Section 6.2 Geometry of Linear Operators

NORM-PRESERVING LINEAR OPERATORS

In the last section we studied three kinds of operators on R^2: rotations about the origin, reflections about lines through the origin, and orthogonal projections onto lines through the origin; and we showed that the standard matrices for these operators are

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}, \quad P_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{1–3}$$

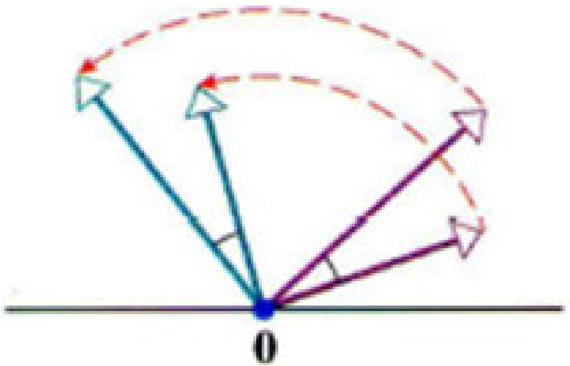
Rotation about the origin through an angle θ

Reflection about the line through the origin making an angle θ with the positive x-axis

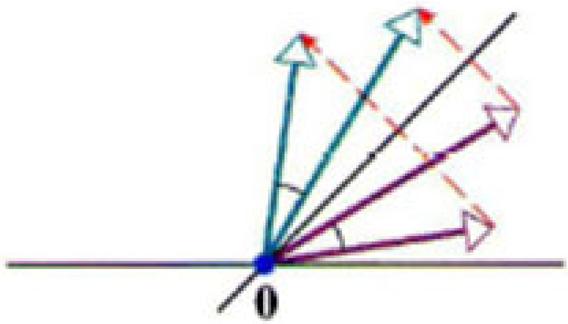
Orthogonal projection onto the line through the origin making an angle θ with the positive x-axis

As suggested in Figure 6.2.1, rotations about the origin and reflections about lines through the origin do not change the lengths of vectors or the angles between vectors; thus, we say that these operators are *length preserving* and *angle preserving*.

In contrast, an orthogonal projection onto a line through the origin can change the length of a vector and the angle between vectors.



A rotation about the origin does not change lengths of vectors or angles between vectors.



A reflection about a line through the origin does not change lengths of vectors or angles between vectors.

Figure 6.2.1

In general, a linear operator $T: R^n \to R^n$ with the length-preserving property $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ is called an ***orthogonal operator*** or a ***linear isometry*** (from the Greek *isometros*, meaning "equal measure"). Thus, for example, rotations about the origin and reflections about lines through the origin of R^2 are orthogonal operators.

Theorem 6.2.1 *If $T : R^n \to R^n$ is a linear operator on R^n, then the following statements are equivalent.*

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n. **[*T* orthogonal (i.e., length preserving)]**

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n. **[*T* is dot product preserving.]**

Proof (a) ⇒ (b) Suppose that T is length preserving, and let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in R^n. We leave it for you to derive the relationship

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) \tag{4}$$

by writing $\|\mathbf{x}+\mathbf{y}\|^2$ and $\|\mathbf{x}-\mathbf{y}\|^2$ as

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \quad \text{and} \quad \|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})$$

and then expanding the dot products. It now follows from (4) that

$$T(\mathbf{x}) \cdot T(\mathbf{y}) = \frac{1}{4}\left(\|T(\mathbf{x})+T(\mathbf{y})\|^2 - \|T(\mathbf{x})-T(\mathbf{y})\|^2\right)$$

$$= \frac{1}{4}\left(\|T(\mathbf{x}+\mathbf{y})\|^2 - \|T(\mathbf{x}-\mathbf{y})\|^2\right) \qquad \text{[Additivity and Theorem 6.1.3]}$$

$$= \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) \qquad \text{[T is length preserving.]}$$

$$= \mathbf{x} \cdot \mathbf{y} \qquad \text{[Formula (4)]}$$

Theorem 6.2.1 *If $T : R^n \to R^n$ is a linear operator on R^n, then the following statements are equivalent.*

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n. **[T orthogonal (i.e., length preserving)]**

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n. **[T is dot product preserving.]**

***Proof* (b) ⇒ (a)** Conversely, suppose that T is dot product preserving, and let $\mathbf{x}$ be any vector in R^n. Since we can express $\|\mathbf{x}\|$ as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} \tag{5}$$

it follows that

$$\|T(\mathbf{x})\| = \sqrt{T(\mathbf{x}) \cdot T(\mathbf{x})} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \|\mathbf{x}\| \quad \blacksquare$$

REMARK Formulas (4) and (5) are "flip sides of a coin" in that (5) provides a way of expressing norms in terms of dot products, whereas (4), which is sometimes called the ***polarization identity***, provides a way of expressing dot products in terms of norms.

$$\mathbf{x} \cdot \mathbf{y} = \tfrac{1}{4}\left(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2\right) \tag{4}$$

ORTHOGONAL OPERATORS PRESERVE ANGLES AND ORTHOGONALITY

Recall that the angle between two nonzero vectors **x** and **y** in R^n is given by the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) \tag{6}$$

Thus, if $T : R^n \rightarrow R^n$ is an orthogonal operator, the fact that T is length preserving and dot product preserving implies that

$$\cos^{-1}\left(\frac{T(\mathbf{x})\cdot T(\mathbf{y})}{\|T(\mathbf{x})\|\,\|T(\mathbf{y})\|}\right) = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) \tag{7}$$

which implies that an orthogonal operator preserves angles.

In particular, an orthogonal operator preserves orthogonality in the sense that the images of two vectors are orthogonal if and only if the original vectors are orthogonal.

ORTHOGONAL MATRICES

Our next goal is to explore the relationship between the orthogonality of an operator and properties of its standard matrix.

As a first step, suppose that A is the standard matrix for an orthogonal linear operator $T : R^n \to R^n$. Since $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in R^n, and since $\|T(\mathbf{x})\| = \|\mathbf{x}\|$, it follows that

$$\|A\mathbf{x}\| = \|\mathbf{x}\| \tag{8}$$

for all $\mathbf{x}$ in R^n.

Thus, the problem of determining whether a linear operator is orthogonal reduces to determining whether its standard matrix satisfies (8) for all $\mathbf{x}$ in R^n.

The following definition will be useful in our investigation of this problem.

Definition 6.2.2 A square matrix A is said to be ***orthogonal*** if $A^{-1} = A^T$.

EXAMPLE 1 An Orthogonal Matrix

The matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

is orthogonal since

$$A^T A = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

and hence

$$A^{-1} = A^T = \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \qquad \blacksquare$$

The following theorem states some of the basic properties of orthogonal matrices.

