Section 6.1 Matrices as Transformations

Definition 6.1.1 Given a set D of allowable inputs, a ***function*** f is a rule that associates a unique output with each input from D; the set D is called the ***domain*** of f. If the input is denoted by x, then the corresponding output is denoted by $f(x)$ (read, "f of x"). The output is also called the ***value*** of f at x or the ***image*** of x under f, and we say that f ***maps*** x into $f(x)$. It is common to denote the output by the single letter y and write $y = f(x)$. The set of all outputs y that results as x varies over the domain is called the ***range*** of f.

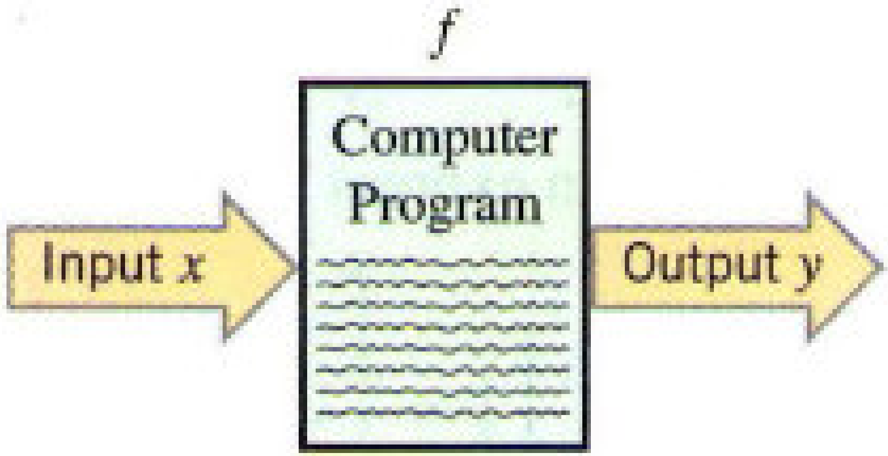


Figure 6.1.1

think of a function as a computer program that takes an ***input*** x, operates on it in some way, and produces exactly one ***output*** y.

source: *Contemporary Linear Algebra, H. Anton / R.C. Busby*

A function whose inputs and outputs are vectors is called a ***transformation***, and it is standard to denote transformations by capital letters such as F, T, or L. If T is a transformation that maps the vector $\mathbf{x}$ into the vector $\mathbf{w}$, then the relationship $\mathbf{w} = T(\mathbf{x})$ is sometimes written as

$$\mathbf{x} \xrightarrow{T} \mathbf{w}$$

which is read, "T maps $\mathbf{x}$ into $\mathbf{w}$."

EXAMPLE 1

A Scaling Transformation

Let T be the transformation that maps a vector $\mathbf{x} = (x_1, x_2)$ in R^2 into the vector $2\mathbf{x} = (2x_1, 2x_2)$ in R^2. This relationship can be expressed in various ways:

$$T(\mathbf{x}) = 2\mathbf{x}, \qquad T(x_1, x_2) = (2x_1, 2x_2)$$

$$\mathbf{x} \xrightarrow{T} 2\mathbf{x}, \qquad (x_1, x_2) \xrightarrow{T} (2x_1, 2x_2)$$

In particular, if $\mathbf{x} = (-1, 3)$, then $T(\mathbf{x}) = 2\mathbf{x} = (-2, 6)$, which we can express as

$$T(-1, 3) = (-2, 6) \quad \text{or equivalently,} \quad (-1, 3) \xrightarrow{T} (-2, 6)$$

■

EXAMPLE 2

A Component-Squaring Transformation

Let T be the transformation that maps a vector $\mathbf{x}$ in R^3 into the vector in R^3 whose components are the squares of the components of $\mathbf{x}$. Thus, if $\mathbf{x} = (x_1, x_2, x_3)$, then

$$T(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2) \quad \text{or equivalently,} \quad (x_1, x_2, x_3) \xrightarrow{T} (x_1^2, x_2^2, x_3^2) \qquad \blacksquare$$

EXAMPLE 3

A Matrix Multiplication Transformation

Consider the 3×2 matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} \qquad (1)$$

→ A transformation expressed by a matrix.

and let T_A be the transformation that maps a 2×1 column vector $\mathbf{x}$ in R^2 into the 3×1 column vector $A\mathbf{x}$ in R^3. This relationship can be expressed as

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{or as} \quad \mathbf{x} \xrightarrow{T_A} A\mathbf{x}$$

If we write $A\mathbf{x}$ in component form as

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

then we can express the transformation T_A in component form as

$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 4x_2 \end{bmatrix} \tag{2}$$

This formula can also be expressed more compactly in comma-delimited form as

$$T_A(x_1, x_2) = (x_1 - x_2, 2x_1 + 5x_2, 3x_1 + 4x_2) \tag{3}$$

which emphasizes T_A as a mapping from points into points. For example, Formulas (2) and (3) yield

$$T_A\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 13 \\ 9 \end{bmatrix} \quad \text{or} \quad T_A(-1, 3) = (-4, 13, 9) \qquad \blacksquare$$

