Section 6.3 Kernel and Range

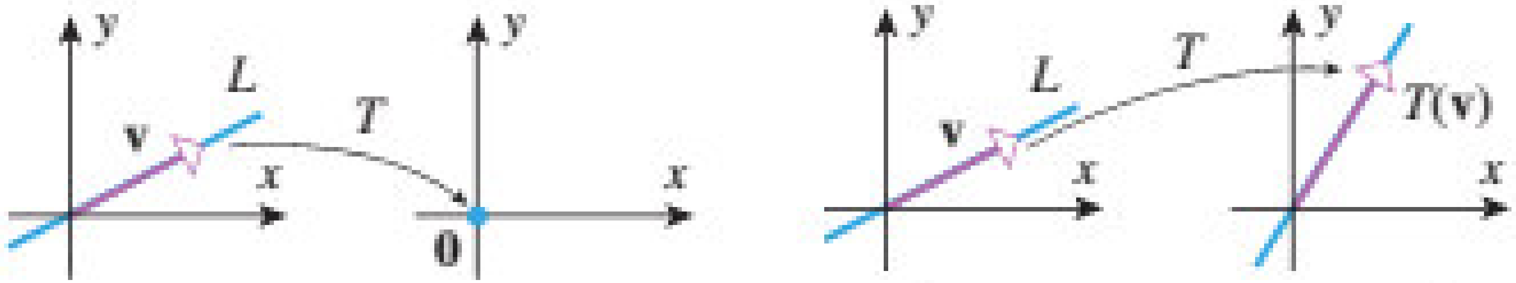
If $\mathbf{x} = t\mathbf{v}$ is a line through the origin of R^n, and if T is a linear operator on R^n, then the image of the line under the transformation T is the set of vectors of the form

$$T(\mathbf{x}) = T(t\mathbf{v}) = tT(\mathbf{v})$$

Geometrically, there are two possibilities for this image:

1. If $T(\mathbf{v}) = \mathbf{0}$, then $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, so the image is the single point $\mathbf{0}$.
2. If $T(\mathbf{v}) \neq \mathbf{0}$, then the image is the line through the origin determined by $T(\mathbf{v})$.

(See Figure 6.3.1.)



T maps L into the point $\mathbf{0}$ if $T(\mathbf{v}) = \mathbf{0}$.

T maps L into the line spanned by $T(\mathbf{v})$ if $T(\mathbf{v}) \neq \mathbf{0}$.

Figure 6.3.1

Similarly, if $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ is a plane through the origin of R^n, then the image of this plane under the transformation T is the set of vectors of the form

$$T(\mathbf{x}) = T(t_1\mathbf{v}_1 + t_2\mathbf{v}_2) = t_1T(\mathbf{v}_1) + t_2T(\mathbf{v}_2)$$

There are three possibilities for this image:

1. If $T(\mathbf{v}_1) = \mathbf{0}$ and $T(\mathbf{v}_2) = \mathbf{0}$, then $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, so the image is the single point $\mathbf{0}$.
2. If $T(\mathbf{v}_1) \neq \mathbf{0}$ and $T(\mathbf{v}_2) \neq \mathbf{0}$, and if $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are not scalar multiples of one another, then the image is a plane through the origin.
3. The image is a line through the origin in the remaining cases.

Definition 6.3.1 If $T: R^n \to R^m$ is a linear transformation, then the set of vectors in R^n that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$.

nuc(T)

EXAMPLE 1 Kernels of Some Basic Operators

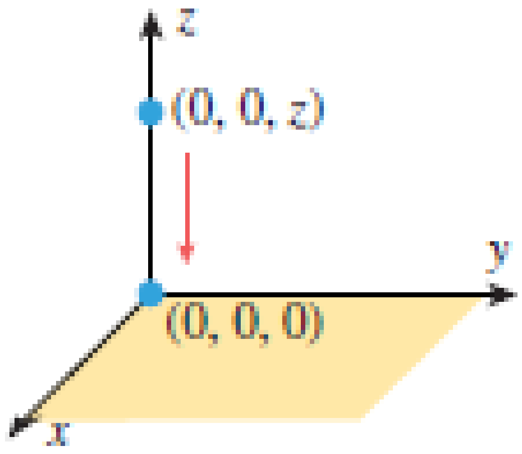


Figure 6.3.2

In each part, find the kernel of the stated linear operator on R^3.

