-\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}$, $\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$, $0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Rules for Matrix Addition and Scalar Multiplication. From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

$$
\text{(3)} \quad
\begin{aligned}
&\text{(a)} && \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\
&\text{(b)} && (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) && \text{(written } \mathbf{A} + \mathbf{B} + \mathbf{C}) \\
&\text{(c)} && \mathbf{A} + \mathbf{0} = \mathbf{A} \\
&\text{(d)} && \mathbf{A} + (-\mathbf{A}) = \mathbf{0}.
\end{aligned}
$$

Here **0** denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero. → see previous slide.

Hence matrix addition is *commutative* and *associative* [by (3a) and (3b)].

Similarly, for scalar multiplication we obtain the rules

$$
\text{(4)} \quad
\begin{aligned}
&\text{(a)} && c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \\
&\text{(b)} && (c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A} \\
&\text{(c)} && c(k\mathbf{A}) = (ck)\mathbf{A} && \text{(written } ck\mathbf{A}) \\
&\text{(d)} && 1\mathbf{A} = \mathbf{A}.
\end{aligned}
$$

Matrix Multiplication

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$\text{(1)} \qquad c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{matrix} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{matrix}$$

The condition $r = n$ means that the second factor, **B,** must have as many rows as the first factor has columns, namely n.

c_{jk} in (1) is obtained by multiplying each entry in the jth row of **A** by the corresponding entry in the kth column of **B** and then adding these n products. For instance, $c_{21} = a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2n}b_{n1}$, and so on. One calls this briefly a ***"multiplication of rows into columns."*** See the illustration in Fig. 155, where $n = 3$.

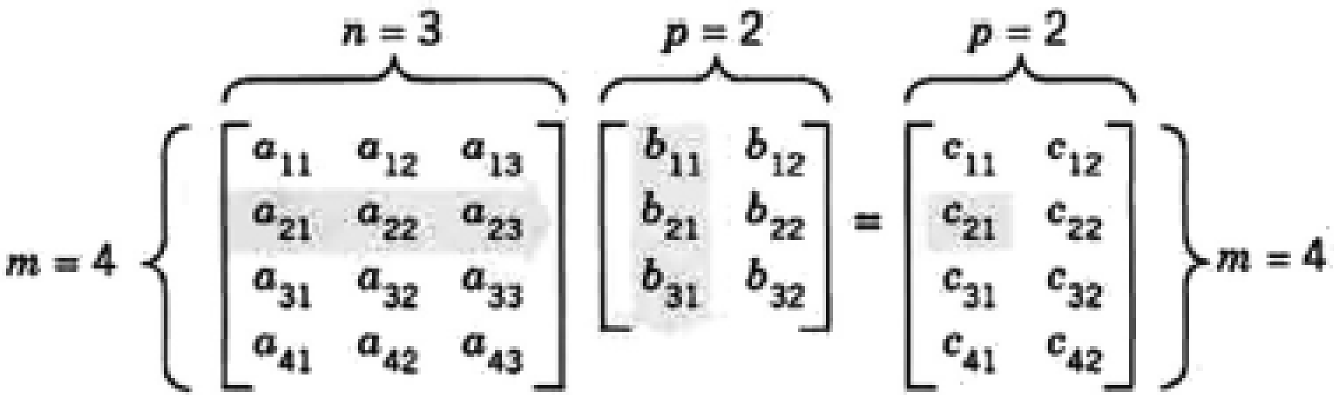


Fig. 155. Notations in a product **AB** = **C**

→ In matrix multiplication, are then the corresponding entries directly multiplied?
No! They are not and that has to do with the use of matrices (eg, linear systems).

Matrix Multiplication

$$AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Multiplication of a Matrix and a Vector

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix} \quad \text{whereas} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \quad \text{is undefined.}$$

Products of Row and Column Vectors

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

CAUTION! Matrix Multiplication Is Not Commutative, AB $\neq$ BA in General

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

AB = 0 but A and B ≠ 0!

Our examples show that the ***order of factors*** in matrix products ***must always be observed very carefully.*** Otherwise matrix multiplication satisfies rules similar to those for numbers, namely.

(2)

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$ *written* $k\mathbf{AB}$ *or* $\mathbf{A}k\mathbf{B}$

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ *written* $\mathbf{ABC}$

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

provided **A, B,** and **C** are such that the expressions on the left are defined; here, k is any scalar. (2b) is called the **associative law.** (2c) and (2d) are called the **distributive laws.**

Linear Systems of Equations

A **linear system of m equations in n unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line. $a_{11}, \cdots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \cdots, b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a **homogeneous system.** If at least one b_j is not zero, then (1) is called a **nonhomogeneous system.**

A **solution** of (1) is a set of numbers $x_1, \cdots, x_n$ that satisfies all the m equations. A **solution vector** of (1) is a vector $\mathbf{x}$ whose components form a solution of (1). If the system (1) is homogeneous, it has at least the **trivial solution** $x_1 = 0, \cdots, x_n = 0$.