If T is a transformation whose domain *is* R^n and whose range is *in* R^m, then we will write

$$T : R^n \rightarrow R^m \tag{4}$$

(read, "T maps R^n into R^m") when we want to emphasize the spaces involved. Depending on your geometric point of view, you can think of a transformation $T : R^n \rightarrow R^m$ as mapping points into points or vectors into vectors (Figure 6.1.2). For example, the scaling transformation in Example 1 maps the vectors (or points) in R^2 into vectors (or points) in R^2, so

$$T : R^2 \rightarrow R^2 \tag{5}$$

and the transformation T_A in Example 3 maps the vectors (or points) in R^2 into vectors (or points) in R^3, so

$$T_A : R^2 \rightarrow R^3 \tag{6}$$

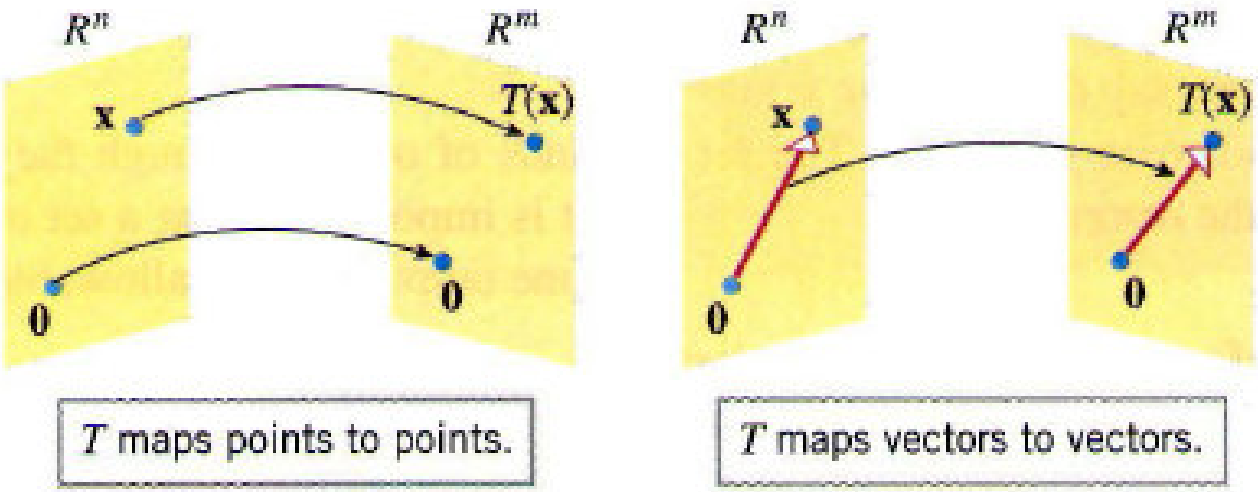


Figure 6.1.2

$$T : R^n \rightarrow R^m \tag{4}$$

Keep in mind that the set R^n in (4) is the domain of T but that R^m may not be the range of T. The set R^m, which is called the ***codomain*** of T, is intended only to describe the space in which the image vectors lie and may actually be larger than the range of T (Figure 6.1.3).

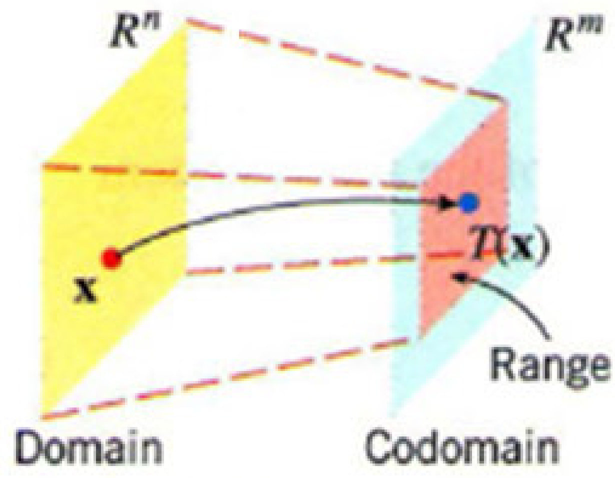


Figure 6.1.3

$$T : R^2 \rightarrow R^2 \tag{5}$$

$$T_A : R^2 \rightarrow R^3 \tag{6}$$

Note that the transformation T in (5) maps the space R^2 back into itself, whereas the transformation T_A in (6) maps R^2 into a different space. In general, if $T : R^n \rightarrow R^n$, then we will refer to the transformation T as an ***operator*** on R^n to emphasize that it maps R^n back into R^n.

MATRIX TRANSFORMATIONS

Example 3 is a special case of a general class of transformations, called ***matrix transformations***. Specifically, if A is an $m \times n$ matrix, and if $\mathbf{x}$ is a column vector in R^n, then the product $A\mathbf{x}$ is a vector in R^m, so multiplying $\mathbf{x}$ by A creates a transformation that maps vectors in R^n into vectors in R^m. We call this transformation ***multiplication by A*** or the ***transformation A*** and denote it by T_A to emphasize the matrix A. Thus,

$$T_A : R^n \rightarrow R^m$$

and

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{or equivalently,} \quad \mathbf{x} \xrightarrow{T_A} A\mathbf{x}$$

In the special case where A is square, say $n \times n$, we have $T_A : R^n \rightarrow R^n$, and we call T_A a ***matrix operator*** on R^n.