(a) The zero operator $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$.

(b) The identity operator $T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$.

(c) The orthogonal projection T on the xy-plane.

(d) A rotation T about a line through the origin through an angle θ.

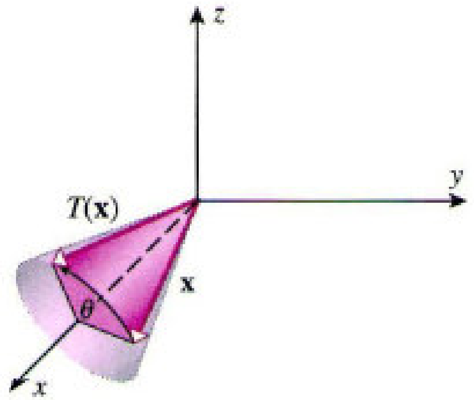
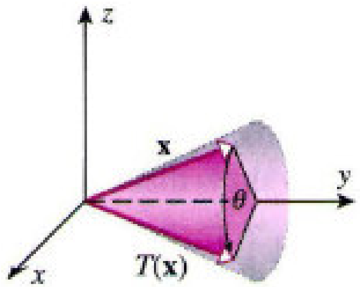
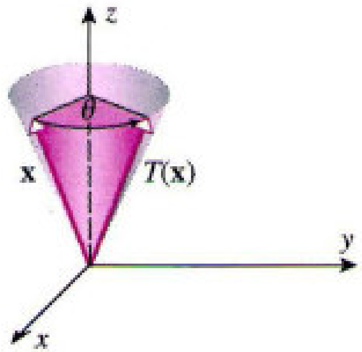
Solution (a) The transformation maps every vector $\mathbf{x}$ into $\mathbf{0}$, so the kernel is all of R^3; that is, $\ker(T_0) = R^3$.

Solution (b) Since $T_I(\mathbf{x}) = \mathbf{x}$, it follows that $T_I(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. This implies that $\ker(T_I) = \{\mathbf{0}\}$.

Solution (c) The orthogonal projection on the xy-plane maps a general point $\mathbf{x} = (x, y, z)$ into $(x, y, 0)$, so the points that get mapped into $\mathbf{0} = (0, 0, 0)$ are those for which $x = 0$ and $y = 0$. Thus, the kernel of the projection T is the z-axis (Figure 6.3.2).

Solution (d) The only vector whose image under the rotation is **0** is the vector **0** itself; that is, the kernel of the rotation T is $\{\mathbf{0}\}$. ■

Table 6.2.6

Operator	Illustration	Standard Matrix
Rotation about the positive x-axis through an angle θ		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Rotation about the positive y-axis through an angle θ		$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Rotation about the positive z-axis through an angle θ		$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

T(**0**) = **0**

It is important to note that the kernel of a linear transformation always contains the vector **0** by Theorem 6.1.3; the following theorem shows that the kernel of a linear transformation is always a subspace.

If a nonempty set of vectors is a subspace, it must be closed under scalar multiplication and addition.

Theorem 6.3.2 *If $T : R^n \to R^m$ is a linear transformation, then the kernel of T is a subspace of R^n.*

Proof The kernel of T is a nonempty set since it contains the zero vector in R^n. To show that it is a subspace of R^n we must show that it is closed under scalar multiplication and addition. For this purpose, let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\ker(T)$, and let c be any scalar. Then

$$T(c\mathbf{v}) = cT(\mathbf{v}) = c\mathbf{0} = \mathbf{0}$$

so $c\mathbf{v}$ is in $\ker(T)$, which shows that $\ker(T)$ is closed under scalar multiplication. Also,

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}+\mathbf{0} = \mathbf{0}$$

so $\mathbf{u}+\mathbf{v}$ is in $\ker(T)$, which shows that $\ker(T)$ is closed under addition. ■

KERNEL OF A MATRIX TRANSFORMATION

If A is an $m \times n$ matrix and $T_A: R^n \to R^m$ is the corresponding linear transformation, then $T_A(\mathbf{x}) = A\mathbf{x}$, so that $\mathbf{x}$ is in the kernel of T_A if and only if $A\mathbf{x} = \mathbf{0}$. Thus, we have the following result.

