CHAPTER 7

Notions of dimension and structure in *n*-space make it possible to visualize and interpret data using familiar geometric ideas. Virtually all applications of linear algebra use these ideas in some way.

Dimension and Structure

Section 7.1 Basis and Dimension

If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ is a subspace of R^n, and if the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ are linearly dependent, then at least one of the vectors in S can be deleted, and the remaining vectors will still span V.

For example, suppose that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set. The linear dependence of S implies that at least one vector in that set is a linear combination of the others, say

$$\mathbf{v}_3 = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \tag{1}$$

Thus, every vector $\mathbf{w}$ in V can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ alone by first writing it as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, say

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

and then substituting (1) to obtain

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = (c_1 + c_3k_1)\mathbf{v}_1 + (c_2 + c_3k_2)\mathbf{v}_2$$

This discussion suggests that spanning sets of linearly independent vectors are special in that they do not contain superfluous vectors.

Definition 7.1.1 A set of vectors in a subspace V of R^n is said to be a ***basis*** for V if it is linearly independent and spans V.

EXAMPLE 1 Some Simple Bases

- If V is a line through the origin of R^n, then any nonzero vector on the line forms a basis for V.
- If V is a plane through the origin of R^n, then any two nonzero vectors in the plane that are not scalar multiples of one another form a basis for V.
- If $V = \{\mathbf{0}\}$ is the zero subspace of R^n, then V has no basis since it does not contain any linearly independent vectors.*

EXAMPLE 2 The Standard Basis for R^n

The standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent, for if we write

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = \mathbf{0} \tag{2}$$

in component form, then we obtain $(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0)$, which implies that all of the coefficients in (2) are 0.

Furthermore, these vectors span R^n because an arbitrary vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in R^n can be expressed as

$$\mathbf{x} = x_1(1, 0, \ldots, 0) + x_2(0, 1, \ldots, 0) + \cdots + x_n(0, 0, \ldots, 1) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

We call $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ the ***standard basis*** for R^n. ■

Theorem 7.1.2 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of two or more nonzero vectors in R^n, then S is linearly dependent if and only if some vector in S is a linear combination of its predecessors.*

Proof If some vector in S is a linear combination of predecessors in S, then the linear dependence of S follows from Theorem 3.4.6.

Conversely, assume that S is a linearly dependent set. This implies that there exist scalars, not all zero, such that

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0} \tag{3}$$

so let t_j be that nonzero scalar that has the largest index. Since this implies that all terms in (3) beyond the jth (if any) are zero, we can rewrite this equation as

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_j\mathbf{v}_j = \mathbf{0}$$

Since $t_j \neq 0$, we can multiply this equation through by $1/t_j$ and solve for $\mathbf{v}_j$ as a linear combination of its predecessors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$, which completes the proof. ■

Theorem 3.4.6 *A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ in R^n with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.*

EXAMPLE 3 Linear Independence Using Theorem 7.1.2

Show that the vectors

$$\mathbf{v}_1 = (0, 2, 0), \quad \mathbf{v}_2 = (3, 0, 3), \quad \mathbf{v}_3 = (-4, 0, 4)$$

are linearly independent by showing that no vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combination of predecessors.

Solution The vector $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$ and hence is not a linear combination of $\mathbf{v}_1$. The vector $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, for if there were a relationship of the form $\mathbf{v}_3 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$, then we would have to have $t_1 = 0$ in order to produce the zero second coefficient of $\mathbf{v}_3$. This would then imply that $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_2$, which it is not. Thus, no vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combination of predecessors. ■

EXAMPLE 4 Independence of Nonzero Row Vectors in a Row Echelon Form

The nonzero row vectors of a matrix in row echelon form are linearly independent. To visualize why this is true, consider the following typical matrices in row echelon form, where the $*$'s denote arbitrary real numbers:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

If we list the nonzero row vectors of such matrices in the order $\mathbf{v}_1, \mathbf{v}_2, \ldots,$ *starting at the bottom and working up*, then no row vector in the list can be expressed as a linear combination of predecessors in the list because there is no way to produce its leading 1 by such a linear combination. The linear independence of the nonzero row vectors now follows from Theorem 7.1.2. ■

→ According to Theorem 7.1.2, if there is no linear combination, S is linearly independent.