EXAMPLE 3

A Matrix Multiplication Transformation

Consider the 3×2 matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} \tag{1}$$

and let T_A be the transformation that maps a 2×1 column vector $\mathbf{x}$ in R^2 into the 3×1 column vector $A\mathbf{x}$ in R^3. This relationship can be expressed as

$$T_A(\mathbf{x}) = A\mathbf{x} \quad \text{or as} \quad \mathbf{x} \xrightarrow{T_A} A\mathbf{x}$$

EXAMPLE 4

Zero Transformations

If 0 is the $m \times n$ zero matrix, then

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

so multiplication by 0 maps every vector in R^n into the zero vector in R^m. Accordingly, we call T_0 the ***zero transformation*** from R^n to R^m. ■

EXAMPLE 5

Identity Operators

If I is the $n \times n$ identity matrix, then for every vector $\mathbf{x}$ in R^n we have

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in R^n back into itself. Accordingly, we call T_I the ***identity operator*** on R^n. ■

Thus far, much of our work with matrices has been in the context of solving a linear system

$$A\mathbf{x} = \mathbf{b} \tag{7}$$

Now we will focus on the transformation aspect of matrix multiplication. For example, if A has size $m \times n$, then multiplication by A defines a matrix transformation from R^n to R^m, so the problem of solving (7) can be viewed geometrically as the problem of finding a vector $\mathbf{x}$ in R^n whose image under the transformation T_A is the vector $\mathbf{b}$ in R^m.

EXAMPLE 6 Let $T_A : R^2 \to R^3$ be the matrix transformation in Example 3.

A Matrix Transformation

(a) Find a vector $\mathbf{x}$ in R^2, if any, whose image under T_A is

$$\mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$$

Solution (a) The stated problem is equivalent to finding a solution $\mathbf{x}$ of the linear system

$$\begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix} \tag{8}$$

We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

It follows from this that (8) has the unique solution

$$\mathbf{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and hence that $T_A(\mathbf{x}) = \mathbf{b}$. This shows that the vector $\mathbf{b}$ is in the range of T_A.

To be in ***reduced row echelon form***, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading* 1**.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A matrix that has the first three properties is said to be in ***row echelon form***.

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition 6.1.2 A function $T: R^n \to R^m$ is called a ***linear transformation*** from R^n to R^m if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for all scalars c:

(i) $T(c\mathbf{v}) = cT(\mathbf{v})$ **[Homogeneity property]**

(ii) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

In the special case where $m = n$, the linear transformation T is called a ***linear operator*** on R^n.

The two properties in this definition can be used in combination to show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in R^n and c_1 and c_2 are any scalars, then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

(verify). More generally, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in R^n and $c_1, c_2, \ldots, c_k$ are any scalars, then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k) \tag{11}$$

Engineers and physicists sometimes call this the ***superposition principle***.

EXAMPLE 7 if A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are column vectors in R^n, and c is a scalar, then $A(c\mathbf{v}) = c(A\mathbf{v})$ and $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$. Thus, the matrix transformation $T_A : R^n \to R^m$ is linear since

$$T_A(c\mathbf{v}) = A(c\mathbf{v}) = c(A\mathbf{v}) = cT_A(\mathbf{v})$$

$$T_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v}) \qquad \blacksquare$$

EXAMPLE 8 In Example 2 we considered the transformation

$$T(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2)$$

This transformation is not linear since it violates both conditions in Definition 6.1.2 (although the violation of either condition would have been sufficient for nonlinearity). The homogeneity condition is violated since

$$T(c\mathbf{v}) = T(cv_1, cv_2, cv_3) = (c^2v_1^2, c^2v_2^2, c^2v_3^2) = c^2(v_1^2, v_2^2, v_3^2) = c^2T(\mathbf{v})$$

which means that $T(c\mathbf{v}) \neq cT(\mathbf{v})$ for some scalars and vectors.

The additivity condition is violated since

$$T(\mathbf{u} + \mathbf{v}) = T(u_1 + v_1, u_2 + v_2, u_3 + v_3) = ((u_1 + v_1)^2, (u_2 + v_2)^2, (u_3 + v_3)^2)$$

whereas

$$T(\mathbf{u}) + T(\mathbf{v}) = (u_1^2, u_2^2, u_3^2) + (v_1^2, v_2^2, v_3^2) = (u_1^2 + v_1^2, u_2^2 + v_2^2, u_3^2 + v_3^2)$$

Thus, $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ for some vectors in R^3. ■

SOME PROPERTIES OF LINEAR TRANSFORMATIONS

The next theorem gives some basic properties of linear transformations.

Theorem 6.1.3 *If $T : R^n \to R^m$ is a linear transformation, then:*

(*a*) $T(\mathbf{0}) = \mathbf{0}$

(*b*) $T(-\mathbf{u}) = -T(\mathbf{u})$

(*c*) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

Proof To prove (*a*) set $c = 0$ in the formula $T(c\mathbf{v}) = cT(\mathbf{v})$, and to prove (*b*) set $c = -1$ in this formula. To prove part (*c*) replace $\mathbf{v}$ by $-\mathbf{v}$ in the formula $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and apply part (*b*) on the right side of the resulting equation. ■

ALL LINEAR TRANSFORMATIONS FROM R^n TO R^m ARE MATRIX TRANSFORMATIONS

We saw in Example 7 that every matrix transformation from R^n to R^m is linear. We will now show that matrix transformations are the *only* linear transformations from R^n to R^m in the sense that if $T : R^n \to R^m$ is a linear transformation, then there is a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n (assuming, of course, that $\mathbf{x}$ is expressed in column form). This is an extremely important result because it means

$\Downarrow$

all linear transformations from R^n to R^m can be performed by matrix multiplications, even if they don't arise in that way.