Theorem 6.2.3

(*a*) *The transpose of an orthogonal matrix is orthogonal.*

(*b*) *The inverse of an orthogonal matrix is orthogonal.*

(*c*) *A product of orthogonal matrices is orthogonal.*

(*d*) *If A is orthogonal, then* $\det(A) = 1$ *or* $\det(A) = -1$.

***Proof* (*a*)** If A is orthogonal, then $A^TA = I$. We can rewrite this as $A^T(A^T)^T = I$, which implies that $(A^T)^{-1} = (A^T)^T$. Thus, A^T is orthogonal.

***Proof* (*b*)** If A is orthogonal, then $A^{-1} = A^T$. Transposing both sides of this equation yields

$$(A^{-1})^T = (A^T)^T = A = (A^{-1})^{-1}$$

which implies that A^{-1} is orthogonal.

Theorem 6.2.3

(*a*) *The transpose of an orthogonal matrix is orthogonal.*

(*b*) *The inverse of an orthogonal matrix is orthogonal.*

(*c*) *A product of orthogonal matrices is orthogonal.*

(*d*) *If A is orthogonal, then* $\det(A) = 1$ *or* $\det(A) = -1$.

***Proof* (*c*)** We will give the proof for a product of two orthogonal matrices. If A and B are orthogonal matrices, then

$$(AB)^{-1} = B^{-1}A^{-1} = B^TA^T = (AB)^T$$

see Kreyszig, sc. 7.8 for inverse
and transpose of a matrix product.

Thus, AB is orthogonal.

***Proof* (*d*)** If A is orthogonal, then $A^TA = I$. Taking the determinant of both sides, and using properties of determinants yields

$$\det(A)\det(A^T) = \det(A)\det(A) = 1$$

see Kreyszig, sc. 7.8 for
properties of determinants.

which implies that $\det(A) = 1$ or $\det(A) = -1$. ■

Theorem 6.2.4 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A^TA = I$.

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(d) *The column vectors of A are orthonormal.*

We will prove the chain of implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

***Proof* (a) ⇒ (b)** It follows from Formula (12) of Section 3.2 that

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = \mathbf{x} \cdot A^TA\mathbf{x} = \mathbf{x} \cdot I\mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$$

from which part (b) follows.

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v} \qquad (12)$$

If 2 vectors u and v are column vectors, then $u.v = v^Tu$.

$Au.v = v^T(Au) = (v^TA)u = (A^Tv)^Tu = u.A^Tv$.

***Proof* (b) ⇒ (c)** This follows from Theorem 6.2.1 with $T(\mathbf{x}) = A\mathbf{x}$.

Proof (c) ⇒ (d) Define $T: R^n \to R^n$ to be the matrix operator $T(\mathbf{x}) = A\mathbf{x}$. By hypothesis, $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n, so Theorem 6.2.1 implies that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

This tells us that T preserves lengths and orthogonality, so T must map every set of orthonormal vectors into another set of orthonormal vectors. This is true, in particular, for the set of standard unit vectors, so

$$T(\mathbf{e}_1) = A\mathbf{e}_1, \quad T(\mathbf{e}_2) = A\mathbf{e}_2, \ldots, \quad T(\mathbf{e}_n) = A\mathbf{e}_n$$

must be an orthonormal set. However, these are the column vectors of A (why?), which proves part (d).

Proof (d) ⇒ (a) Assume that the column vectors of A are orthonormal, and denote these vectors by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. It follows from Formula (9) of Section 3.6 that $A^TA = I$ (verify). ■

$$A^TA = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \|\mathbf{a}_1\|^2 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_1 \cdot \mathbf{a}_2 & \|\mathbf{a}_2\|^2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_1 \cdot \mathbf{a}_n & \mathbf{a}_2 \cdot \mathbf{a}_n & \cdots & \|\mathbf{a}_n\|^2 \end{bmatrix} \quad (9)$$

If A is square, then the condition $A^TA = I$ in part (a) of Theorem 6.2.4 is equivalent to saying that $A^{-1} = A^T$ (i.e., A is orthogonal). Thus, in the case of a square matrix, Theorems 6.2.4 and 6.2.3 together yield the following theorem about orthogonal matrices.

Theorem 6.2.5 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A is orthogonal.*

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(d) *The column vectors of A are orthonormal.*

(e) *The row vectors of A are orthonormal.*

Recall that a linear operator $T : R^n \to R^n$ is defined to be orthogonal if $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. Thus, T is orthogonal if and only if its standard matrix has the property $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n. This fact and parts (a) and (b) of Theorem 6.2.5 yield the following result about standard matrices of orthogonal operators.

⇓

Theorem 6.2.6 *A linear operator $T : R^n \to R^n$ is orthogonal if and only if its standard matrix is orthogonal.*

EXAMPLE 2 Standard Matrices of Rotations and Reflections Are Orthogonal

Since rotations about the origin and reflections about lines through the origin of R^2 are orthogonal operators, the standard matrices of these operators must be orthogonal. This is indeed the case, since Formula (1) implies that

$$R_\theta^T R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and, similarly, Formula (2) implies that $H_\theta^T H_\theta = I$ (verify). ■

EXAMPLE 3 Identifying Orthogonal Matrices

We showed in Example 1 that the matrix

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} \tag{9}$$

is orthogonal by confirming that $A^TA = I$. In light of Theorem 6.2.5, we can also establish the orthogonality of A by showing that the row vectors or the column vectors are orthonormal. We leave it for you to check both. ■

ALL ORTHOGONAL LINEAR OPERATORS ON R^2 ARE ROTATIONS OR REFLECTIONS

We have seen that rotations about the origin and reflections about lines through the origin of R^2 are orthogonal (i.e., length preserving) operators. We will now show that these are the *only* orthogonal operators on R^2.

Theorem 6.2.7 *If $T : R^2 \rightarrow R^2$ is an orthogonal linear operator, then the standard matrix for T is expressible in the form*

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad or \quad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{10}$$

That is, T is either a rotation about the origin or a reflection about a line through the origin.

A 2x2 orthogonal matrix represents a rotation if det(A) = 1 and it represents a reflection if det(A) = −1.

EXAMPLE 4 Geometric Properties of Orthogonal Matrices

In each part, describe the linear operator on R^2 corresponding to the matrix A.