Theorem 7.1.3 **(*Existence of a Basis*)** *If V is a nonzero subspace of R^n, then there exists a basis for V that has at most n vectors.*

Proof Let V be a nonzero subspace of R^n. We will give a procedure for constructing a set in V with at most n vectors that spans V and in which no vector is a linear combination of predecessors (and hence is linearly independent). Here is the construction:

- Let $\mathbf{v}_1$ be any nonzero vector in V. If $V = \text{span}\{\mathbf{v}_1\}$, then we have our linearly independent spanning set.
- If $V \neq \text{span}\{\mathbf{v}_1\}$, then choose any vector $\mathbf{v}_2$ in V that is not a linear combination of $\mathbf{v}_1$ (i.e., is not a scalar multiple of $\mathbf{v}_1$). If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then we have our linearly independent spanning set.
- If $V \neq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then choose any vector $\mathbf{v}_3$ in V that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. If $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then we have our linearly independent spanning set. If $V \neq \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then choose any $\mathbf{v}_4$ in V that is not a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and repeat the process in the preceding steps.
- If we continue this construction process, then there are two logical possibilities: At some stage we will produce a linearly independent set that spans V or, if not, we will encounter a linearly independent set of $n + 1$ vectors. But the latter is impossible since a linearly independent set in R^n can contain at most n vectors (Theorem 3.4.8). ■

Theorem 3.4.8 *A set with more than n vectors in R^n is linearly dependent.*

Theorem 7.1.4 *All bases for a nonzero subspace of R^n have the same number of vectors.*

Proof Let V be a nonzero subspace of R^n, and suppose that the sets $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ and $B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are bases for V. Our goal is to prove that $k = m$, which we will do by assuming that $k \neq m$ and obtaining a contradiction. Since the cases $k < m$ and $m < k$ differ only in notation, it will be sufficient to give the proof in the case where $k < m$.

Since B_1 spans V, and since the vectors in B_2 are in V, each $\mathbf{w}_i$ in B_2 can be expressed as a linear combination of the vectors in B_1, say

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{k1}\mathbf{v}_k \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{k2}\mathbf{v}_k \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \\
\mathbf{w}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{km}\mathbf{v}_k
\end{aligned} \tag{4}$$

Now consider the homogeneous linear system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{km} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

of k equations in the m unknowns $c_1, c_2, \ldots, c_m$. Since $k < m$, this system has more unknowns than equations and hence has a nontrivial solution (Theorem 2.2.3).

This implies that there exist numbers $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{bmatrix} + \cdots + c_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{km} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Theorem 2.2.3 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

or equivalently,

$$\begin{aligned} c_1a_{11} + c_2a_{12} + \cdots + c_ma_{1m} &= 0 \\ c_1a_{21} + c_2a_{22} + \cdots + c_ma_{2m} &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\ c_1a_{k1} + c_2a_{k2} + \cdots + c_ma_{km} &= 0 \end{aligned} \tag{5}$$

To complete the proof, we will show that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_m\mathbf{w}_m = \mathbf{0} \tag{6}$$

which will contradict the linear independence of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$. For this purpose, we first use (4) to rewrite the left side of (6) as

$$\begin{aligned} c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_m\mathbf{w}_m = \\ & c_1(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{k1}\mathbf{v}_k) \\ & \quad + c_2(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{k2}\mathbf{v}_k) \\ & \qquad \ddots \\ & \qquad\quad + c_m(a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{km}\mathbf{v}_k) \end{aligned} \tag{7}$$

Next we multiply out on the right side of (7) and regroup the terms to form a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. The resulting coefficients in this linear combination match up with the expressions on the left side of (5) (verify), so it follows from (5) that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_m\mathbf{w}_m = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$$

which is the contradiction we were looking for. ■

EXAMPLE 5 Number of Vectors in a Basis

- Every basis for a line through the origin of R^n has one vector.
- Every basis for a plane through the origin of R^n has two vectors.
- Every basis for R^n has n vectors (since the standard basis has n vectors). ■

Definition 7.1.5 If V is a nonzero subspace of R^n, then the ***dimension*** of V, written $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the zero subspace to have dimension 0.