To prove this result, let us assume that $\mathbf{x}$ is written in column form and express it as a linear combination of the standard unit vectors by writing

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Thus, it follows from (11) that

$$T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n) \tag{12}$$

If we now create the matrix A that has $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$ as successive column vectors, then it follows from Formula (10) of Section 3.1 that (12) can be expressed as

$$T(\mathbf{x}) = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}$$

Thus, we have established the following result.

Theorem 6.1.4 *Let $T : R^n \to R^m$ be a linear transformation, and suppose that vectors are expressed in column form. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors in R^n, and if $\mathbf{x}$ is any vector in R^n, then $T(\mathbf{x})$ can be expressed as*

$$T(\mathbf{x}) = A\mathbf{x} \tag{13}$$

where

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

The matrix A in this theorem is called the ***standard matrix for T***, and we say that T is the ***transformation corresponding to A***, or that T is the ***transformation represented by A***, or sometimes simply that ***T is the transformation A***.

When it is desirable to emphasize the relationship between T and its standard matrix, we will denote A by $[T]$; that is, we will write

$$[T] = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

With this notation, the relationship in (13) becomes

$$\boxed{T(\mathbf{x}) = [T]\mathbf{x}} \tag{14}$$

$$\Downarrow$$

REMARK Theorem 6.1.4 shows that *a linear transformation* $T: R^n \to R^m$ *is completely determined by its values at the standard unit vectors* in the sense that once the images of the standard unit vectors are known, the standard matrix $[T]$ can be constructed and then used to compute images of all other vectors using (14).

EXAMPLE 10

Standard Matrix for a Scaling Operator

In Example 1 we considered the scaling operator $T: R^2 \rightarrow R^2$ defined by $T(\mathbf{x}) = 2\mathbf{x}$. Show that this operator is linear, and find its standard matrix.

Solution The transformation T is homogeneous since

$$T(c\mathbf{v}) = 2(c\mathbf{v}) = c(2\mathbf{v}) = cT(\mathbf{v})$$

and it is additive since

$$T(\mathbf{u} + \mathbf{v}) = 2(\mathbf{u} + \mathbf{v}) = 2\mathbf{u} + 2\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

From Theorem 6.1.4, the standard matrix for T is

$$[T] = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = [2\mathbf{e}_1 \quad 2\mathbf{e}_2] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

As a check,

$$[T]\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\mathbf{x} = T(\mathbf{x})$$

■

Transformations from R^n to R^m are often specified by formulas that relate the components of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in R^n with those of its image $\mathbf{w} = T(\mathbf{x}) = (w_1, w_2, \dots, w_m)$ in R^m.

It follows from Theorem 6.1.4 that such a transformation is linear if and only if the relationship between $\mathbf{w}$ and $\mathbf{x}$ is expressible as $\mathbf{w} = A\mathbf{x}$, where $A = [a_{ij}]$ is the standard matrix for T.

If we write out the individual equations in this matrix equation, we obtain

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

$$\Downarrow$$

Thus, it follows that $T(\mathbf{x}) = (w_1, w_2, \dots, w_m)$ *is a linear transformation if and only if the equations relating the components of* $\mathbf{x}$ *and* $\mathbf{w}$ *are linear equations.*

EXAMPLE 11

Standard Matrix for a Linear Transformation

Show that the transformation $T : R^3 \to R^2$ defined by the formula

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3) \tag{15}$$

is linear and find its standard matrix.

Solution Let $\mathbf{x} = (x_1, x_2, x_3)$ be a vector in R^3, and let $\mathbf{w} = (w_1, w_2)$ be its image under the transformation T. It follows from (15) that

$$w_1 = x_1 + x_2 \quad \text{and} \quad w_2 = x_2 - x_3$$

Since these are linear equations, the transformation T is linear.

To find the standard matrix for T we compute the images of the standard unit vectors under the transformation. These are

$$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0)$$
$$T(\mathbf{e}_2) = T(0, 1, 0) = (1, 1) \qquad T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3) \quad (15)$$
$$T(\mathbf{e}_3) = T(0, 0, 1) = (0, -1)$$

Writing these vectors in column form yields the standard matrix

$$[T] = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

As a check,

$$[T]\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$$

which agrees with (15), except for the use of matrix notation. ■

ROTATIONS ABOUT THE ORIGIN

Some of the most important linear operators on R^2 and R^3 are rotations, reflections, and projections. In this section we will show how to find standard matrices for such operators on R^2, and in the next section we will use these matrices to study the operators in more detail.

Let θ be a fixed angle, and consider the operator T that rotates each vector $\mathbf{x}$ in R^2 about the origin through the angle θ^* (Figure 6.1.7*a*). It is not hard to visualize that T is linear by drawing some appropriate pictures.