Theorem 6.3.3 *If A is an $m \times n$ matrix, then the kernel of the corresponding linear transformation is the solution space of $A\mathbf{x} = \mathbf{0}$.*

EXAMPLE 2 Kernel of a Matrix Operator

In part (c) of Example 1 we showed that the kernel of the orthogonal projection of R^3 onto the xy-plane is the z-axis. This can also be deduced from Theorem 6.3.3 by considering the standard matrix for this projection, namely

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is evident from this matrix that a general solution of the system $A\mathbf{x} = \mathbf{0}$ is

$$x = 0, \quad y = 0, \quad z = t$$

which are parametric equations for the z-axis. ■

Definition 6.3.4 If A is an $m \times n$ matrix, then the solution space of the linear system $A\mathbf{x} = \mathbf{0}$, or, equivalently, the kernel of the transformation T_A, is called the ***null space*** of the matrix A and is denoted by null(A).

nuc(A)

EXAMPLE 3 Finding the Null Space of a Matrix

Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

GAUSS–JORDAN AND GAUSSIAN ELIMINATION

We have seen how easy it is to solve a linear system once its augmented matrix is in reduced row echelon form.

use of elementary row operations.

It can be proved that elementary row operations, when applied to the augmented matrix of a linear system, do not change the solution set of the system. Thus, we are assured that the linear system corresponding to the reduced row echelon form will have the same solutions as the original system.

The procedure (or algorithm) for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***. This algorithm consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and then a ***backward phase*** in which zeros are introduced above the leading 1's.

If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***.

EXAMPLE 7

Homogeneous System with Nontrivial Solutions

Use Gauss–Jordan elimination to solve the homogeneous linear system

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0\\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0\\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}
$$

The augmented matrix for the given homogeneous system is

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
2 & 6 & -5 & -2 & 4 & -3 & 0\\
0 & 0 & 5 & 10 & 0 & 15 & 0\\
2 & 6 & 0 & 8 & 4 & 18 & 0
\end{bmatrix}
$$

The reduced row echelon form is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 0 \tag{13}$$

We leave it for you to show that the solution set can be expressed in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0) \tag{14}$$

or alternatively, as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{15}$$

■

EXAMPLE 3 Find the null space of the matrix

Finding the Null Space of a Matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Solution We will solve the problem by producing a set of vectors that spans the subspace. The null space of A is the solution space of $A\mathbf{x} = \mathbf{0}$, so the stated problem boils down to solving this linear system. The computations were performed in Example 7 of Section 2.2, where we showed that the solution space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = -3x_2 - 4x_4 - 2x_5$;
$x_3 = -2x_4$;
$x_6 = 0$.

Thus, $\text{null}(A) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. ■

Theorem 6.3.5 *If $T: R^n \to R^m$ is a linear transformation, then T maps subspaces of R^n into subspaces of R^m.*

Proof Let S be any subspace of R^n, and let $W = T(S)$ be its image under T. We want to show that W is closed under scalar multiplication and addition; so we must show that if $\mathbf{u}$ and $\mathbf{v}$ are any vectors in W, and if c is any scalar, then $c\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are images under T of vectors in S. To find vectors with these images, suppose that $\mathbf{u}$ and $\mathbf{v}$ are the images of the vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ in S, respectively; that is,

$$\mathbf{u} = T(\mathbf{u}_0) \quad \text{and} \quad \mathbf{v} = T(\mathbf{v}_0)$$