EXAMPLE 6 Dimensions of Subspaces of R^n

It follows from Definition 7.1.5 and Example 5 that:

- A line through the origin of R^n has dimension 1.
- A plane through the origin of R^n has dimension 2.
- R^n has dimension n. ■

DIMENSION OF A SOLUTION SPACE

We stated that the general solution of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ that results from Gauss–Jordan elimination is of the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s \tag{8}$$

in which the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are linearly independent. We will call these vectors the ***canonical solutions*** of $A\mathbf{x} = \mathbf{0}$.

Note that these are the solutions that result from (8) by setting one of the parameters to 1 and the others to zero. Since the canonical solution vectors span the solution space and are linearly independent, they form a basis for the solution space; we call that basis the ***canonical basis*** for the solution space.

Theorem 2.2.2 (***Dimension Theorem for Homogeneous Systems***) *If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.*

EXAMPLE 7 Basis and Dimension of the Solution Space of $A\mathbf{x} = \mathbf{0}$

Find the canonical basis for the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

$x_1 = -3x_2 - 4x_4 - 2x_5$;
$x_3 = -2x_4$;
$x_6 = 0$.

$x_2 = r$, $x_4 = s$, $x_5 = t$.

and state the dimension of that solution space.

Solution We showed in Example 7 of Section 2.2 that the general solution produced by Gauss–Jordan elimination is

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

Thus, the canonical basis vectors are

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

and the solution space is a three-dimensional subspace of R^6. ■

DIMENSION OF A HYPERPLANE

Recall from Section 3.5 that if $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a nonzero vector in R^n, then the hyperplane $\mathbf{a}^\perp$ through the origin of R^n is given by the equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Let us view this as a linear system of one equation in n unknowns. Since this system has one leading variable and $n-1$ free variables, its solution space has dimension $n-1$, and this implies that $\dim(\mathbf{a}^\perp) = n-1$. For example, hyperplanes through the origin of R^2 (lines) have dimension 1, and hyperplanes through the origin of R^3 (planes) have dimension 2.

Theorem 7.1.6 *If* $\mathbf{a}$ *is a nonzero vector in* R^n, *then* $\dim(\mathbf{a}^\perp) = n - 1$.

Section 7.2 Properties of Bases

PROPERTIES OF BASES

In absence of restrictive conditions, there will generally be many ways to express a vector in span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ as a linear combination of the spanning vectors. For example, let us consider how we might express the vector $\mathbf{v} = (3, 4, 5)$ as a linear combination of the vectors

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (0, 1, 0), \quad \mathbf{v}_3 = (0, 0, 1), \quad \mathbf{v}_4 = (1, 1, 1)$$

One obvious possibility is to discount the presence of $\mathbf{v}_4$ and write

$$(3, 4, 5) = 3\mathbf{v}_1 + 4\mathbf{v}_2 + 5\mathbf{v}_3 + 0\mathbf{v}_4$$

Other ways can be discovered by expressing the vectors in column form and writing the vector equation

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \tag{1}$$

as the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Solving this system yields (verify)

$$c_1 = 3 - t, \quad c_2 = 4 - t, \quad c_3 = 5 - t, \quad c_4 = t$$

so substituting in (1) and writing the vectors in component form yields

$$(3, 4, 5) = (3 - t)(1, 0, 0) + (4 - t)(0, 1, 0) + (5 - t)(0, 0, 1) + t(1, 1, 1)$$

Thus, for example, taking $t = 1$ yields

$$(3, 4, 5) = 2(1, 0, 0) + 3(0, 1, 0) + 4(0, 0, 1) + (1, 1, 1)$$

and taking $t = -1$ yields

$$(3, 4, 5) = 4(1, 0, 0) + 5(0, 1, 0) + 6(0, 0, 1) - (1, 1, 1)$$

The following theorem shows that it was the linear dependence of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ that made it possible to express $\mathbf{v}$ as a linear combination of these vectors in more than one way.