For example, Figure 6.1.7*b* makes it evident that the rotation T is homogeneous because the same image of a vector $\mathbf{v}$ results whether one first multiplies $\mathbf{v}$ by c and then rotates or first rotates $\mathbf{v}$ and then multiplies by c; also, Figure 6.1.7*c* makes it evident that T is additive because the same image results whether one first adds $\mathbf{u}$ and $\mathbf{v}$ and then rotates the sum or first rotates the vectors $\mathbf{u}$ and $\mathbf{v}$ and then forms the sum of the rotated vectors.

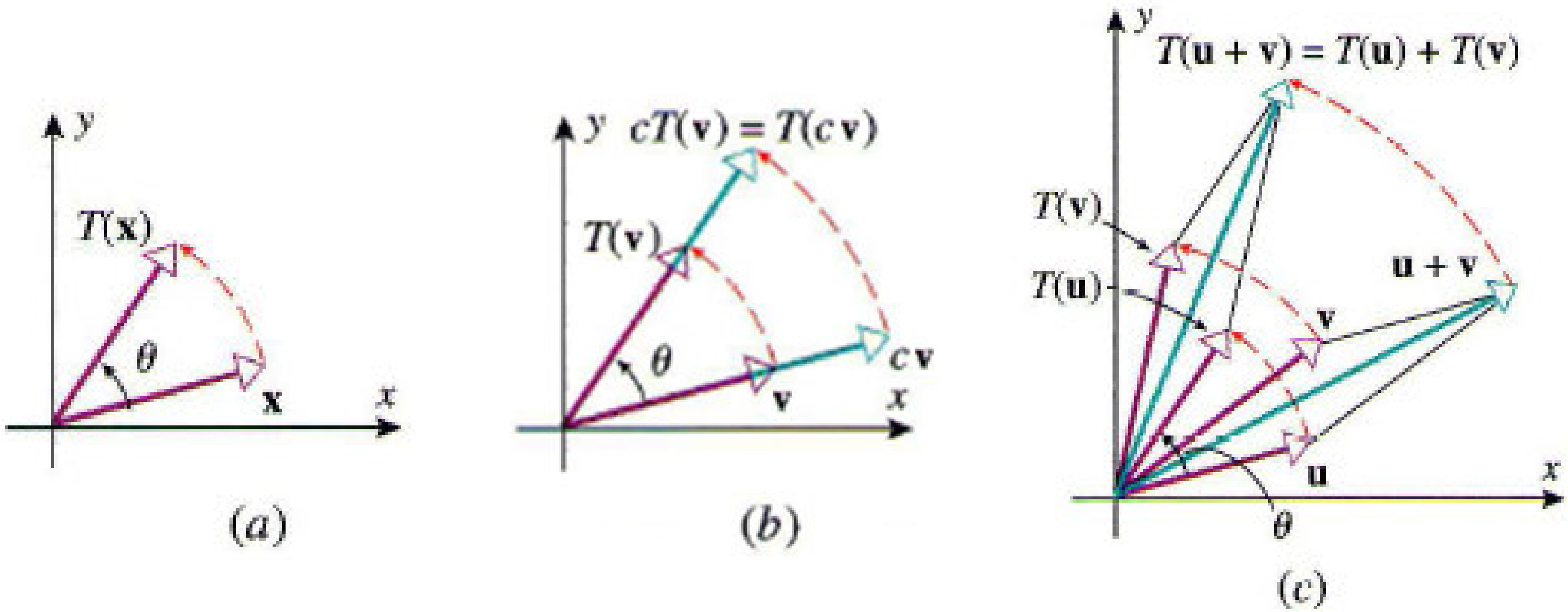


Figure 6.1.7

Let us now try to find the standard matrix for the rotation operator. In keeping with standard usage, we will denote the standard matrix for the rotation about the origin through an angle θ by R_θ. From Figure 6.1.8 we see that this matrix is