Since S is a subspace of R^n, it is closed under scalar multiplication and addition, so $c\mathbf{v}_0$ and $\mathbf{u}_0 + \mathbf{v}_0$ are also vectors in S. These are the vectors we are looking for, since

$$T(c\mathbf{v}_0) = cT(\mathbf{v}_0) = c\mathbf{v} \quad \text{and} \quad T(\mathbf{u}_0+\mathbf{v}_0) = T(\mathbf{u}_0) + T(\mathbf{v}_0) = \mathbf{u}+\mathbf{v}$$

which shows that $c\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are images of vectors in S. ■

→ In other words, vector c**v** is the result of T in c$\mathbf{v}_0$ and **u** + **v** is the result of T in $\mathbf{u}_0 + \mathbf{v}_0$.

RANGE OF A LINEAR TRANSFORMATION

We will now shift our focus from the kernel to the range of a linear transformation. The following definition is a reformulation of Definition 6.1.1 in the context of transformations.

Definition 6.3.6 If $T : R^n \to R^m$ is a linear transformation, then the ***range*** of T, denoted by $\text{ran}(T)$, is the set of all vectors in R^m that are images of at least one vector in R^n. Stated another way, $\text{ran}(T)$ is the image of the domain R^n under the transformation T.

im(T)

EXAMPLE 4 Ranges of Some Basic Operators on R^3

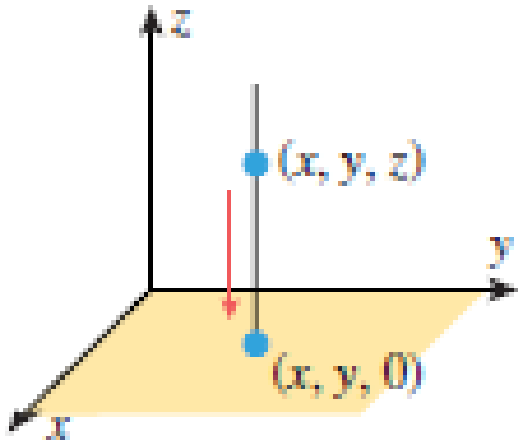


Figure 6.3.3

Describe the ranges of the following linear operators on R^3.

(a) The zero operator $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$.

(b) The identity operator $T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$.

(c) The orthogonal projection T on the xy-plane.

(d) A rotation T about a line through the origin through an angle θ.

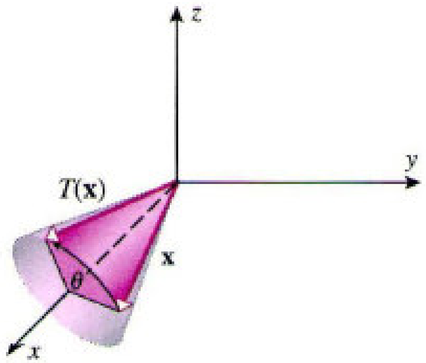
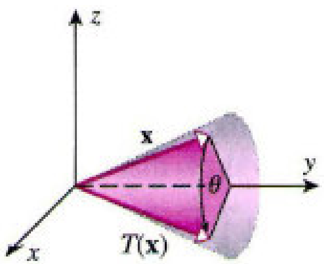
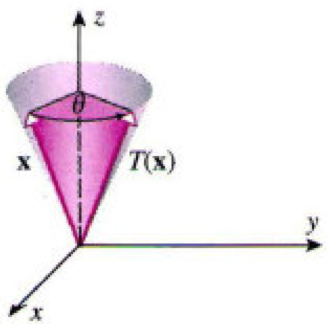
Solution (a) This transformation maps every vector in R^3 into $\mathbf{0}$, so $\text{ran}(T_0) = \{\mathbf{0}\}$.

Solution (b) This transformation maps every vector into itself, so every vector in R^3 is the image of some vector. Thus, $\text{ran}(T_I) = R^3$.

Solution (c) This transformation maps a general point $\mathbf{x} = (x, y, z)$ into $(x, y, 0)$, so the range consists of all points with a z-component of zero. Geometrically, $\text{ran}(T)$ is the xy-plane (Figure 6.3.3).