Theorem 7.2.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for subspace V of R^n, then every vector $\mathbf{v}$ in V can be expressed in exactly one way as a linear combination of the vectors in S.*

Proof Let $\mathbf{v}$ be any vector in V. Since S spans V, there is at least one way to express $\mathbf{v}$ as a linear combination of the vectors in S. To see that there is exactly one way to do this, suppose that

$$\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k \quad \text{and} \quad \mathbf{v} = t_1'\mathbf{v}_1 + t_2'\mathbf{v}_2 + \cdots + t_k'\mathbf{v}_k \tag{2}$$

Subtracting the second equation from the first yields

$$\mathbf{0} = (t_1 - t_1')\mathbf{v}_1 + (t_2 - t_2')\mathbf{v}_2 + \cdots + (t_k - t_k')\mathbf{v}_k$$

Since the right side of this equation is a linear combination of the vectors in S, and since these vectors are linearly independent, each of the coefficients in the linear combination must be zero. Thus, the two linear combinations in (2) are the same. ■

The following theorem reveals two important facts about bases:

1. Every spanning set for a subspace is either a basis for that subspace or has a basis as a subset.
2. Every linearly independent set in a subspace is either a basis for the subspace or can be extended to a basis for the subspace.

Theorem 7.2.2 *Let S be a finite set of vectors in a nonzero subspace V of R^n.*

(*a*) *If S spans V, but is not a basis for V, then a basis for V can be obtained by removing appropriate vectors from S.*

(*b*) *If S is a linearly independent set, but is not a basis for V, then a basis for V can be obtained by adding appropriate vectors from V to S.*

Proof (a) If S spans V but is not a basis for V, then S must be a linearly dependent set. This means that some vector $\mathbf{v}$ in S is a linear combination of predecessors. Remove this vector from S to obtain a set S'.

The set S' must still span V, since any linear combination of the vectors in S can be rewritten as a linear combination of the vectors in S' by expressing $\mathbf{v}$ in terms of its predecessors. If S' is linearly independent, then it is a basis for V, and we are done.

If S' is not linearly independent, then some vector in S' can be expressed as a linear combination of predecessors. Remove this vector from S' to obtain a set S''. As before, this new set will still span V. If S'' is linearly independent, then it is a basis for V and we are done; otherwise, we continue the process of removing vectors until we reach a basis.

Proof (b) If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$ is a linearly independent set of vectors in V but is not a basis for V, then S does not span V. Thus, there is some vector $\mathbf{v}_1$ in V that is not a linear combination of the vectors in S. Add this vector to S to obtain the set $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s, \mathbf{v}_1\}$. This set must still be linearly independent since none of the vectors in S' can be linear combinations of predecessors. If S' spans V, then it is a basis for V, and we are done.

If S' does not span V, then there is some vector $\mathbf{v}_2$ in V that is not a linear combination of the vectors in S'. Add this vector to S' to obtain the set $S'' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s, \mathbf{v}_1, \mathbf{v}_2\}$. As before, this set will still be linearly independent. If S'' spans V, then it is a basis and we are done; otherwise we continue the process until we reach a basis or a linearly independent set with n vectors.

But in the latter case the set also has to be a basis for V; otherwise, it would not span V, and the procedure we have been following would allow us to add another vector to the set and create a linearly independent set with $n + 1$ vectors—an impossibility by Theorem 3.4.8. Thus, the procedure eventually produces a basis in all cases. ■

→ If a linearly independent set with n vectors is reached and as V is a subspace of R^n,
it must span V at that stage.

$\boxed{\textbf{Theorem 3.4.8}\ \textit{A set with more than } n \textit{ vectors in } R^n \textit{ is linearly dependent.}}$

By definition, the dimension of a nonzero subspace V of R^n is the number of vectors in a basis for V; however, the dimension can also be viewed as the maximum number of linearly independent vectors in V, for if $\dim(V) = k$, and if we could produce more than k linearly independent vectors, then by part (*b*) of Theorem 7.2.2, that set of vectors would either have to be a basis for V or part of a basis for V, contradicting the fact that all bases for V have k vectors.

Theorem 7.2.3 *If V is a nonzero subspace of R^n, then* $\dim(V)$ *is the maximum number of linearly independent vectors in V.*

REMARK Engineers use the term ***degrees of freedom*** as a synonym for dimension, the idea being that a space with k degrees of freedom allows freedom of motion or variation in at most k independent directions.