Coefficient Matrix

Matrix Form of the Linear System (1). From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation

(2) $$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Definition of matrix multiplication allows the description of a linear system by (2).

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

are column vectors. We assume that the coefficients a_{jk} are not all zero, so that $\mathbf{A}$ is not a zero matrix. Note that $\mathbf{x}$ has n components, whereas $\mathbf{b}$ has m components.

Augmented Matrix

The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1). The dashed vertical line could be omitted (as we shall do later); it is merely a reminder that the last column of $\tilde{\mathbf{A}}$ does not belong to **A**.

The augmented matrix $\tilde{\mathbf{A}}$ ***determines the system* (1) *completely*** because it contains all the given numbers appearing in (1).

Geometric Interpretation. Existence and Uniqueness of Solutions

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2.$$

If we interpret x_1, x_2 as coordinates in the x_1x_2-plane,
then each of the two equations represents a straight line,
and (x_1, x_2) is a solution if and only if the point P with coordinates x_1, x_2 lies on both lines.

Hence there are three possible cases:

(a) Precisely one solution if the lines intersect.

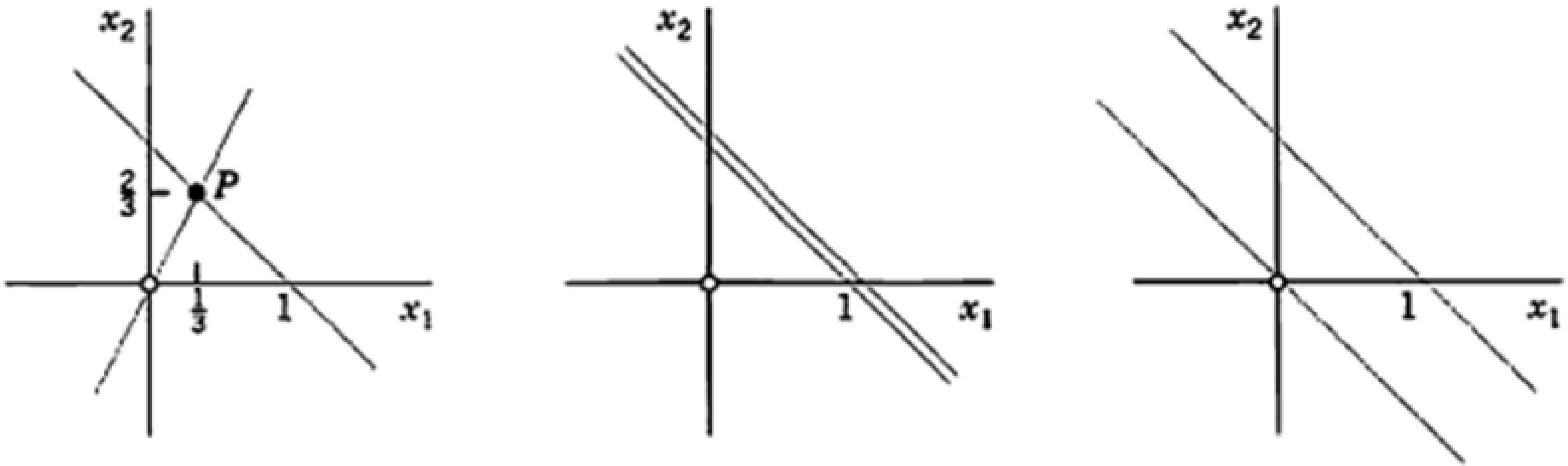
(b) Infinitely many solutions if the lines coincide.

(c) No solution if the lines are parallel

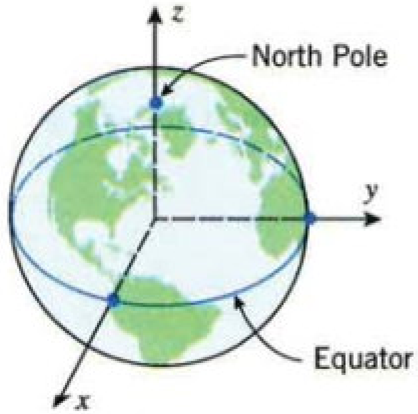
existence: (a) & (b);
uniqueness: (a).