Solution (d) Every vector in R^3 is the image of some vector under the rotation T. For example, to find a vector whose image is $\mathbf{x}$, rotate $\mathbf{x}$ about the line through the angle $-\theta$ to obtain a vector $\mathbf{w}$; the image of $\mathbf{w}$, when rotated through the angle θ, will be $\mathbf{x}$. Thus, $\operatorname{ran}(T) = R^3$. ■

Table 6.2.6

Operator	Illustration	Standard Matrix
Rotation about the positive x-axis through an angle θ		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Rotation about the positive y-axis through an angle θ		$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Rotation about the positive z-axis through an angle θ		$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 6.3.7 *If $T : R^n \to R^m$ is a linear transformation, then* $\operatorname{ran}(T)$ *is a subspace of R^m.*

→ special case of Theorem 6.3.5.

RANGE OF A MATRIX TRANSFORMATION

If A is an $m \times n$ matrix and $T_A : R^n \to R^m$ is the corresponding linear transformation, then $T_A(\mathbf{x}) = A\mathbf{x}$, so that a vector $\mathbf{b}$ in R^m is in the range of T_A if and only if there is a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. Stated another way, $\mathbf{b}$ is in the range of T_A if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

Theorem 6.3.8 *If A is an $m \times n$ matrix, then the range of the corresponding linear transformation is the column space of A.*

If $T_A : R^n \to R^m$ is the linear transformation corresponding to the matrix A, then the range of T_A and the column space of A are the same object from different points of view—the first emphasizes the transformation and the second the matrix.

EXAMPLE 5

Range of a Matrix Operator

In part (c) of Example 4 we showed that the range of the orthogonal projection of R^3 onto the xy-plane is the xy-plane. This can also be deduced from Theorem 6.3.8 by considering the standard matrix for this projection, namely

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The range of the projection is the column space of A, which consists of all vectors of the form

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Thus, the range of the projection, in comma-delimited notation, is the set of points of the form $(x, y, 0)$, which is the xy-plane. ■

It is important in many kinds of problems to be able to determine whether a given vector $\mathbf{b}$ in R^m is in the range of a linear transformation $T: R^n \to R^m$. If A is the standard matrix for T, then this problem reduces to determining whether $\mathbf{b}$ is in the column space of A.

EXAMPLE 6 Column Space

Suppose that

$$A = \begin{bmatrix} 1 & -8 & -7 & -4 \\ 2 & -3 & -1 & 5 \\ 3 & 2 & 5 & 14 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 8 \\ -10 \\ -28 \end{bmatrix}$$

Determine whether $\mathbf{b}$ is in the column space of A, and, if so, express it as a linear combination of the column vectors of A.

Solution The problem can be solved by determining whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If the answer is "yes," then $\mathbf{b}$ is in the column space, and the components of any solution $\mathbf{x}$ can be used as coefficients for the desired linear combination; if the answer is "no," then $\mathbf{b}$ is not in the column space of A. We leave it for you to confirm that the reduced row echelon form of the augmented matrix for the system is

$$\begin{bmatrix} 1 & 0 & 1 & 4 & -8 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 4 & -8 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -8 - x_3 - 4x_4$;
$x_2 = -2 - x_3 - x_4$;

$x_3 = s$; $x_4 = t$.

$$A = \begin{bmatrix} 1 & -8 & -7 & -4 \\ 2 & -3 & -1 & 5 \\ 3 & 2 & 5 & 14 \end{bmatrix}$$

We can see from this matrix that the system is consistent, and we leave it for you to show that a general solution is

$$x_1 = -8 - s - 4t, \quad x_2 = -2 - s - t, \quad x_3 = s, \quad x_4 = t$$

Since the parameters s and t are arbitrary, there are infinitely many ways to express **b** as a linear combination of the column vectors of A.

