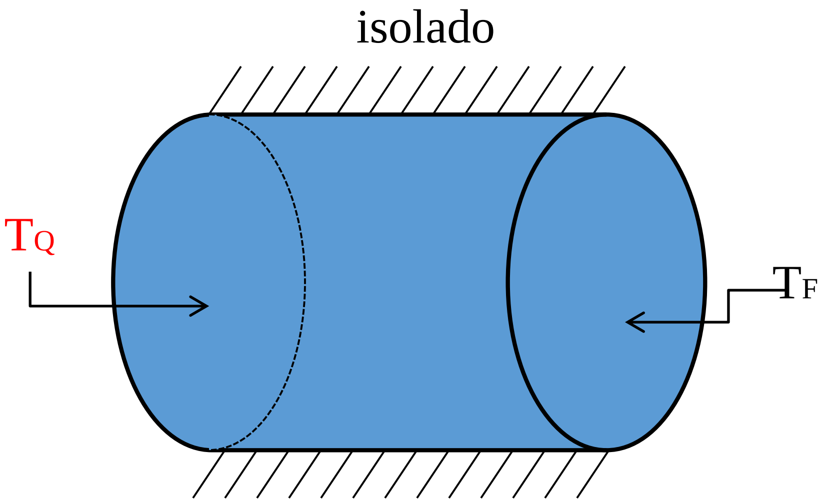
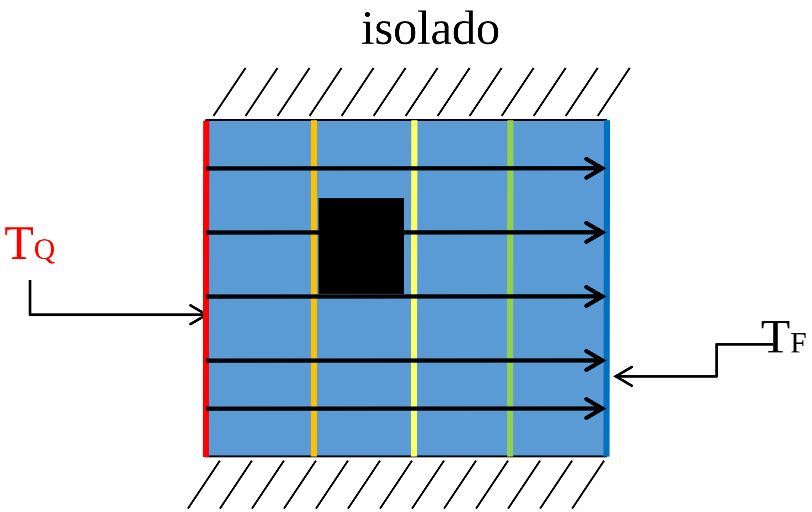
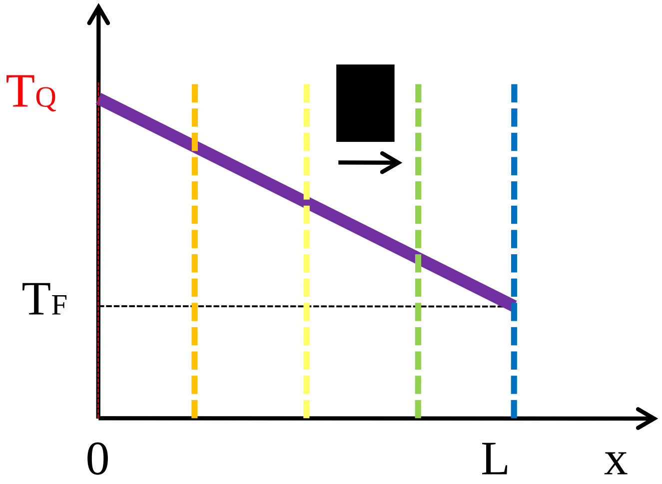
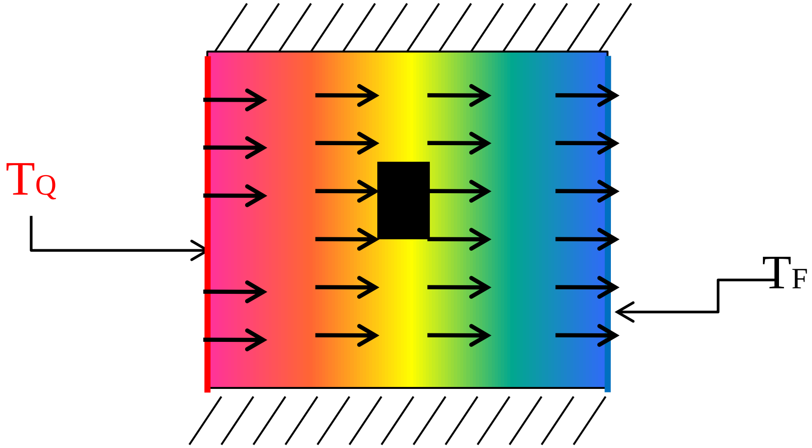
Condução Bidimensional em Regime Permanente