For instance.

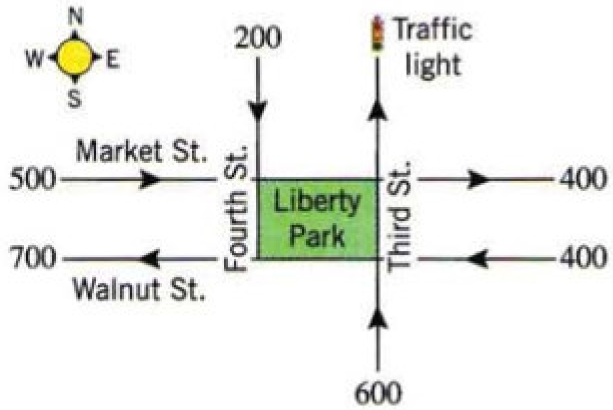
Case (a)	Case (b)	Case (c)
$x_1 + x_2 = 1$	$x_1 + x_2 = 1$	$x_1 + x_2 = 1$
$2x_1 - x_2 = 0$	$2x_1 + 2x_2 = 2$	$x_1 + x_2 = 0$



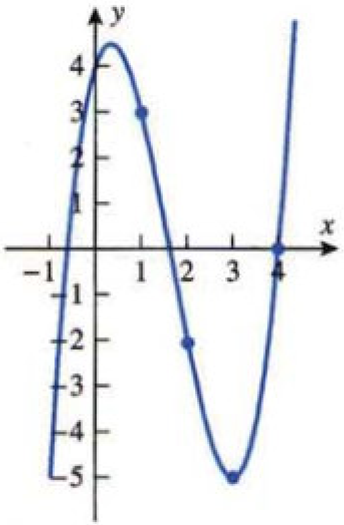
Applications of Linear Systems



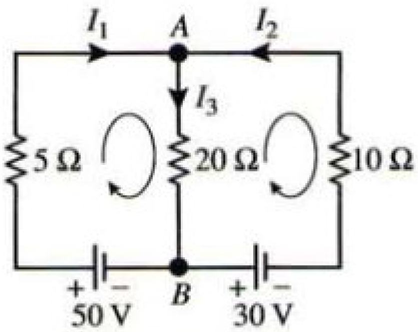
GLOBAL POSITIONING



NETWORK ANALYSIS



POLYNOMIAL INTERPOLATION



ELECTRICAL CIRCUITS

$$x_1\,(CH_4) + x_2\,(O_2) \longrightarrow$$

$$x_3\,(CO_2) + x_4\,(H_2O)$$

BALANCING CHEMICAL EQUATIONS

source: Contemporary Linear Algebra, H. Anton / R.C. Busby

Solving Linear Systems by Row Reduction

If by a sequence of elementary row operations,
the augmented matrix for a system of linear equations
is put in *reduced* row echelon form,
then the solution set can be obtained either by inspection,
or by converting certain linear equations to parametric form.

Elementary Row Operations. Row-Equivalent Systems

Elementary Row Operations for Matrices:

Interchange of two rows

Addition of a constant multiple of one row to another row

*Multiplication of a row by a **nonzero** constant c.*

CAUTION! These operations are for rows, ***not for columns!*** They correspond to

Elementary Operations for Equations:

Interchange of two equations

Addition of a constant multiple of one equation to another equation

*Multiplication of an equation by a **nonzero** constant c.*

Elementary Operations for Equations:

Interchange of two equations

Addition of a constant multiple of one equation to another equation

Multiplication of an equation by a ***nonzero*** *constant c.*

Clearly, the interchange of two equations does not alter the solution set.

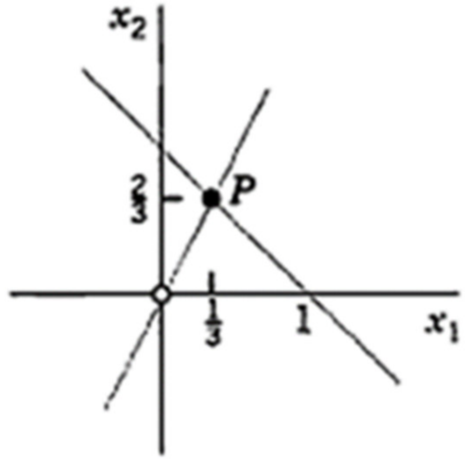
Neither does that addition because we can undo it by a corresponding subtraction.

Similarly for that multiplication, which we can undo by multiplying the new equation by $1/c$ (since $c \neq 0$), producing the original equation.