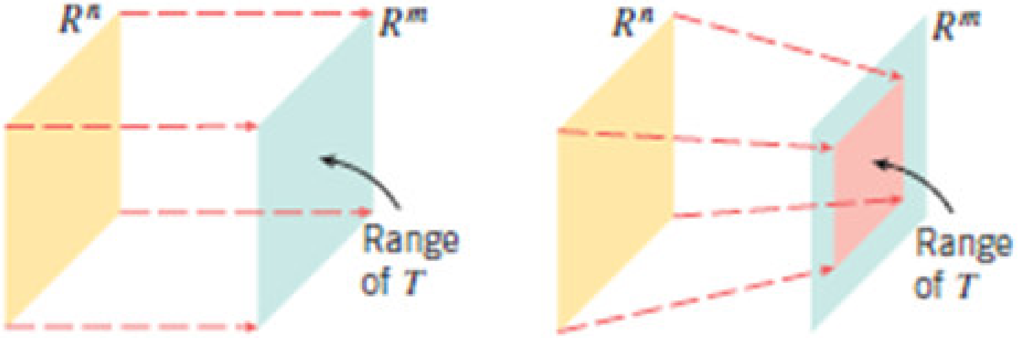
A particularly simple way is to take $s = 0$ and $t = 0$, in which case we obtain $x_1 = -8$, $x_2 = -2$, $x_3 = 0$, $x_4 = 0$. This yields the linear combination

$$\mathbf{b} = \begin{bmatrix} 8 \\ -10 \\ -28 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -7 \\ -1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 5 \\ 14 \end{bmatrix}$$

You may find it instructive to express **b** as a linear combination of the column vectors of A in some other ways by choosing different values for the parameters s and t. ■

There are many problems in which one is concerned with the following questions about a linear transformation $T : R^n \to R^m$:

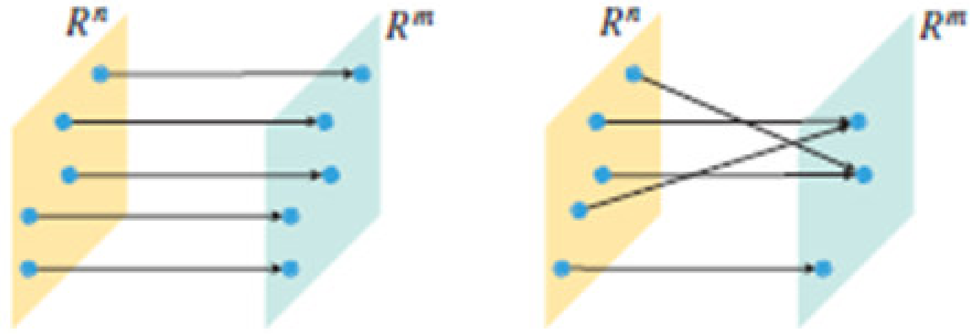
- **The Existence Question**—Is every vector in R^m the image of at least one vector in R^n; that is, is the range of T all of R^m? (See the schematic diagram in Figure 6.3.4.)
- **The Uniqueness Question**—Can two different vectors in R^n have the same image in R^m? (See the schematic diagram in Figure 6.3.5.)



The range is R^m, so every vector in R^m is the image of at least one vector in R^n.

The range is not all of R^m, so there are vectors in R^m that are not images of any vectors in R^n.

Figure 6.3.4



Distinct vectors in R^n have distinct images in R^m.

There are distinct vectors in R^n that have the same image in R^m.

Figure 6.3.5

Definition 6.3.9 A transformation $T : R^n \to R^m$ is said to be ***onto*** if its range is the entire codomain R^m; that is, every vector in R^m is the image of at least one vector in R^n.

Definition 6.3.10 A transformation $T : R^n \to R^m$ is said to be ***one-to-one*** (sometimes written 1–1) if T maps distinct vectors in R^n into distinct vectors in R^m.

→ onto: “sobrejetora”; one-to-one: “injetora”; both: “bijetora”.