Recapitulação: Isotermas e Isofluxos – 1D





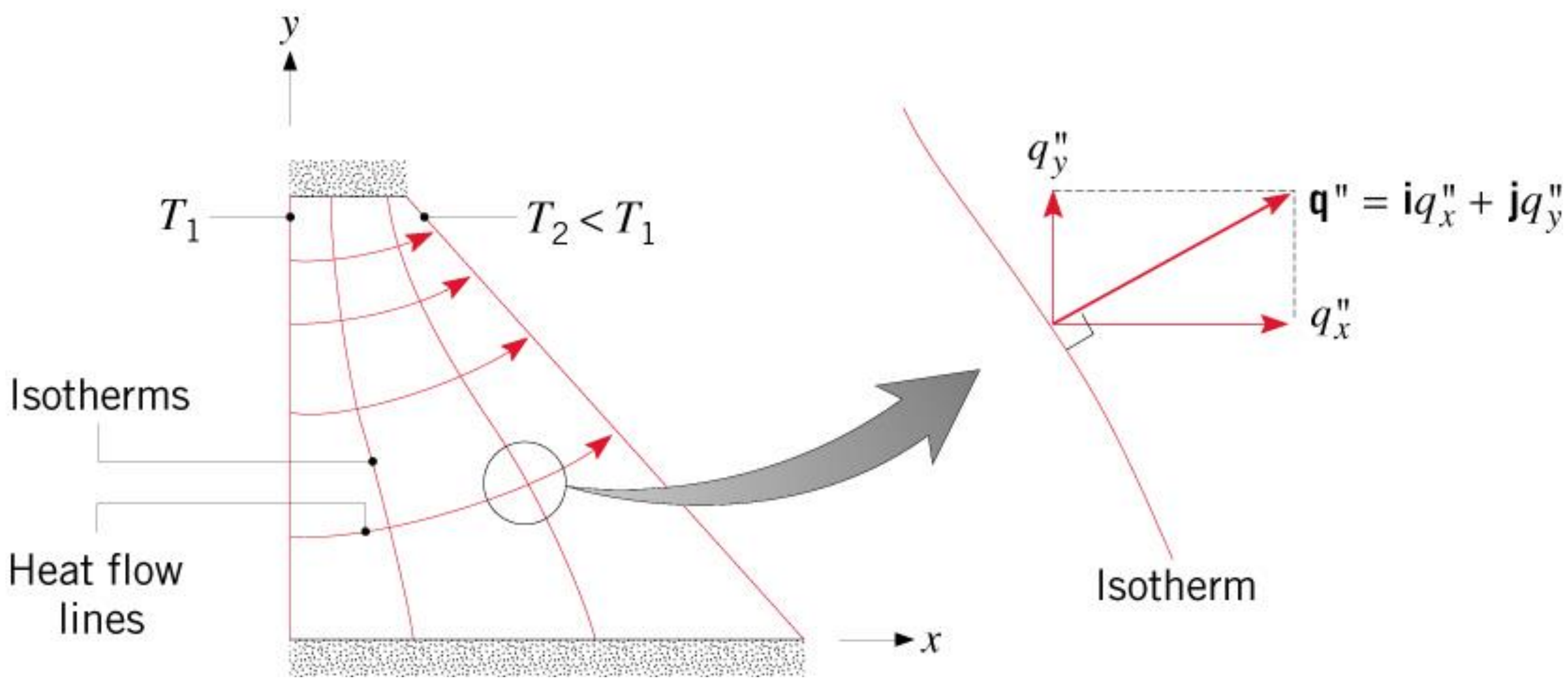




Introdução

-
  - $T\,(x,y)$
  - q''_x , q''_y

•Isotermas e isofluxos



Métodos de Solução da Equação da Difusão de Calor

- Considerando regime permanente com propriedades constantes e com geração em coord. cartesianas