(a) $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (b) $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Solution (a) The column vectors of A are orthonormal (verify), so the matrix A is orthogonal. This implies that the operator is either a rotation about the origin or a reflection about a line through the origin. Since $\det(A) = 1$, we know definitively that the operator is a rotation. We can determine the angle of rotation by comparing A to the general rotation matrix R_θ in (1). This yields

$$\cos\theta = 1/\sqrt{2} \quad \text{and} \quad \sin\theta = 1/\sqrt{2}$$

from which we conclude that the angle of rotation is $\theta = \pi/4\ (= 45^\circ)$.

Solution (b) The matrix A is orthogonal and $\det(A) = -1$, so the corresponding operator is a reflection about a line through the origin. We can determine the angle that the line makes with the positive x-axis by comparing A to the general reflection matrix H_θ in (2). This yields

$$\cos 2\theta = 1/\sqrt{2} \quad \text{and} \quad \sin 2\theta = 1/\sqrt{2}$$

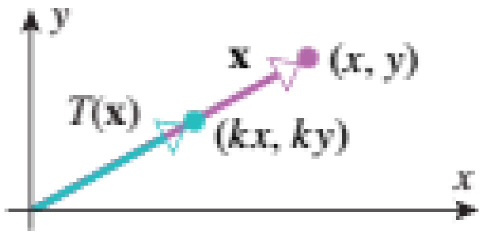
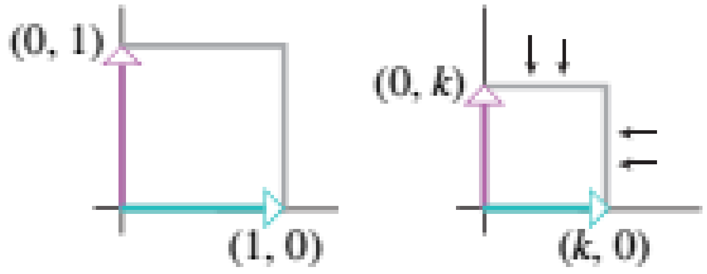
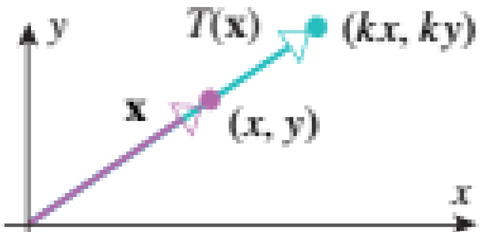
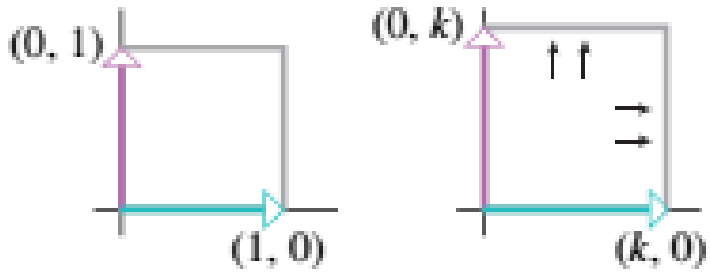
from which we conclude that $\theta = \pi/8\ (= 22.5^\circ)$. ■

CONTRACTIONS AND DILATIONS OF R^2

Up to now we have focused primarily on length-preserving linear operators; now we will consider some important linear operators that are not length preserving.

If k is a nonnegative scalar, then the linear operator $T(x, y) = (kx, ky)$ is called the ***scaling operator with factor k***. In particular, this operator is called a ***contraction*** if $0 \leq k < 1$ and a ***dilation*** if $k > 1$. Contractions preserve the directions of vectors but reduce their lengths by the factor k, and dilations preserve the directions of vectors but increase their lengths by the factor k. Table 6.2.1 provides the basic information about scaling operators on R^2.

Table 6.2.1

<table>
<tr><th>Operator</th><th>Illustration
$T(x, y) = (kx, ky)$</th><th>Effect on the
Unit Square</th><th>Standard
Matrix</th></tr>
<tr><td>Contraction with factor k on R^2
$(0 \leq k < 1)$</td><td>
</td><td>
</td><td rowspan="2">$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$</td></tr>
<tr><td>Dilation with factor k on R^2
$(k > 1)$</td><td>
</td><td>
</td></tr>
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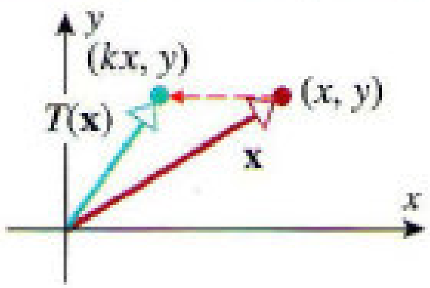
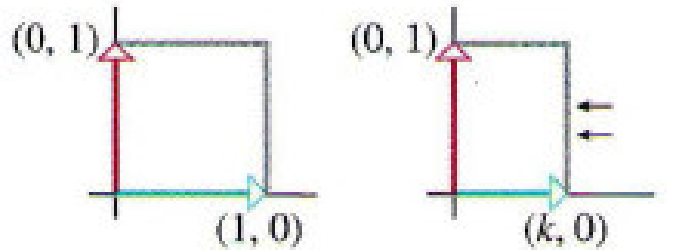
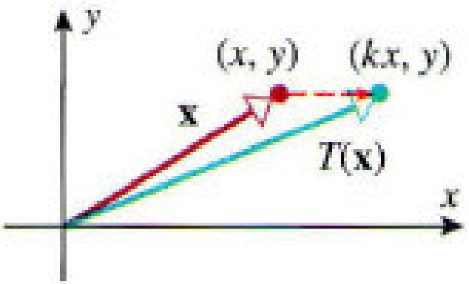
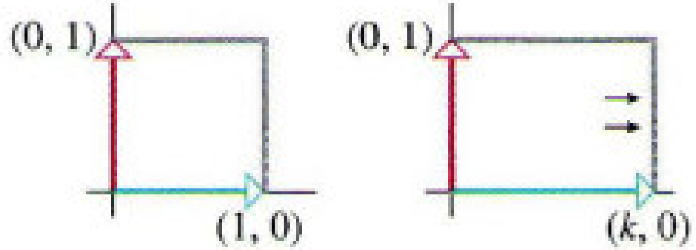
VERTICAL AND HORIZONTAL COMPRESSIONS AND EXPANSIONS OF R^2