SUBSPACES OF SUBSPACES

Up to now we have focused on subspaces of R^n. However, if V and W are subspaces of R^n, and if V is a subset of W, then we also say that V is a ***subspace*** of W. For example, in Figure 7.2.1 the space $\{\mathbf{0}\}$ is a subspace of the line, which in turn is a subspace of the plane, which in turn is a subspace of R^3.

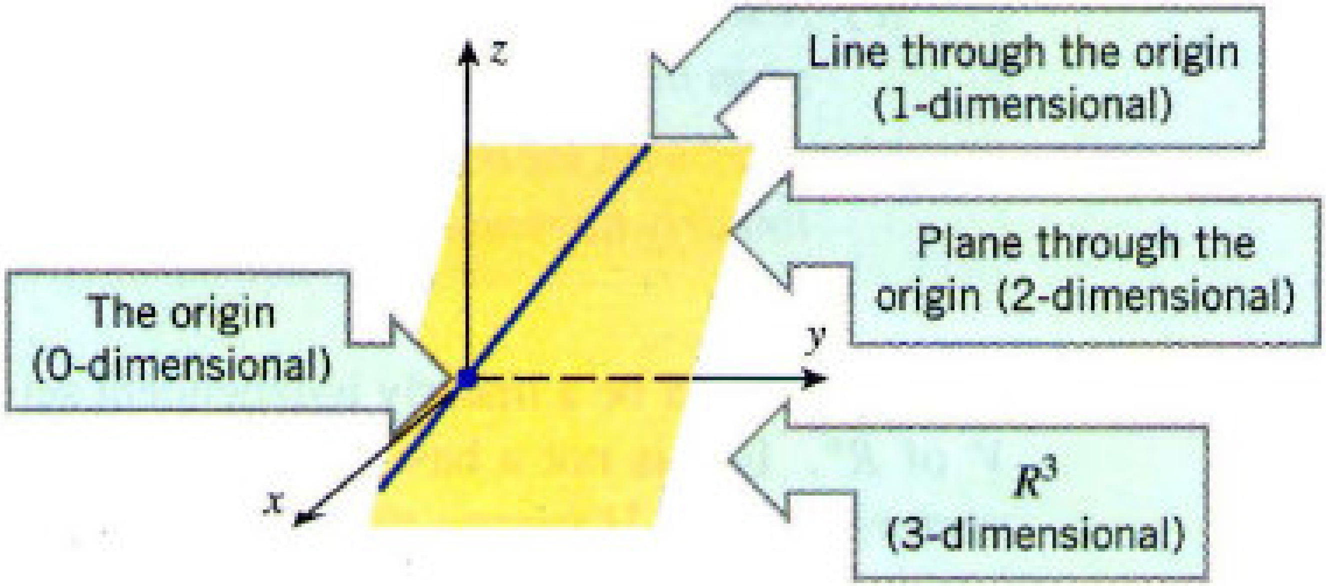

Figure 7.2.1

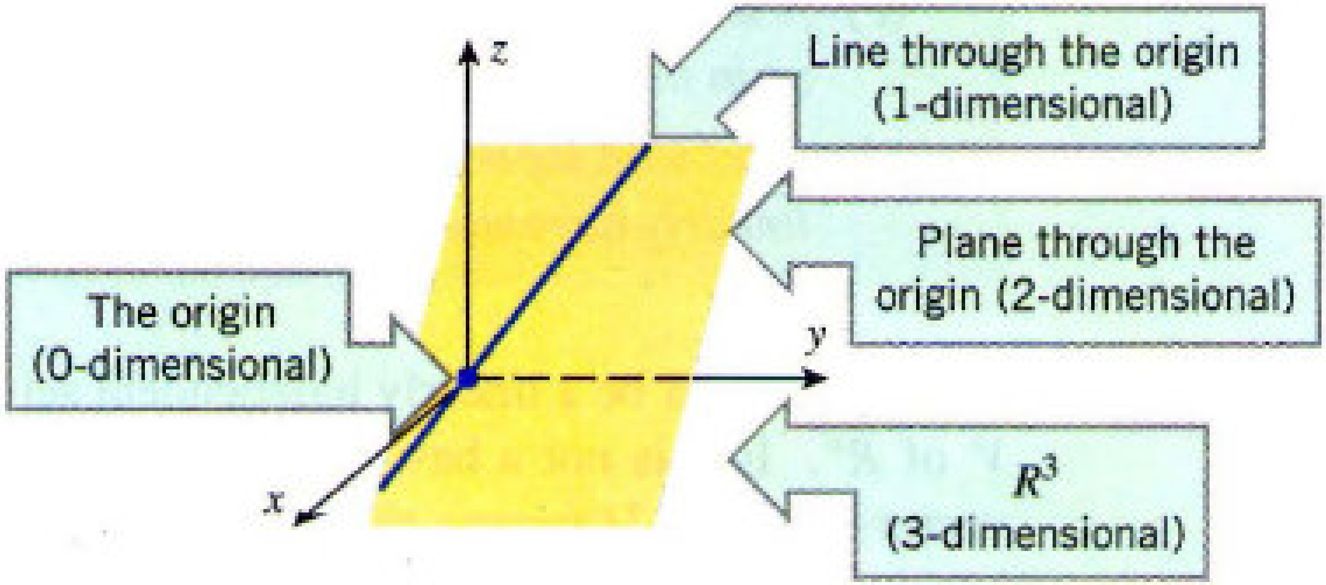

Figure 7.2.1

Since the dimension of a subspace of R^n is the maximum number of linearly independent vectors that the subspace can have, it follows that if V is a subspace of W, then the dimension of V cannot exceed the dimension of W. In particular, the dimension of a subspace of R^n can be at most n, just as you would suspect. Further, if V is a subspace of W, and if the two spaces have the same dimension, then they must be the same space (Exercise P8).

Theorem 7.2.4 *If V and W are subspaces of R^n, and if V is a subspace of W, then:*

(a) $0 \leq \dim(V) \leq \dim(W) \leq n$

(b) $V = W$ *if and only if* $\dim(V) = \dim(W)$

→ For part (b) of theorem 7.2.4, consider theorem 7.2.2, particularly its part (b).

There will be occasions on which we are given some nonempty set S, and we will be interested in knowing how the subspace $\text{span}(S)$ is affected by adding additional vectors to S. The following theorem deals with this question.

Theorem 7.2.5 *Let S be a nonempty set of vectors in R^n, and let S' be a set that results by adding additional vectors in R^n to S.*

(*a*) *If the additional vectors are in* $\text{span}(S)$, *then* $\text{span}(S') = \text{span}(S)$.

(*b*) *If* $\text{span}(S') = \text{span}(S)$, *then the additional vectors are in* $\text{span}(S)$.