EXAMPLE 7 One-to-One and Onto

Let $T : R^2 \to R^2$ be the operator that rotates each vector in the xy-plane about the origin through an angle θ. This operator is one-to-one because rotating distinct vectors through the same angle produces distinct vectors; it is onto because any vector $\mathbf{x}$ in R^2 is the image of some vector $\mathbf{w}$ under the rotation (rotate $\mathbf{x}$ through the angle $-\theta$ to obtain $\mathbf{w}$). ■

EXAMPLE 8 Neither One-to-One nor Onto

Let $T : R^3 \to R^3$ be the orthogonal projection on the xy-plane. This operator is not one-to-one because distinct points on any vertical line map into the same point in the xy-plane; it is not onto because its range (the xy-plane) is not all of R^3. ■

EXAMPLE 9 One-to-One but Not Onto

Let $T : R^2 \rightarrow R^3$ be the linear transformation defined by the formula $T(x, y) = (x, y, 0)$. To show that this linear transformation is one-to-one, consider the images of two points $\mathbf{x}_1 = (x_1, y_1)$ and $\mathbf{x}_2 = (x_2, y_2)$. If $T(\mathbf{x}_1) = T(\mathbf{x}_2)$, then $(x_1, y_1, 0) = (x_2, y_2, 0)$, which implies that $x_1 = x_2$ and $y_1 = y_2$. Thus if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$, which means T maps distinct vectors into distinct vectors. The transformation is not onto because its range is not all of R^3. For example, there is no vector in R^2 that maps into $(0, 0, 1)$. ■

Above, it is $x_1 = x_2$ and $y_1 = y_2$
(it is incorrect in the textbook).

EXAMPLE 10 Onto but Not One-to-One

Let $T : R^3 \rightarrow R^2$ be the linear transformation defined by the formula $T(x, y, z) = (x, y)$. This transformation is onto because each vector $\mathbf{w} = (x, y)$ in R^2 is the image of at least one vector in R^3; in fact, it is the image of any vector $\mathbf{x} = (x, y, z)$ whose first two components are the same as those of $\mathbf{w}$. The transformation is not one-to-one because two distinct vectors of the form $\mathbf{x}_1 = (x, y, z_1)$ and $\mathbf{x}_2 = (x, y, z_2)$ map into the same point (x, y). ■

Theorem 6.3.11 *If $T : R^n \rightarrow R^m$ is a linear transformation, then the following statements are equivalent.*

(*a*) *T is one-to-one.*

(*b*) $\ker(T) = \{\mathbf{0}\}$.

***Proof* (*a*) ⇒ (*b*)** Assume that T is one-to-one. Since T is linear, we know that $T(\mathbf{0}) = \mathbf{0}$ by Theorem 6.1.3. The fact that T is one-to-one implies that $\mathbf{x} = \mathbf{0}$ is the only vector for which $T(\mathbf{x}) = \mathbf{0}$, so $\ker(T) = \{\mathbf{0}\}$.

***Proof* (*b*) ⇒ (*a*)** Assume that $\ker(T) = \{\mathbf{0}\}$. To prove that T is one-to-one we will show that if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$. But if $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{x}_1 - \mathbf{x}_2 \neq \mathbf{0}$, which means that $\mathbf{x}_1 - \mathbf{x}_2$ is not in $\ker(T)$. This being the case,

$$T(\mathbf{x}_1 - \mathbf{x}_2) = T(\mathbf{x}_1) - T(\mathbf{x}_2) \neq \mathbf{0}$$

Thus, $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$. ■

ONE-TO-ONE AND ONTO FROM THE VIEWPOINT OF LINEAR SYSTEMS

If A is an $m \times n$ matrix and $T_A: R^n \rightarrow R^m$ is the corresponding linear transformation, then $T_A(\mathbf{x}) = A\mathbf{x}$. Thus, to say that $\ker(T_A) = \{\mathbf{0}\}$ (i.e., that T_A is one-to-one) is the same as saying that the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Also, to say that T_A is onto is the same as saying that for each vector $\mathbf{b}$ in R^m there is at least one vector $\mathbf{x}$ in R^n such that $A\mathbf{x} = \mathbf{b}$. This establishes the following theorems.