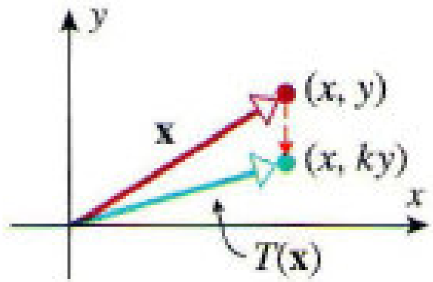
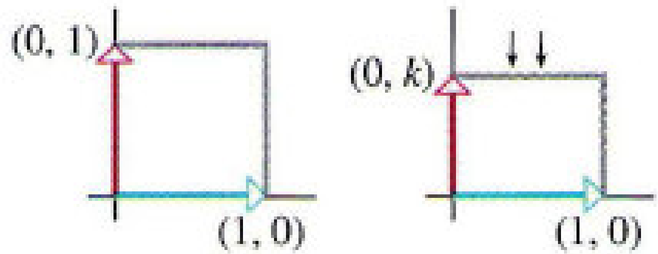
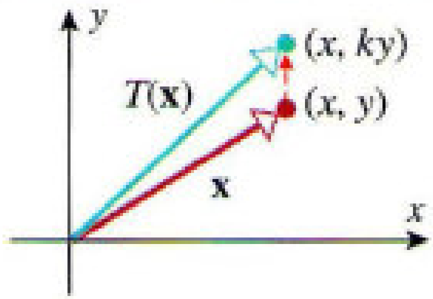
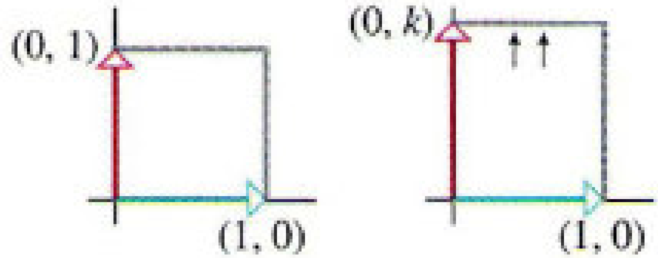
An operator $T(x, y) = (kx, y)$ that multiplies the x-coordinate of each point in the xy-plane by a nonnegative constant k has the effect of expanding or compressing every figure in the plane in the x-direction—it compresses if $0 \le k < 1$ and expands if $k > 1$. Accordingly, we call T the ***expansion*** (or ***compression***) ***in the x-direction with factor k***. Similarly, $T(x, y) = (x, ky)$ is the ***expansion*** (or ***compression***) ***in the y-direction with factor k***. Table 6.2.2 provides the basic information about expansion and compression operators on R^2.

Table 6.2.2

Operator	Illustration $T(x, y) = (kx, y)$	Effect on the Unit Square	Standard Matrix
Compression of R^2 in the x-direction with factor k $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of R^2 in the x-direction with factor k $(k > 1)$			

VERTICAL AND HORIZONTAL COMPRESSIONS AND EXPANSIONS OF R^2

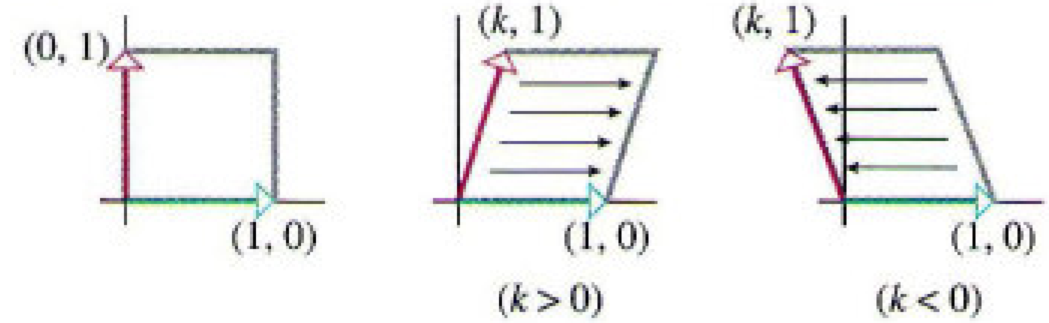
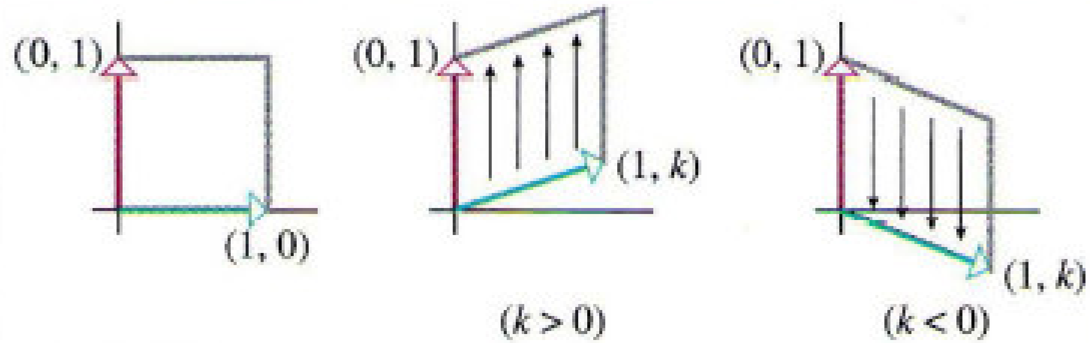
Table 6.2.2

Operator	Illustration $T(x, y) = (x, ky)$	Effect on the Unit Square	Standard Matrix
Compression of R^2 in the y-direction with factor k ($0 \leq k < 1$)			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of R^2 in the y-direction with factor k ($k > 1$)			

SHEARS

A linear operator of the form $T(x, y) = (x + ky, y)$ translates a point (x, y) in the xy-plane parallel to the x-axis by an amount ky that is proportional to the y-coordinate of the point. This operator leaves the points on the x-axis fixed (since $y = 0$), but as we progress away from the x-axis, the translation distance increases. We call this operator the ***shear in the x-direction with factor k***. Similarly, a linear operator of the form $T(x, y) = (x, y + kx)$ is called the ***shear in the y-direction with factor k***. Table 6.2.3 provides the basic information about shears in R^2.

Table 6.2.3

Operator	Effect on the Unit Square	Standard Matrix
Shear of R^2 in the x-direction with factor k $T(x, y) = (x + ky, y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of R^2 in the y-direction with factor k $T(x, y) = (x, y + kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

EXAMPLE 5

Some Basic Linear Operators on R^2

In each part describe the linear operator corresponding to A, and show its effect on the unit square.