$$R_\theta = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{16}$$

so the image of a vector $\mathbf{x} = (x, y)$ under this rotation is

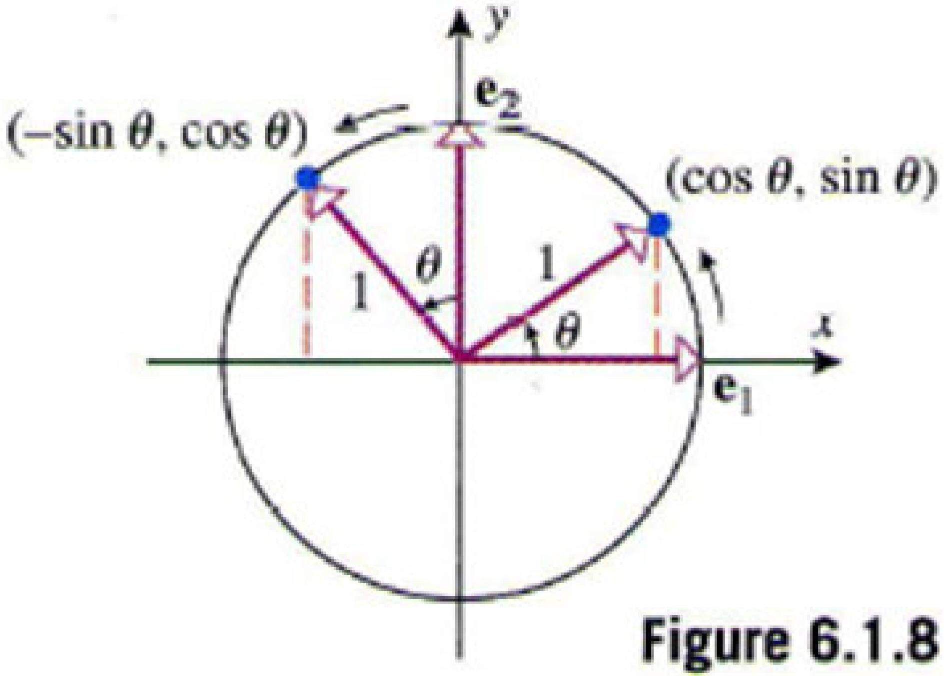


Figure 6.1.8

$$R_\theta\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{17}$$

EXAMPLE 12

A Rotation Operator

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ radians ($= 30°$) about the origin.

Solution It follows from (17) with $\theta = \pi/6$ that

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

or in comma-delimited notation this image is approximately $(0.37, 1.37)$. ■

$$R_{\theta}\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{17}$$

REFLECTIONS ABOUT LINES THROUGH THE ORIGIN

Now let us consider the operator $T : R^2 \rightarrow R^2$ that reflects each vector $\mathbf{x}$ about a line through the origin that makes an angle θ with the positive x-axis (Figure 6.1.9). The same kind of geometric argument that we used to establish the linearity of rotation operators can be used to establish linearity of reflection operators.

In keeping with a common convention of associating the letter H with reflections, we will denote the standard matrix for the reflection in Figure 6.1.9 by H_θ.

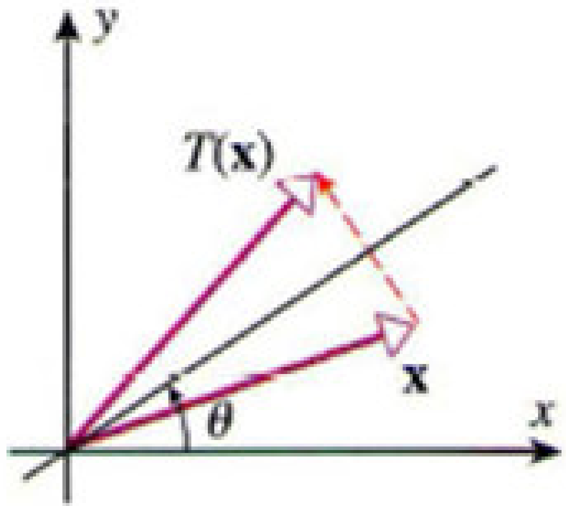


Figure 6.1.9

From Figure 6.1.10 we see that this matrix is

$$H_\theta = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} \cos 2\theta & \cos\left(\frac{\pi}{2} - 2\theta\right) \\ \sin 2\theta & -\sin\left(\frac{\pi}{2} - 2\theta\right) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{18}$$