- Métodos de Solução:
  - Exata/Analítica: Método de Separação de Variáveis (possível transformada de Laplace ou Fourier
    - ➢ Limitado para problemas mais simples
  - Approximado/Gráfico $\left(\dot{q} = 0\right)$
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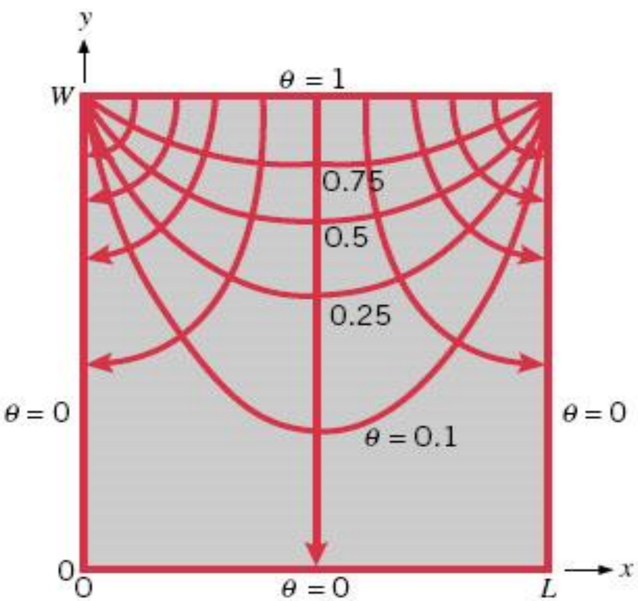
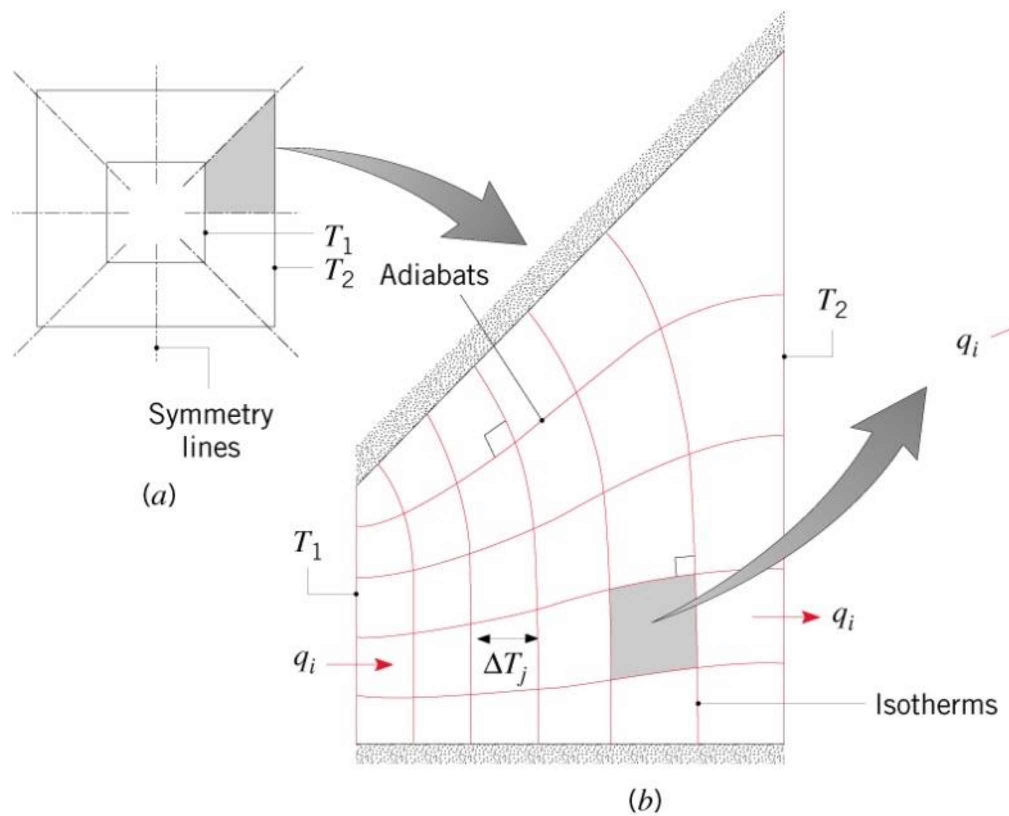
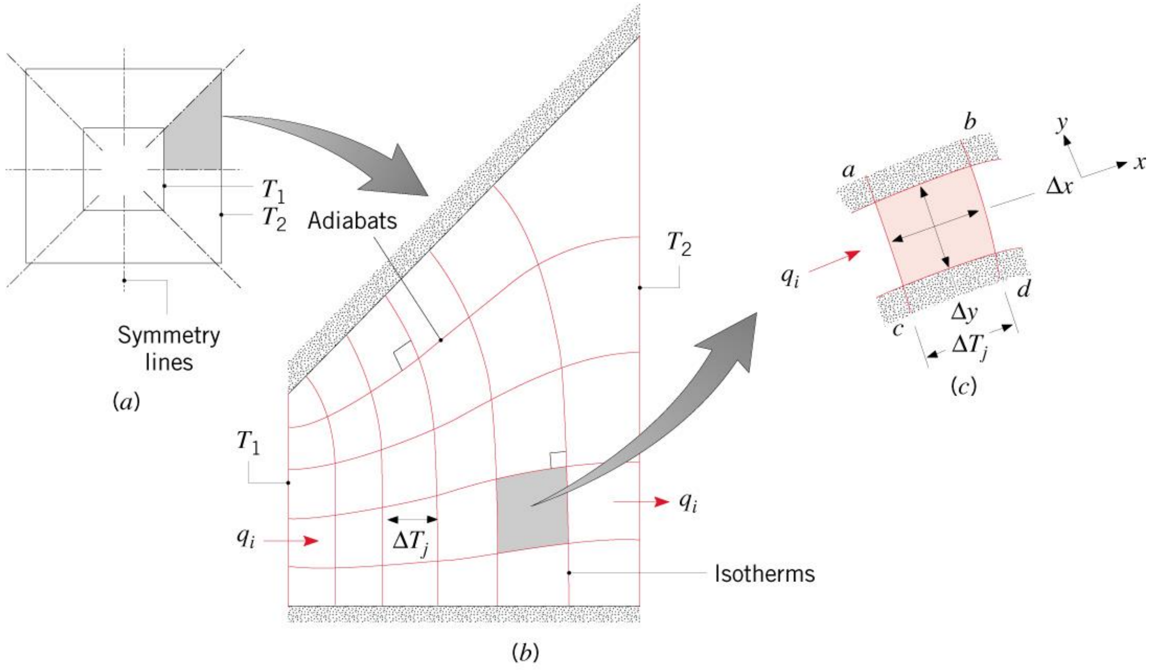
Condições de Simetria para problemas gráficos e numéricos





Método Gráfico



– Determinação da taxa transferida de calor:

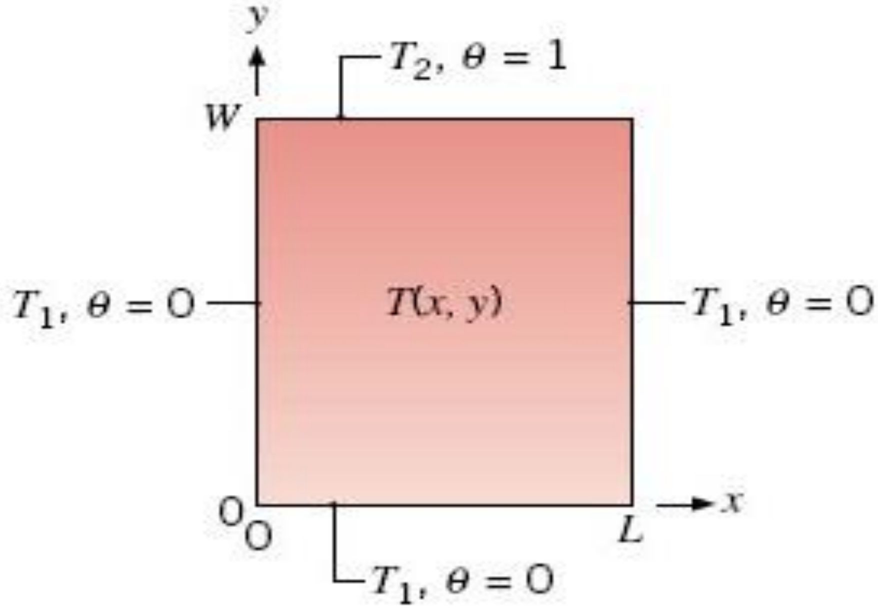
$$q \approx Mq_i \approx M\left[k\left(\Delta y \cdot \ell\right)\frac{\Delta T_j}{\Delta x}\right] \approx \frac{M\ell}{N}k\Delta T_{1-2}$$

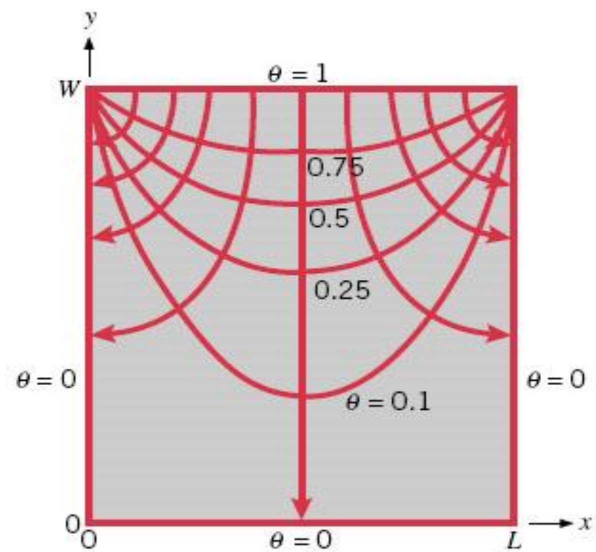
$$q' \approx \frac{M}{N}k\Delta T_{1-2} \qquad (4\ S.5)$$

M é o número de isofluxos e N o número de isotermas

Método Analítico

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0; \quad \theta=\frac{T-T_1}{T_2-T_1}; \quad \frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0$$





Condições de contorno

$$\theta(0,y)=0 \qquad \theta(x,0)=0$$
$$\theta(L,y)=0 \qquad \theta(x,W)=1$$

MÉTODO DE SEPARAÇÃO DE VARIÁVEIS

$$\theta(x, y) = X(x).Y(y)$$

Substituindo

$$X''.Y + Y''.X = 0 \qquad \div XY$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y}$$

MÉTODO DE SEPARAÇÃO DE VARIÁVEIS

$$\theta(x, y) = X(x).Y(y)$$

Substituindo

$$X''.Y + Y''.X = 0 \qquad \div XY$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda^2$$

Autovalor

$$-\frac{X''}{X}=\frac{Y''}{Y}=\lambda^2$$

Resultando

$$\begin{cases} Y''-\lambda^2 Y=0 & (I) \\ X''+\lambda^2 X=0 & (II) \end{cases}$$

De (I)

$$Y''-\lambda^2 Y=0$$

$$\left(D^2-\lambda^2\right)Y=0$$

$$\left(D-\lambda\right)\left(D+\lambda\right)Y=0$$

$$Y(y)=c_1 e^{\lambda y}+c_2 e^{-\lambda y}$$

De (II) $X''+\lambda^2 X=0$

$$\left(D^2+\lambda^2\right)X=0 \quad ; m\sqrt{-\lambda^2}=\pm\lambda\, i$$

$$\alpha_1 \ \ e \ \ \alpha_2 \ \ são \ \ complexos$$

$$\alpha_1=\lambda\, i \ ; \ \alpha_2=-\lambda\, i$$

$$X(x)=d_1e^{\alpha_1 x}+d_2e^{\alpha_2 x}=d_1e^{(a+bi)x}+d_2e^{(a-bi)x}$$

$$X(x)=e^{ax}\left(d_1e^{bix}+d_2e^{-bix}\right);$$

$$fórmula\ de\ Euler(Moivre):\ e^{\pm ix}=\cos x\pm i\, sen\, x$$

$$X(x)=e^{ax}\left(d_1\left(\cos bx+i\, sen\, bx\right)+d_2\left(\cos bx-i\, sen\, bx\right)\right)$$

$$X(x)=e^{ax}\left(\left(d_1+d_2\right)\cos bx+i\left(d_1-d_2\right)sen\, bx\right)$$

$$X(x)=e^{ax}\left(c_3\cos bx+c_4 sen\, bx\right) \quad ; como\ a=0$$

$$X(x)=c_3\cos\lambda x+c_4 sen\,\lambda x$$

Da solução Geral:

$$\theta(x,y) = X(x).Y(y)$$

$$\theta(x,y) = (c_3 \cos\lambda x + c_4 sen\, \lambda x)\left(c_1 e^{\lambda y} + c_2 e^{-\lambda y}\right)$$