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Solution By comparing the forms of these matrices to those in Tables 6.2.1, 6.2.2, and 6.2.3, we see that the matrix A_1 corresponds to a shear in the x-direction with factor 2, the matrix A_2 corresponds to a dilation with factor 2, and A_3 corresponds to an expansion in the x-direction with factor 2. The effects of these operators on the unit square are shown in Figure 6.2.3. ■

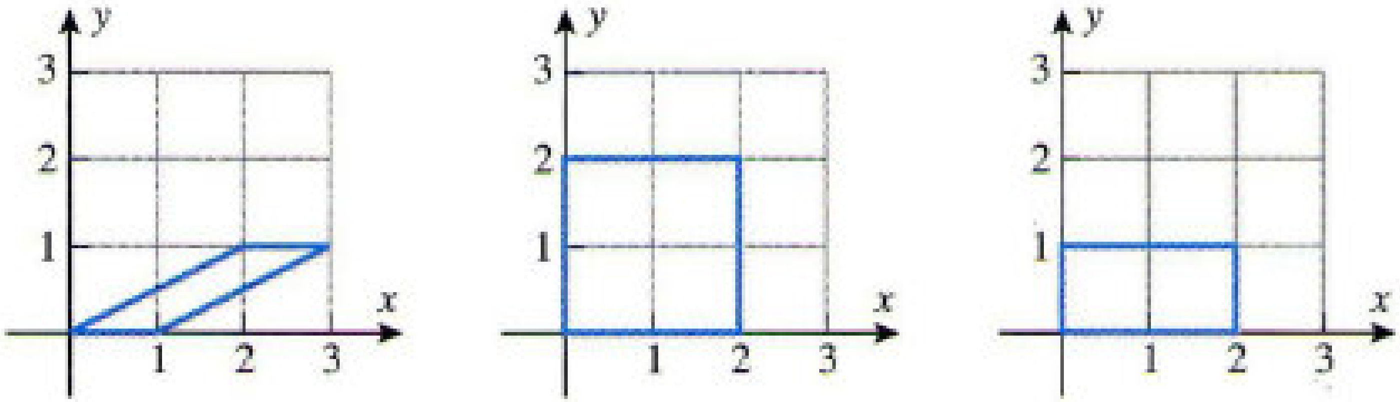


Figure 6.2.3

standard matrices → $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ (a) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ (b) $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ (c)

EXAMPLE 6 Application to Computer Graphics

Figure 6.2.4 shows a famous picture of Albert Einstein and three computer-generated linear transformations of that picture. The original picture was scanned and then digitized to decompose it into a rectangular array of pixels. The transformed picture was then obtained as follows:

- The program MATLAB was used to assign coordinates and a gray level to each pixel.
- The coordinates of the pixels were transformed by matrix multiplication.
- The images were then assigned their original gray levels to produce the transformed picture. ■

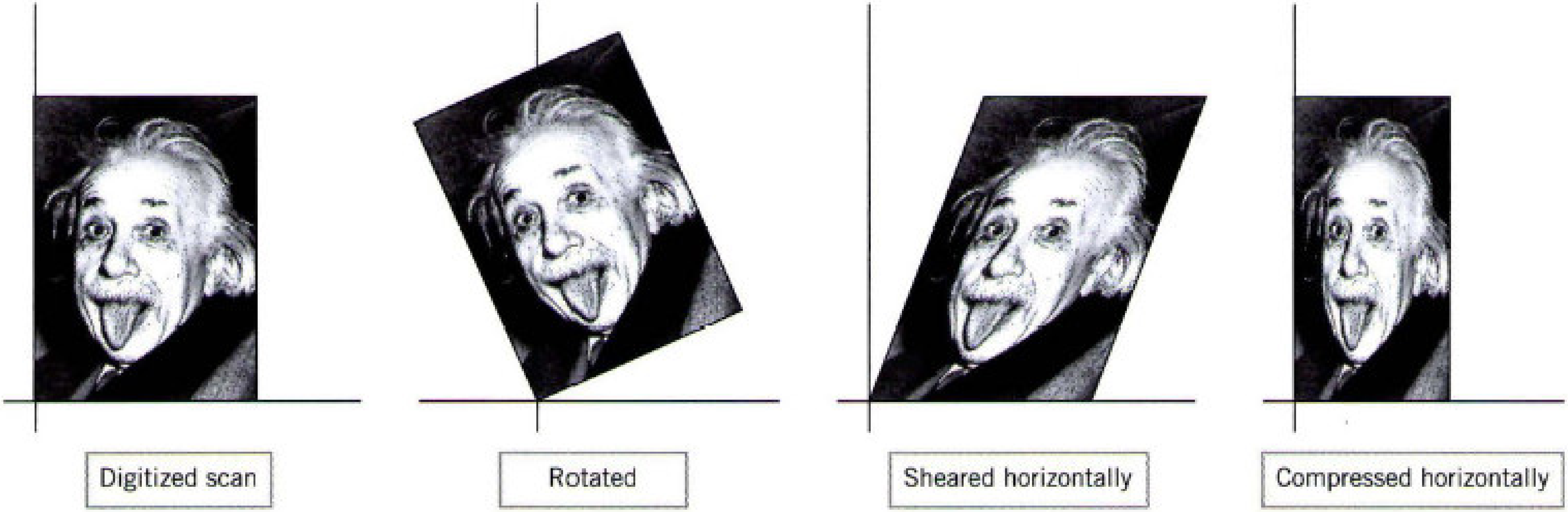
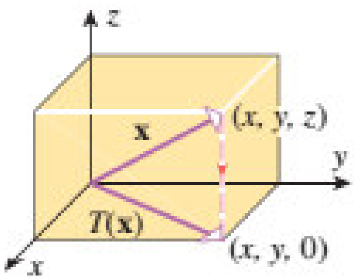
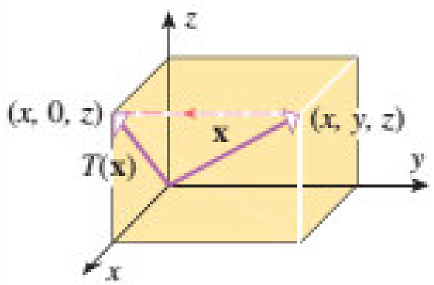
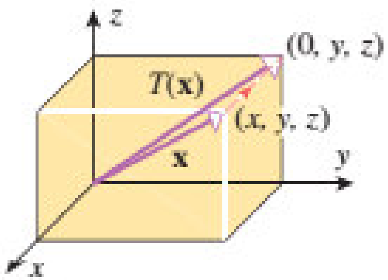


Figure 6.2.4

LINEAR OPERATORS ON R^3

We now turn our attention to linear operators on R^3. As in R^2, we will want to distinguish between operators that preserve lengths (orthogonal operators) and those that do not. The most important linear operators that are not length preserving are orthogonal projections onto subspaces, and the simplest of these are the orthogonal projections onto the coordinate planes of an xyz-coordinate system. Table 6.2.4 provides the basic information about such operators.