so the image of a vector $\mathbf{x} = (x, y)$ under this reflection is

$$H_\theta \mathbf{x} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{19}$$

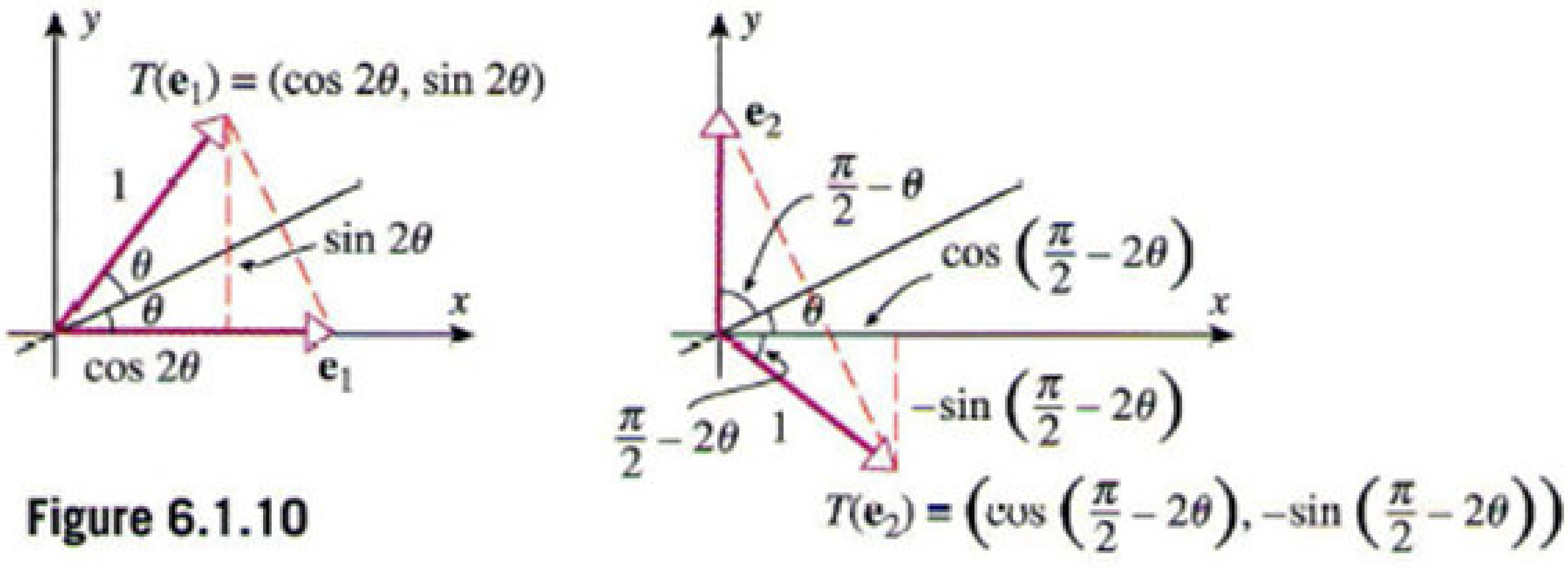


Figure 6.1.10

EXAMPLE 13

A Reflection Operator

Find the image of the vector $\mathbf{x} = (1, 1)$ under a reflection about the line through the origin that makes an angle of $\pi/6\ (= 30°)$ with the positive x-axis.

Solution Substituting $\theta = \pi/6$ in (19) yields

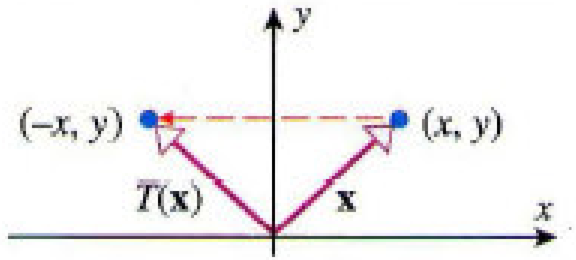
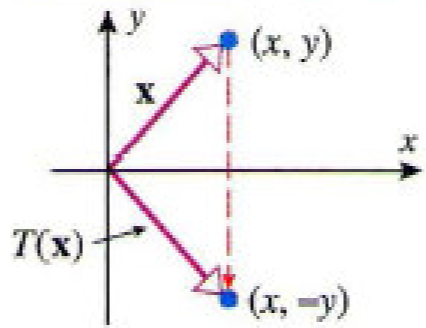
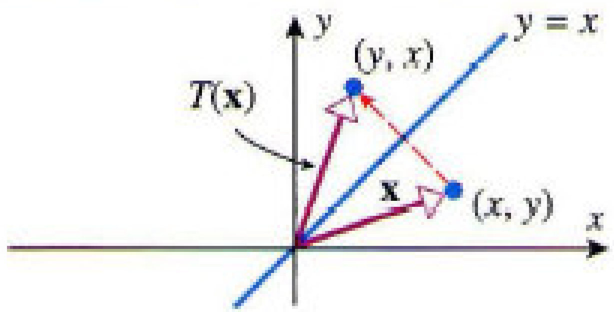
$$H_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{\sqrt{3}-1}{2} \end{bmatrix} \approx \begin{bmatrix} 1.37 \\ 0.37 \end{bmatrix}$$

or in comma-delimited notation this image is approximately $(1.37, 0.37)$. ■

$$H_\theta \mathbf{x} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (19)$$

The most basic reflections in an xy-coordinate system are about the x-axis ($\theta = 0$), the y-axis ($\theta = \pi/2$), and the line $y = x$ ($\theta = \pi/4$). Some information about these reflections is given in Table 6.1.1.

Table 6.1.1

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

ORTHOGONAL PROJECTIONS ONTO LINES THROUGH THE ORIGIN

Consider the operator $T : R^2 \to R^2$ that projects each vector $\mathbf{x}$ in R^2 onto a line through the origin by dropping a perpendicular to that line as shown in Figure 6.1.11; we call this operator the ***orthogonal projection*** of R^2 onto the line. One can show that orthogonal projections onto lines are linear and hence are matrix operators.

The standard matrix for an orthogonal projection onto a general line through the origin can be obtained using Theorem 6.1.4; however, it will be instructive to consider an alternative approach in which we will express the orthogonal projection in terms of a reflection and then use the known standard matrix for the reflection to obtain the matrix for the projection.

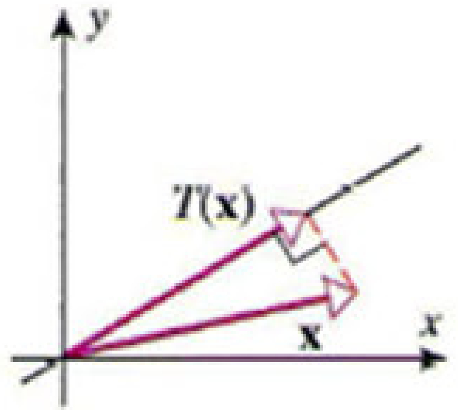


Figure 6.1.11 Orthogonal projection onto a line through the origin

Consider a line through the origin that makes an angle θ with the positive x-axis, and denote the standard matrix for the orthogonal projection by P_θ. It is evident from Figure 6.1.12 that for each $\mathbf{x}$ in R^2 the vector $P_\theta\mathbf{x}$ is related to the vector $H_\theta\mathbf{x}$ by the equation

$$P_\theta\mathbf{x} - \mathbf{x} = \tfrac{1}{2}(H_\theta\mathbf{x} - \mathbf{x})$$