Aplicando as condições de contorno:

$$\theta(0,y) = 0 \;\rightarrow\; X(0) = c_3 \cos(\lambda.0) + c_4\, sen(\lambda.0) \therefore\; c_3 = 0$$

$$\theta(x,0) = 0 \;\rightarrow\; Y(0) = c_1 + c_2 = 0 \rightarrow c_1 = -c_2$$

$$\theta(L,y) = 0 \;\rightarrow\; X(L) = 0 = c_4\, sen(\lambda.L);\; como\;\; c_4 \neq 0$$

$$sen(\lambda.L) = 0$$

$$então\;\; \lambda.L = n\pi;\; n = 1,2,3,\ldots,\infty$$

$$\lambda_n = \frac{n\pi}{L}\;\; (autovalores)$$

$$c_3 = 0$$

$$c_1 = -c_2$$

$$\lambda_n = \frac{n\pi}{L} \quad (autovalores)$$

Da solução Geral:

$$\theta(x,y) = X(x).Y(y)$$

$$\theta_n(x,y) = (c_1 c_4)_n (sen\,\lambda_n x)\left(e^{\lambda_n y} - e^{-\lambda_n y}\right)$$

$$\theta_n(x,y) = c'_n\,(sen\,\lambda_n x)\left(e^{\lambda_n y} - e^{-\lambda_n y}\right)$$

como $\left(e^{\lambda_n y} - e^{-\lambda_n y}\right) = 2\,senh(\lambda_n y)$

$$\theta_n(x,y) = c_n (sen\,\lambda_n x)(senh\,\lambda_n y)$$

Tem-se um infinito número de soluções que satisfazem a equação diferencial e as condições de contorno sendo cada solução linearmente independente. Como o problema é linear a solução geral será a superposição de todas estas soluções na forma:

$$\theta(x,y)=\sum_{n=1}^{\infty}c_n\left(sen\,\lambda_n x\right)\left(senh\,\lambda_n y\right)$$

Para determinar-se a constante c_n, aplica-se a última condição de contorno,

$$\theta(x,W)=1$$

$$\theta(x,W)=1=\sum_{n=1}^{\infty}c_n\left(sen\,\lambda_n x\right)\left(senh\,\lambda_n W\right)$$

Utilizando as propriedades de funções ortogonais:

$$\int_a^b g_m(x) g_n(x) dx = 0 \quad p/ \quad m \neq n$$

Multiplicando de ambos os lados da equação da pag. anterior

$$\int_0^L 1.sen\,\lambda_m x dx = \int_0^L \sum_{n=1}^{\infty} c_n (sen\,\lambda_n x)(senh\,\lambda_n W) sen\,\lambda_m x dx$$

$$c_n = \frac{\int_0^L 1.sen\,\lambda_m x dx}{\int_0^L \sum_{n=1}^{\infty} (sen\,\lambda_n x)^2 (senh\,\lambda_n W) dx} = \frac{\int_0^L 1.sen\,\lambda_m x dx}{(senh\,\lambda_n W) \int_0^L \sum_{n=1}^{\infty} (sen\,\lambda_n x)^2 dx}$$

$$c_n = \frac{2\left[(-1)^{n+1} + 1\right]}{(senh\,\lambda_n W) n\pi}, \; n = 1,2,\ldots,\infty$$

Solução geral

$$\theta(x,y)=\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}(sen\,\lambda_n x)\frac{(senh\,\lambda_n y)}{(senh\,\lambda_n W)}$$

Métodos de Solução da Equação da Difusão de Calor

- Considerando regime permanente com propriedades constantes e com geração em coord. cartesianas

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}(x,y)}{k} = 0$$

- Métodos de Solução:
  - Exata/Analítica: Método de Separação de Variáveis (possível transformada de Laplace ou Fourier
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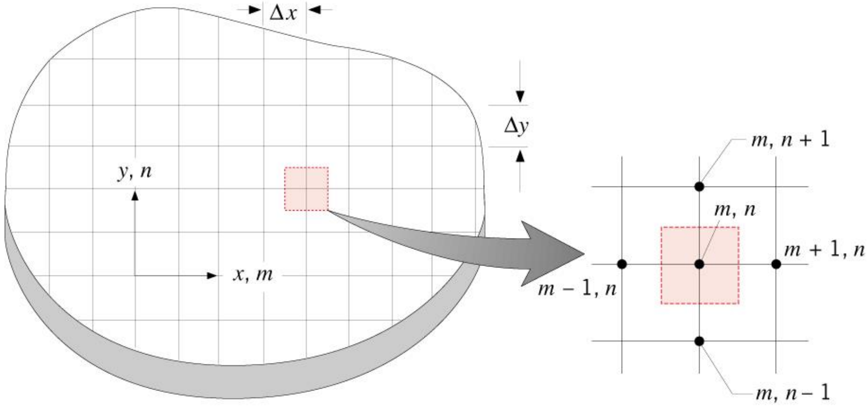
–Métodos Numérico:

–Diferenças Finitas, Método dos Balanços ou Volumes Finitos

–Elementos Finitos

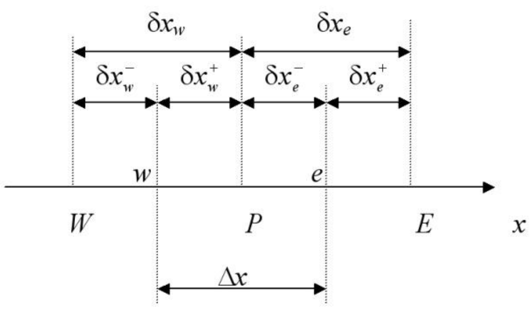


(a)

Método dos Volumes Finitos

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

$$\int_t^{t+\Delta t} dt \int_w^e dx \int_s^n \cdot dy$$

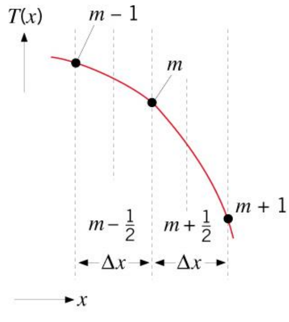


$$\int_w^e \int_s^n \rho u_s \frac{\partial \phi}{\partial s}\, dxdy = \int_w^e \int_s^n \frac{\partial}{\partial x}(\Gamma \frac{\partial \phi}{\partial x})\, dxdy + \int_w^e \int_s^n \partial\partial y(\Gamma \frac{\partial \phi}{\partial y})\, dxdy + \int_w^e \int_s^n S\, dxdy$$

Método de Diferenças Finitas

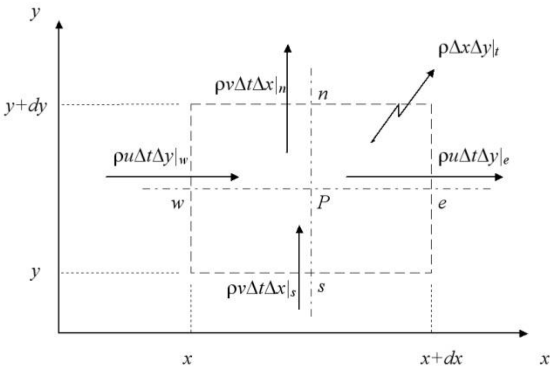
$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,\, n} = \frac{T_{m,\, n} - T_{m-1,\, n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,\, n} = \frac{T_{m+1,\, n} - T_{m,\, n}}{\Delta x}$$



(b)

Método do Balanço de Energia

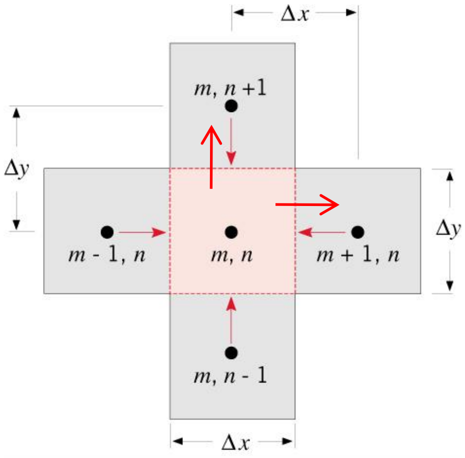


$$\dot{E}_{in} + \dot{E}_g = 0$$

Em Regime Permanente

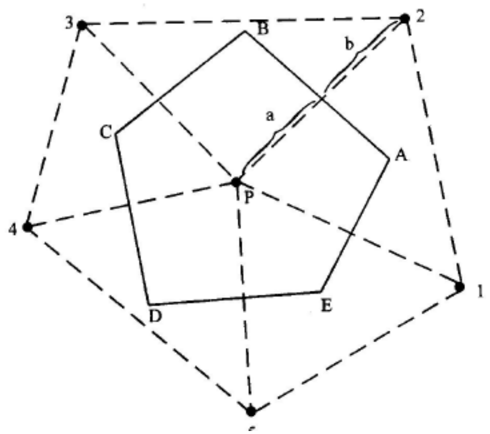
$$\sum_{i=1}^{4} q_{(i)\to(m,n)} + \dot{q}\left(\Delta x \cdot \Delta y \cdot \ell\right) = 0$$

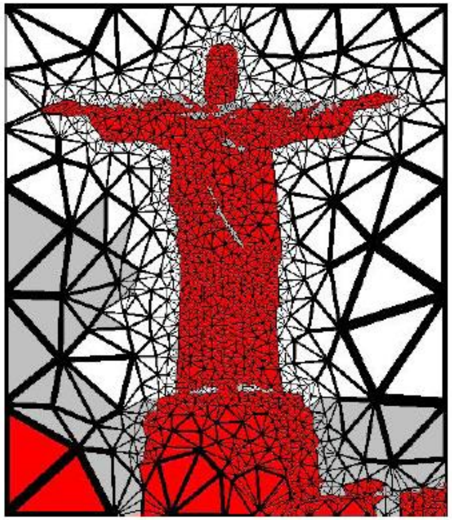
$$q_{(m-1,n)\to(m,n)} = k\left(\Delta y \cdot \ell\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$



(4.31)