Case (α)

$x_1 + x_2 = 1$

$2x_1 - x_2 = 0$



To be in ***reduced row echelon form***, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading* 1**.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A matrix that has the first three properties is said to be in ***row echelon form***.

source: Contemporary Linear Algebra, H. Anton / R.C. Busby

EXAMPLE 1

Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We leave it for you to confirm that each of the matrices in this example satisfies all of the requirements for its stated form. ■

EXAMPLE 2

More on Row Echelon and Reduced Row Echelon Form

all matrices of the following types are in row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Moreover, all matrices of the following types are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

GAUSS–JORDAN AND GAUSSIAN ELIMINATION

Now we will give a step-by-step procedure that can be used to reduce any matrix to reduced row echelon form by elementary row operations. As we state each step, we will illustrate the idea by reducing the following matrix to reduced row echelon form:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \tag{6}$$

It can be proved that elementary row operations, when applied to the augmented matrix of a linear system, do not change the solution set of the system. Thus, we are assured that the linear system corresponding to the reduced row echelon form of (6) will have the same solutions as the original system. Here are the steps for reducing (6) to reduced row echelon form:

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

↑ Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first row of the preceding matrix was multiplied by $\frac{1}{2}$.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the first row of the preceding matrix was added to the third row.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

–5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again to Step 1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

The *entire* matrix is now in row echelon form. To find the reduced row echelon form we need the following additional step.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\frac{7}{2}$ times the third row of the preceding matrix was added to the second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

The last matrix is in reduced row echelon form.

The procedure (or algorithm) we have just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***.

This algorithm consists of two parts,

a ***forward phase*** in which zeros are introduced below the leading 1's

and then a ***backward phase*** in which zeros are introduced above the leading 1's.

If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***. ⇨ end of Step 5

SOME FACTS ABOUT ECHELON FORMS

1. Every matrix has a unique reduced row echelon form;

2. Row echelon forms are not unique;

EXAMPLE 5

Solving a Linear System by Gauss–Jordan Elimination

Solve the following linear system by Gauss–Jordan elimination:

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

Solution The augmented matrix for the system is

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{bmatrix}
$$

Adding -2 times the first row to the second and fourth rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Multiplying the second row by -1 and then adding -5 times the new second row to the third row and -4 times the new second row to the fourth row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ row echelon form;
Gaussian elimination.

Adding −3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ reduced row echelon form;
Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Solving for the leading variables we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

Alternatively, we can express the solution set as a linear combination of column vectors by writing

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

■

GAUSSIAN ELIMINATION AND BACK SUBSTITUTION

In the examples given thus far we solved various linear systems by first transforming the augmented matrix to reduced row echelon form (Gauss–Jordan elimination) and then solving the corresponding linear system.

However, it is possible to use only the forward phase of the reduction algorithm (Gaussian elimination) and solve the system that corresponds to the resulting row echelon form.

With this approach the backward phase of the Gauss–Jordan algorithm is replaced by an algebraic procedure, called ***back substitution***, in which each equation corresponding to the row echelon form is systematically substituted into the equations above, starting at the bottom and working up.

EXAMPLE 6

Gaussian Elimination and Back Substitution

We will solve the linear system in Example 5 using the row echelon form of the augmented matrix produced by Gaussian elimination. In the forward phase of the computations in Example 5, we obtained the following row echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ see slide 33.

To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0 \\ x_3 + 2x_4 \quad\quad + 3x_6 &= 1 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

we proceed as follows:

Step 1.

Solve the equations for the leading variables.

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = 1 - 2x_4 - 3x_6$$
$$x_6 = \frac{1}{3}$$

Step 2.

Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

Step 3.

Assign arbitrary values to the free variables, if any.

If we now assign x_2, x_4, and x_5 the arbitrary values r, s, and t, respectively, we obtain

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = \frac{1}{3}$$

which agrees with the solution obtained in Example 5 by Gauss–Jordan elimination. ■

We now call a linear system S_1 **row-equivalent** to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations.

Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

→ They are also called, by simplicity, equivalent systems.

A linear system is called **overdetermined** if it has more equations than unknowns, **determined** if $m = n$, and **underdetermined** if it has fewer equations than unknowns.

Furthermore, a system is called **consistent** if it has at least one solution (thus, one solution or infinitely many solutions), but **inconsistent** if it has no solutions at all.

Additional aplications of linear systems

- Economic models (Leontief models);
- State changes in systems (Markov chains);
- Distribution of equilibrium temperature;
- Computerized tomography.

source: Elementary Linear Algebra, H. Anton / C. Rorres