Solving for $P_\theta\mathbf{x}$ yields

$$P_\theta\mathbf{x} = \tfrac{1}{2}H_\theta\mathbf{x} + \tfrac{1}{2}\mathbf{x} = \tfrac{1}{2}H_\theta\mathbf{x} + \tfrac{1}{2}I\mathbf{x} = \tfrac{1}{2}(H_\theta + I)\mathbf{x}$$

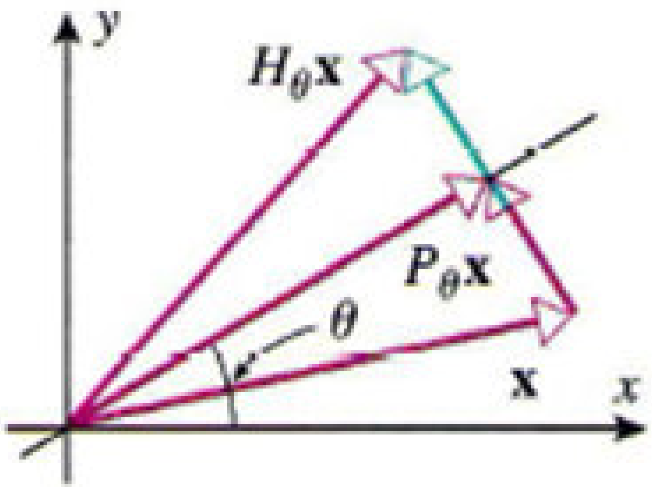


Figure 6.1.12

$$P_\theta = \tfrac{1}{2}(H_\theta + I) \tag{20}$$

We now leave it for you to use this equation and Formula (18) to show that

$$P_\theta = \begin{bmatrix} \tfrac{1}{2}(1+\cos 2\theta) & \tfrac{1}{2}\sin 2\theta \\ \tfrac{1}{2}\sin 2\theta & \tfrac{1}{2}(1-\cos 2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{21}$$

so the image of a vector $\mathbf{x} = (x, y)$ under this projection is

$$P_\theta\mathbf{x} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{22}$$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{18}$$

EXAMPLE 14 An Orthogonal Projection Operator

Find the orthogonal projection of the vector $\mathbf{x} = (1, 1)$ on the line through the origin that makes an angle of $\pi/12\ (= 15^\circ)$ with the x-axis.

Solution Here it is easier to work with the first form of the standard matrix given in (21), since the angle $2\theta = \pi/6$ is nicer to work with than $\theta = \pi/12$. This yields the standard matrix

$$P_{\pi/12} = \begin{bmatrix} \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) & \frac{1}{2}\sin\frac{\pi}{6} \\ \frac{1}{2}\sin\frac{\pi}{6} & \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

for the projection. Thus, the image of $\mathbf{x} = (1, 1)$ under this projection is

$$P_{\pi/12}\mathbf{x} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(3+\sqrt{3}) \\ \frac{1}{4}(3-\sqrt{3}) \end{bmatrix} \approx \begin{bmatrix} 1.18 \\ 0.32 \end{bmatrix}$$

or in comma-delimited notation this projection is approximately (1.18, 0.32). The projection is shown in Figure 6.1.13. ■

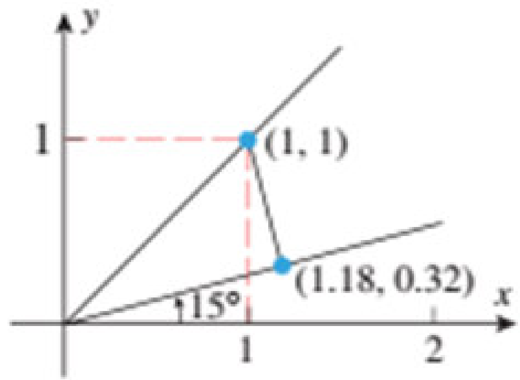
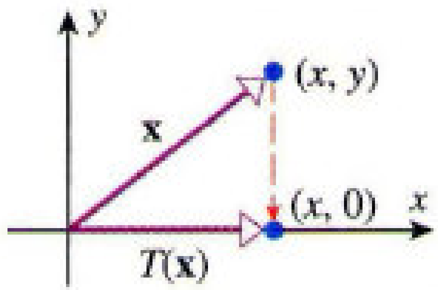
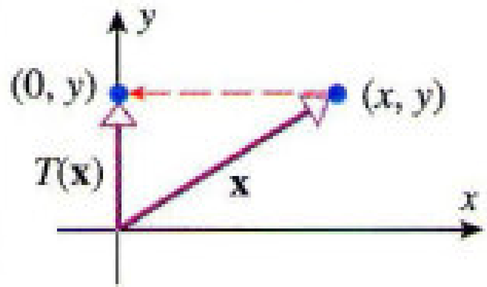


Figure 6.1.13

$$P_\theta = \begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \quad (21)$$

The most basic orthogonal projections in R^2 are onto the coordinate axes. Information about these operators is given in Table 6.1.2.

Table 6.1.2

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection on the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$