Diagrama de Voronoi





Malhas

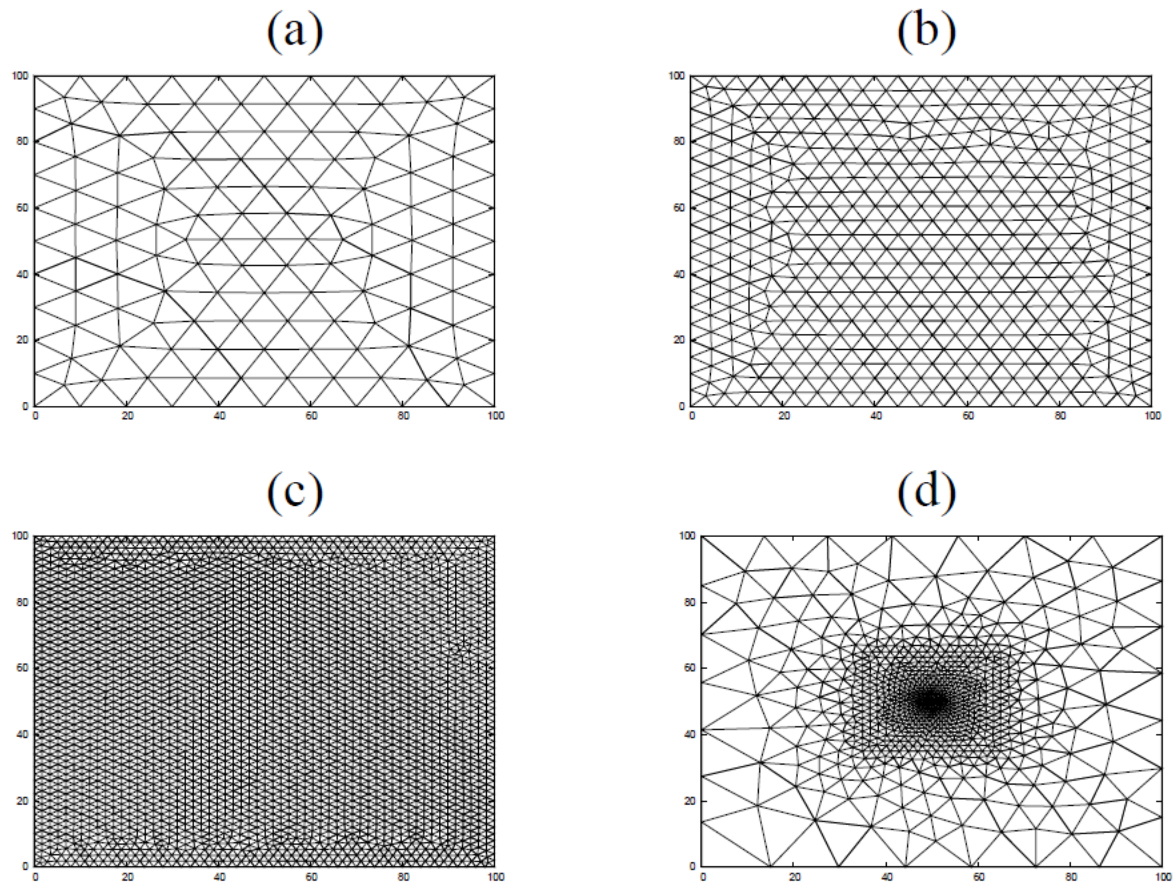
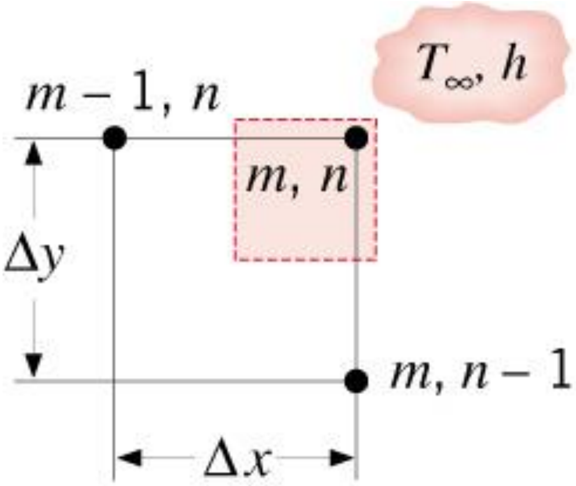


Figura 14: Malhas para a simulação de Volumes Finitos

- A summary of finite-difference equations for common nodal regions is provided in Table 4.2. Consider an *external corner with convection heat transfer*.



$$q_{(m-1,n)\to(m,n)} + q_{(m,n-1)\to(m,n)} + q_{(\infty)\to(m,n)} = 0$$

$$k\left(\frac{\Delta y}{2}\cdot\ell\right)\frac{T_{m-1,n}-T_{m,n}}{\Delta x} + k\left(\frac{\Delta x}{2}\cdot\ell\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}$$
$$+\mathrm{h}\left(\frac{\Delta \mathrm{x}}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) + \mathrm{h}\left(\frac{\Delta \mathrm{y}}{2}\cdot\ell\right)\left(T_\infty - T_{m,n}\right) = 0$$

or, with $\Delta x = \Delta y$,

$$T_{m-1,n} + T_{m,n-1} + 2\frac{\mathrm{h}\Delta x}{k}T_\infty - 2\left(\frac{\mathrm{h}\Delta x}{k}+1\right)T_{m,n} = 0 \qquad (4.43)$$

Métodos de Solução

- Inversão da Matriz: Expression of system of N finite-difference equations for N unknown nodal temperatures as:

$$[A][T]=[C] \qquad (4.48)$$

Coefficient Matrix (NxN)

Solution Vector $(T_1, T_2, \ldots T_N)$

Right-hand Side Vector of Constants $(C_1, C_2 \ldots C_N)$

Solution →

$$[T]=[A]^{-1}[C] \qquad (4.49)$$

Inverse of Coefficient Matrix

- Gauss-Seidel : Each finite-difference equation is written in explicit form, such that its unknown nodal temperature appears alone on the left-hand side:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)} \qquad (4.51)$$