Table 6.2.4

Operator	Illustration	Standard Matrix
Orthogonal projection on the xy-plane $T(x, y, z) = (x, y, 0)$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane $T(x, y, z) = (x, 0, z)$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane $T(x, y, z) = (0, y, z)$		$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We have seen that 2×2 orthogonal matrices correspond to rotations about the origin or reflections about lines through the origin in R^2. One can prove that all 3×3 orthogonal matrices correspond to linear operators on R^3 of the following types:

Type 1: Rotations about lines through the origin.

Type 2: Reflections about planes through the origin.

Type 3: A rotation about a line through the origin followed by a reflection about the plane through the origin that is perpendicular to the line.

Recall that one call tell whether a 2×2 orthogonal matrix A represents a rotation or a reflection by its determinant—a rotation if $\det(A) = 1$ and a reflection if $\det(A) = -1$.

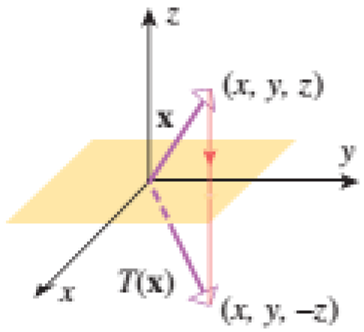
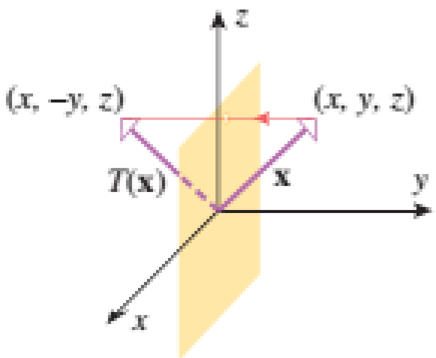
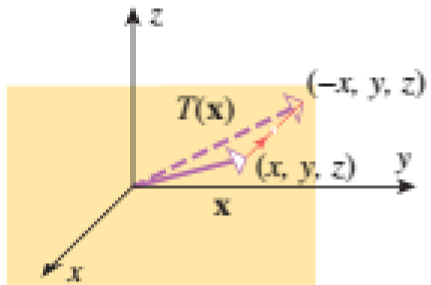
Similarly, if A is a 3×3 orthogonal matrix, then A represents a rotation (i.e., is of type 1) if $\det(A) = 1$ and represents a type 2 or type 3 operator if $\det(A) = -1$. Accordingly, we will frequently refer to 2×2 or 3×3 orthogonal matrices with determinant 1 as ***rotation matrices***.

To tell whether a 3×3 orthogonal matrix with determinant -1 represents a type 2 or a type 3 operator requires an analysis of eigenvectors and eigenvalues.

REFLECTIONS ABOUT COORDINATE PLANES

The most basic reflections in a rectangular xyz-coordinate system are those about the coordinate planes. Table 6.2.5 provides the basic information about such operators on R^3.

Table 6.2.5

Operator	Illustration	Standard Matrix
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ROTATIONS IN R^3

We will now turn our attention to rotations in R^3. To help understand some of the issues involved, we will begin with a familiar example—the rotation of the Earth about its axis through the North and South Poles. For simplicity, we will assume that the Earth is a sphere. Since the Sun rises in the east and sets in the west, we know that the Earth rotates from west to east. However, to an observer above the North Pole the rotation will appear counterclockwise, and to an observer below the South Pole it will appear clockwise (Figure 6.2.5). Thus, when a rotation in R^3 is described as clockwise or counterclockwise, a direction of view along the axis of rotation must also be stated.

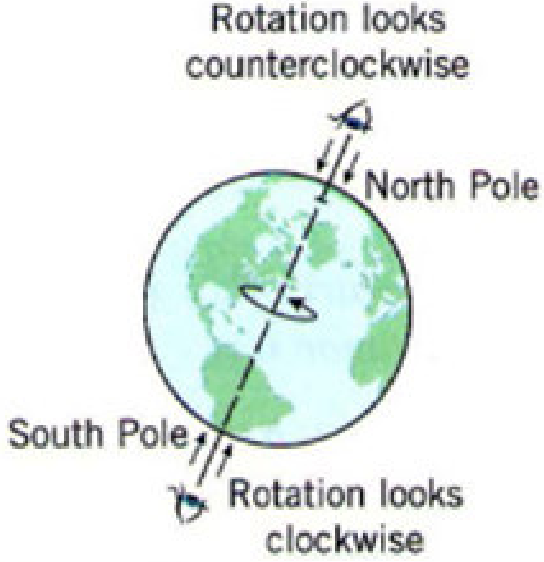


Figure 6.2.5

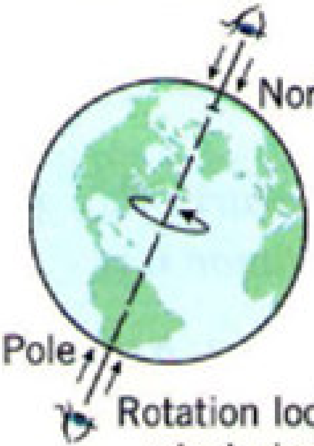


Figure 6.2.5

There are some other facts about the Earth's rotation that are useful for understanding general rotations in R^3. For example, as the Earth rotates about its axis, the North and South Poles remain fixed, as do all other points that lie on the axis of rotation. Thus, the axis of rotation can be thought of as the line of fixed points in the Earth's rotation. Moreover, all points on the Earth that are not on the axis of rotation move in circular paths that are centered on the axis and lie in planes that are perpendicular to the axis. For example, the points in the Equatorial Plane move within the Equatorial Plane in circles about the Earth's center.

A rotation of R^3 is an orthogonal operator with a line of fixed points, called the ***axis of rotation***. In this section we will only be concerned with rotations about lines through the origin, and we will assume for simplicity that an angle of rotation is at most 180° (π radians).

If $T: R^3 \rightarrow R^3$ is a rotation through an angle θ about a line through the origin, and if W is the plane through the origin that is perpendicular to the axis of rotation, then T rotates each nonzero vector $\mathbf{w}$ in W about the origin through the angle θ into a vector $T(\mathbf{w})$ in W (Figure 6.2.6a).