(*c*) *If* $\text{span}(S')$ *and* $\text{span}(S)$ *have the same dimension, then the additional vectors are in* $\text{span}(S)$ *and* $\text{span}(S') = \text{span}(S)$.

We will not formally prove this theorem, but its statements should almost be self-evident. For example, part (a) tells you that if you add vectors to S that are already linear combinations of vectors in S, then you are not going to add anything new to the set of all possible linear combinations of vectors in S.

Part (b) tells you that if the additional vectors do not add anything new to the set of all linear combinations of vectors in S, then the additional vectors must already be linear combinations of vectors in S.

Finally, in part (c) the fact that S is a subset of S' means that $\text{span}(S)$ is a subspace of $\text{span}(S')$ (why?). Thus, if the two spaces have the same dimension, then they must be the same space by Theorem 7.2.4(b); and this means that the additional vectors must be in $\text{span}(S)$.

Theorem 7.2.6

(*a*) *A set of k linearly independent vectors in a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(*b*) *A set of k vectors that span a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(*c*) *A set of fewer than k vectors in a nonzero k-dimensional subspace of R^n cannot span that subspace.*

(*d*) *A set with more than k vectors in a nonzero k-dimensional subspace of R^n is linearly dependent.*

***Proof* (*a*)** Let S be a linearly independent set of k vectors in a nonzero k-dimensional subspace V of R^n. If S is not a basis for V, then S can be extended to a basis by adding appropriate vectors from V. However, this would produce a basis for V with more than k vectors, which is impossible. Thus, S must be a basis.