Theorem 6.3.12 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A: R^n \rightarrow R^m$ is one-to-one if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Theorem 6.3.13 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A: R^n \rightarrow R^m$ is onto if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in R^n.*

Above, it is R^m, not R^n .

EXAMPLE 11 Mapping "Bigger" Spaces into "Smaller" Spaces

Let $T : R^n \to R^m$ be a linear transformation, and suppose that $n > m$.

If A is the standard matrix for T, then the linear system $A\mathbf{x} = \mathbf{0}$ has more unknowns than equations and hence has nontrivial solutions. Accordingly, it follows from Theorem 6.3.12 that T is not one-to-one, and hence we have shown that if a matrix transformation maps a space R^n of higher dimension into a space R^m of smaller dimension, then there must be distinct points in R^n that map into the same point in R^m.

For example, the linear transformation

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_3)$$

maps the higher-dimensional space R^3 into the lower-dimensional space R^2, so you can tell without any computation that T is not one-to-one. ■

We observed earlier in this section that a linear transformation $T : R^n \to R^m$ can be one-to-one and not onto or can be onto and not one-to-one (Examples 9 and 10). The next theorem shows that in the special case where T is a *linear operator*, the two properties go hand in hand—both hold or neither holds.

Theorem 6.3.14 *If $T : R^n \rightarrow R^n$ is a linear operator on R^n, then T is one-to-one if and only if it is onto.*

Proof Let A be the standard matrix for T. By parts (d) and (e) of Theorem 4.4.7, the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n. Combining this with Theorems 6.3.12 and 6.3.13 completes the proof. ■

Alternative proof:

(a)If T is one-to-one, for $\mathbf{x}_1 \neq \mathbf{x}_2$, $T(\mathbf{x}_1) \neq T(\mathbf{x}_2)$. As T is a linear operator mapping the entire domain, any vector in the domain will correspond to a distinct vector in the codomain, covering it entirely. That is, range and codomain are the same. Therefore, T is onto.

(b)If T is onto, range and codomain are the same. So, there is a corresponding vector in the domain for every vector in the codomain. But two distinct vectors in the range are not connected to the same vector in the domain, as T is a function, and a linear operator. Therefore, T is one-to-one.

EXAMPLE 12 Examples 7 and 8 Revisited

We saw in Examples 7 and 8 that a rotation about the origin of R^2 is both one-to-one and onto and that the orthogonal projection on the xy-plane in R^3 is neither one-to-one nor onto. The "both" and "neither" are consistent with Theorem 6.3.14, since the rotation and the projection are both linear *operators*. ■

EXAMPLE 13 Examples 7 and 8 Revisited Using Determinants

The fact that a rotation about the origin R^2 is one-to-one and onto can be established algebraically by showing that the determinant of its standard matrix is not zero. This can be confirmed using Formula (16) of Section 6.1 to obtain

$$\det(R_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

The fact that the orthogonal projection of R^3 on the xy-plane is neither one-to-one nor onto can be established by showing that the determinant of its standard matrix A is zero. This is, in fact, the case, since

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$ ■

If det(A) ≠ 0, A is invertible and Ax = 0 has only the trivial solution. Then T_A is one-to-one, by theorem 6.3.12.

If T is linear operator and one-to-one, it is also onto, by theorem 6.3.14.

In Theorem 4.4.7 we tied together most of the major concepts developed at that point in the text. Theorems 6.3.12, 6.3.13, and 6.3.14 now enable us to add two more results to that theorem.

Theorem 4.4.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* R^n.

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

(*i*) $\det(A) \neq 0$.

(*j*) $\lambda = 0$ *is not an eigenvalue of A.*

A UNIFYING THEOREM

Theorem 6.3.15 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.*

(g) *The column vectors of A are linearly independent.*

(h) *The row vectors of A are linearly independent.*

(i) $\det(A) \neq 0$.

(j) *$\lambda = 0$ is not an eigenvalue of A.*

(k) *T_A is one-to-one.*

(l) *T_A is onto.*