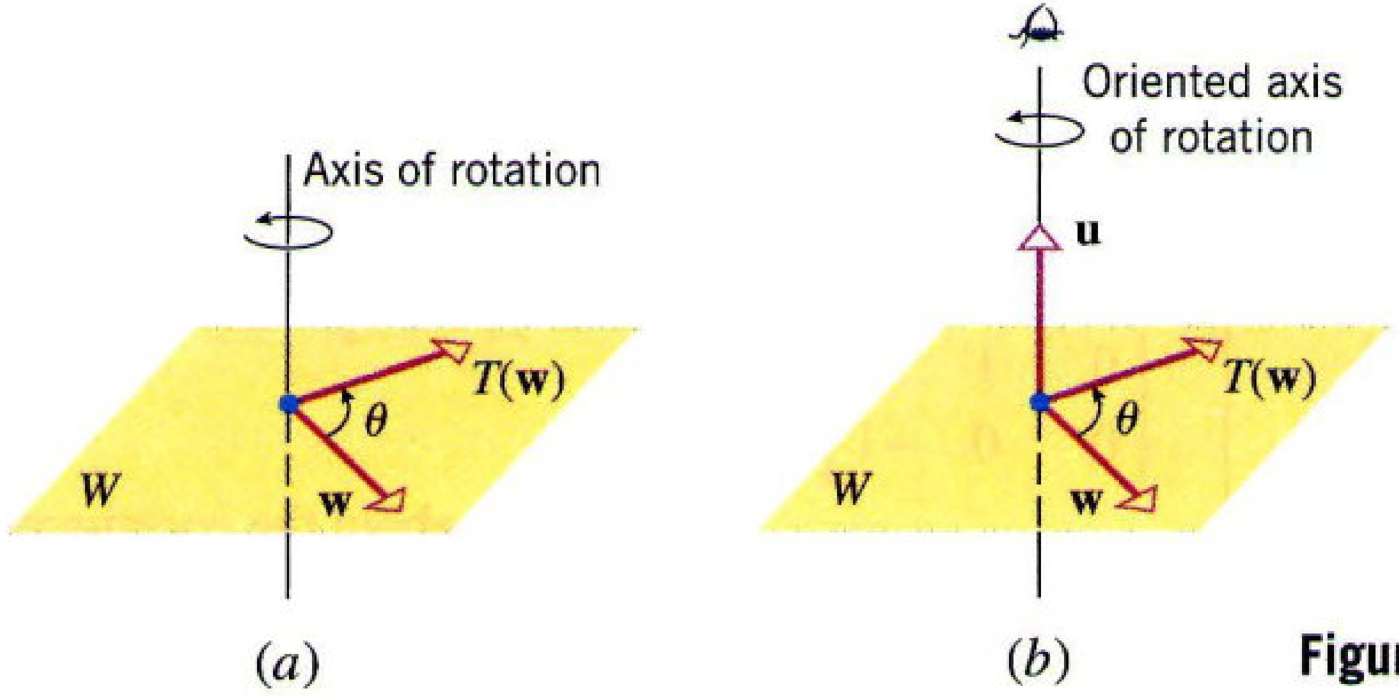


Figure 6.2.6

Thus, within the plane W, the operator T behaves like a rotation of R^2 about the origin. To establish a direction of rotation in W for the angle θ, we need to establish a direction of view along the axis of rotation. We can do this by choosing a nonzero vector **u** on the axis of rotation with its initial point at the origin and agree to view W by looking from the terminal point of **u** toward the origin; we will call **u** an ***orientation*** of the axis of rotation (Figure 6.2.6*b*).

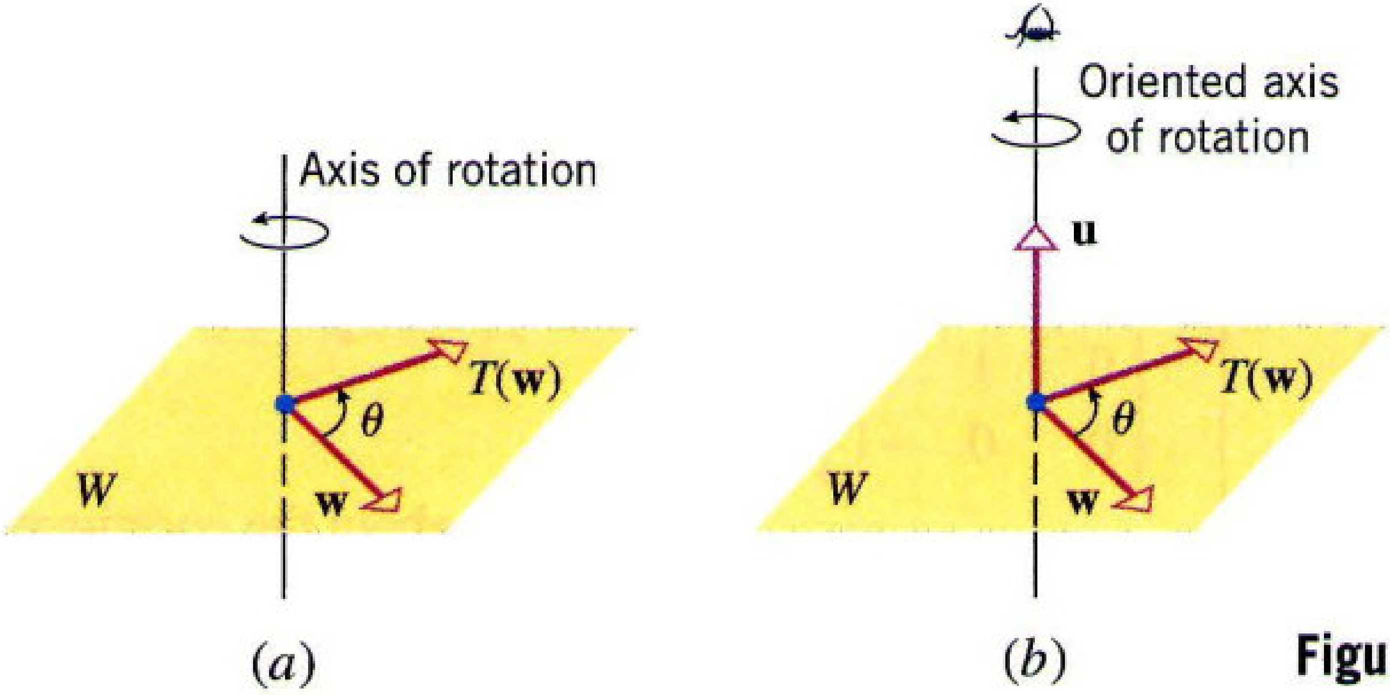


Figure 6.2.6

Now let us see how we might choose the orientation $\mathbf{u}$ so that rotations in the plane W appear counterclockwise when viewed from the terminal point of $\mathbf{u}$. If $\theta \neq 0$ and $\theta \neq \pi$,* then we can accomplish this by taking

$$\mathbf{u} = \mathbf{w} \times T(\mathbf{w}) \tag{12}$$

where $\mathbf{w}$ is any nonzero vector in W. With this choice of $\mathbf{u}$, the right-hand rule holds, and the rotation of $\mathbf{w}$ into $T(\mathbf{w})$ is counterclockwise looking from the terminal point of $\mathbf{u}$ toward the origin (Figure 6.2.7). If we now agree to follow the standard convention of making counterclockwise angles nonnegative, then the angle θ will satisfy the inequalities $0 \leq \theta \leq \pi$.