***Proof* (*b*)** Let S be a set of k vectors in a nonzero k-dimensional subspace V of R^n that span V. If S is not a basis for V, then it can be pared down to a basis for V by removing appropriate vectors. However, this would produce a basis for V with fewer than k vectors, which is impossible. Thus, S must be a basis.

Theorem 7.2.6 (cont.)

(*a*) *A set of k linearly independent vectors in a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(*b*) *A set of k vectors that span a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(*c*) *A set of fewer than k vectors in a nonzero k-dimensional subspace of R^n cannot span that subspace.*

(*d*) *A set with more than k vectors in a nonzero k-dimensional subspace of R^n is linearly dependent.*

Proof (c) Let S be a set with fewer than k vectors that spans a nonzero k-dimensional subspace V of R^n. Then either S is a basis for V or can be made into a basis for V by removing appropriate vectors. In either case we have a basis for V with fewer than k vectors, which is impossible.

Proof (d) Let S be a linearly independent set with more than k vectors from a nonzero k-dimensional subspace of R^n. Then either S is a basis for V or can be made into a basis for V by adding appropriate vectors. In either case we have a basis for V with more than k vectors, which is impossible. ■

EXAMPLE 1 Bases by Inspection

(a) Show that the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2 by inspection.

(b) Show that $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (6, 1, -5)$ form a basis for R^3 by inspection.

Solution (a) We have two vectors in a two-dimensional space, so it suffices to show that the vectors are linearly independent. However, this is obvious, since neither vector is a scalar multiple of the other.

Solution (b) We have three vectors in a three-dimensional space, so again it suffices to show that the vectors are linearly independent. We can do this by showing that none of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is a linear combination of predecessors.

But the vector $\mathbf{v}_2$ is not a linear combination of $\mathbf{v}_1$, since it is not a scalar multiple of $\mathbf{v}_1$, and the vector $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since any such linear combination has a second component of zero, and $\mathbf{v}_3$ does not. ■

EXAMPLE 2 A Determinant Test for Linear Independence

(a) Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (1, -1, 3)$, and $\mathbf{v}_3 = (1, 1, 4)$ form a basis for R^3.

(b) Express $\mathbf{w} = (4, 9, 8)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Solution (a) We have three vectors in a three-dimensional space, so it suffices to show that the vectors are linearly independent. One way to do this is to form the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

that has $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as its column vectors and apply Theorem 6.3.15. The determinant of the matrix A is nonzero [verify that $\det(A) = -7$], so parts (i) and (g) of that theorem imply that the column vectors are linearly independent.

Theorem 6.3.15 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(g) *The column vectors of A are linearly independent.*

(h) *The row vectors of A are linearly independent.*

(i) $\det(A) \neq 0$.

Solution (b) The result in part (a) guarantees that $\mathbf{w}$ can be expressed as a unique linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, but the method used in that part does not tell us what that linear combination is. To find it, we rewrite the vector equation

$$(4, 9, 8) = c_1(1, 2, 1) + c_2(1, -1, 3) + c_3(1, 1, 4) \tag{3}$$

as the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 8 \end{bmatrix}$$

This system has the unique solution $c_1 = 3$, $c_2 = -1$, $c_3 = 2$ (verify), and substituting these values in (3) yields

$$(4, 9, 8) = 3(1, 2, 1) - (1, -1, 3) + 2(1, 1, 4)$$

which expresses $\mathbf{w}$ as the linear combination $\mathbf{w} = 3\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3$. ■

A UNIFYING THEOREM

By combining Theorem 7.2.6 with parts (*g*) and (*h*) of Theorem 6.3.15, we can add four more statements to the latter.

Theorem 7.2.7 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(*e*) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.*

(*f*) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.*

(*g*) $\det(A) \neq 0$.

(*h*) *$\lambda = 0$ is not an eigenvalue of A.*

(*i*) *T_A is one-to-one.*

(*j*) *T_A is onto.*

(*k*) *The column vectors of A are linearly independent.*

(*l*) *The row vectors of A are linearly independent.*

(*m*) *The column vectors of A span R^n.*

(*n*) *The row vectors of A span R^n.*

(*o*) *The column vectors of A form a basis for R^n.*

(*p*) *The row vectors of A form a basis for R^n.*