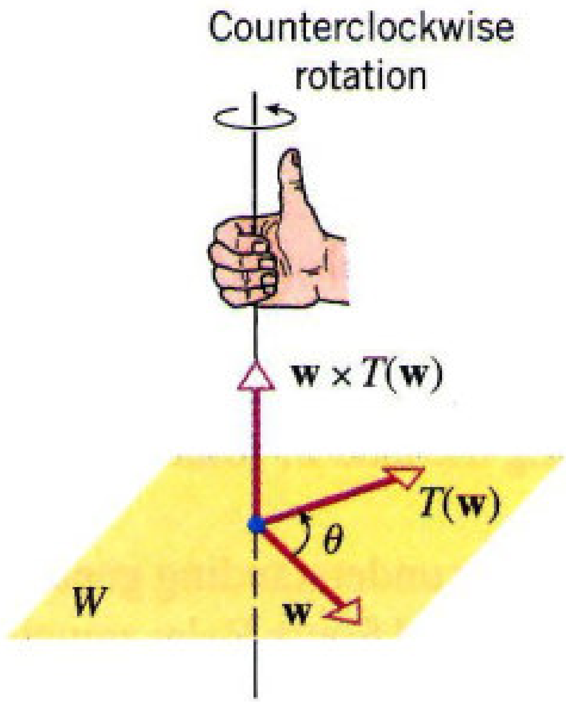
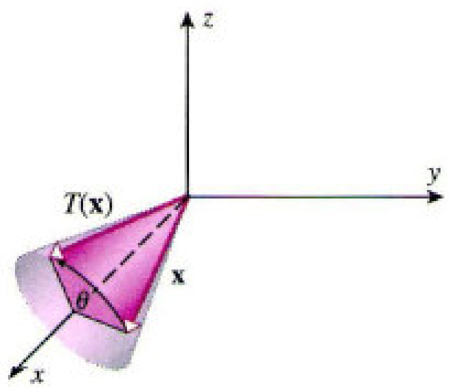
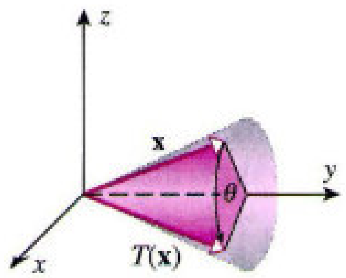
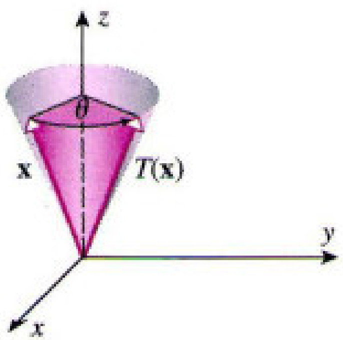


Figure 6.2.7

The most basic rotations in a rectangular xyz-coordinate system are those about the coordinate axes. Table 6.2.6 provides the basic information about these rotations. For each of these rotations, one of the standard unit vectors remains fixed and the images of the other two can be computed by adapting Figure 6.1.8 appropriately.

Table 6.2.6

Operator	Illustration	Standard Matrix
Rotation about the positive x-axis through an angle θ		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Rotation about the positive y-axis through an angle θ		$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Rotation about the positive z-axis through an angle θ		$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For example, in a rotation about the positive y-axis through an angle θ, the vector $\mathbf{e}_2 = (0, 1, 0)$ along the positive y-axis remains fixed, and the vectors $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ undergo rotations through the angle θ in the zx-plane. Thus, if we denote the standard matrix for this rotation by $R_{y,\theta}$, then

$$\mathbf{e}_1 = (1, 0, 0) \xrightarrow{R_{y,\theta}} (\cos\theta, 0, -\sin\theta)$$
$$\mathbf{e}_2 = (0, 1, 0) \xrightarrow{R_{y,\theta}} (0, 1, 0)$$
$$\mathbf{e}_3 = (0, 0, 1) \xrightarrow{R_{y,\theta}} (\sin\theta, 0, \cos\theta) \qquad \text{(see Figure 6.2.8).}$$

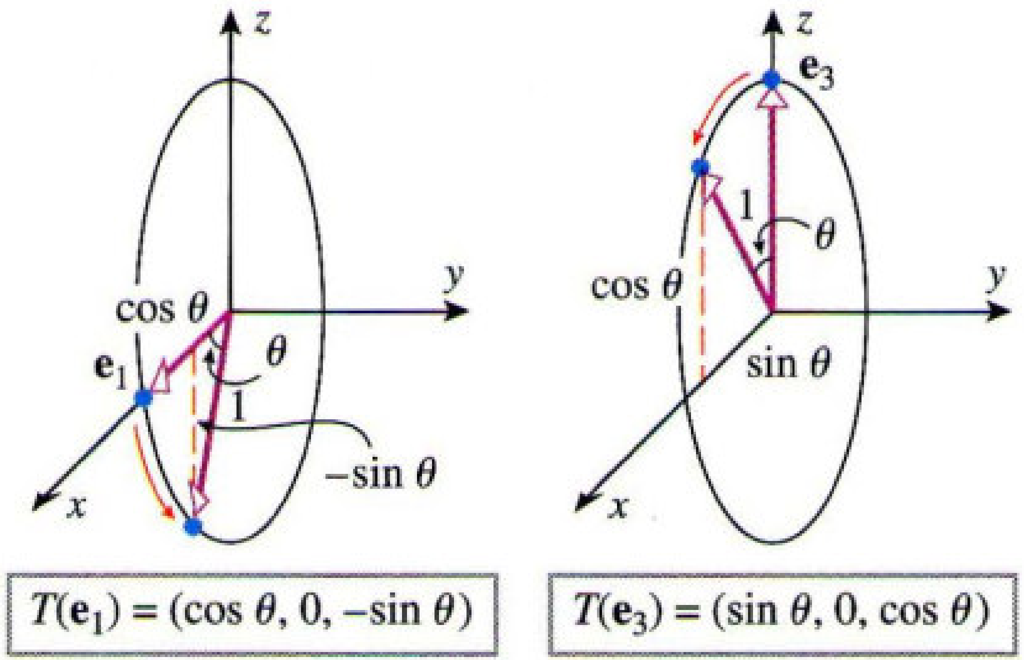


Figure 6.2.8