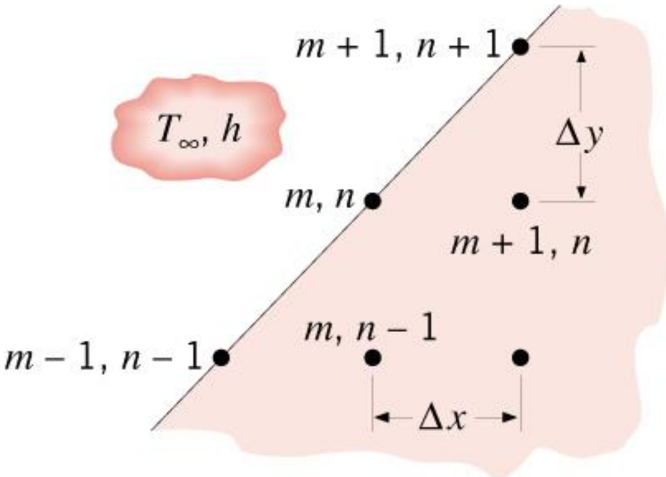
where i =1, 2,…, N and k is the level of iteration.

Iteration proceeds until satisfactory convergence is achieved for all nodes:

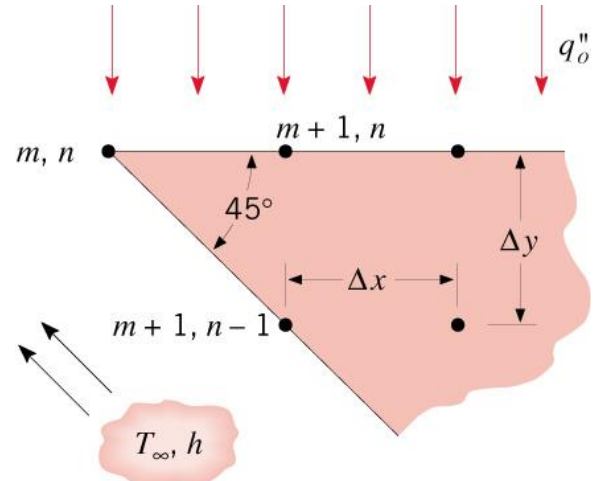
$$\left| T_i^{(k)} - T_i^{(k-1)} \right| \le \varepsilon$$

- What measures may be taken to insure that the results of a finite-difference solution provide an accurate prediction of the temperature field?

Problem 4.39: Finite-difference equations for (a) nodal point on a diagonal surface and (b) tip of a cutting tool.

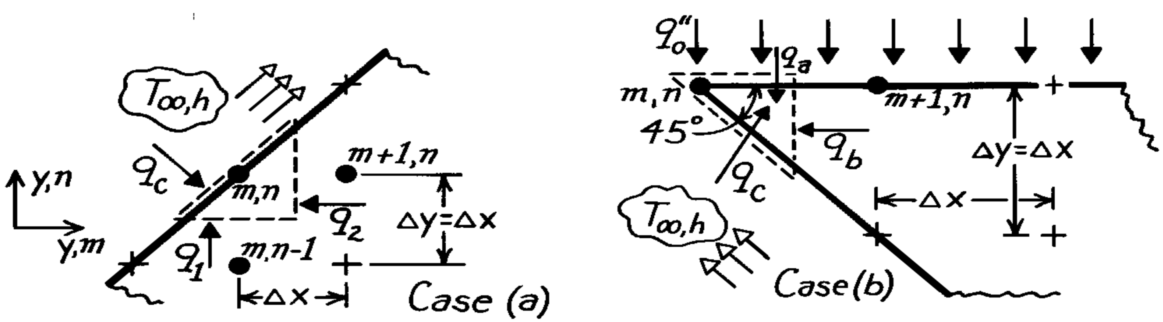


(a) Diagonal surface



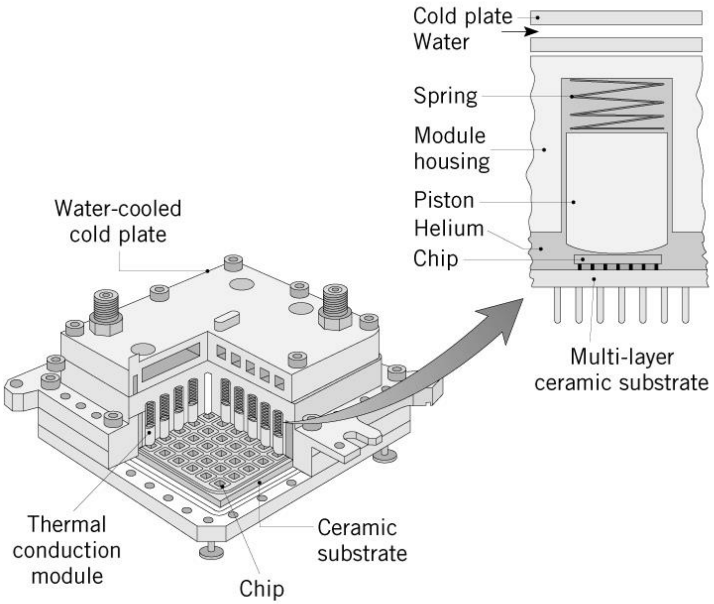
(b) Cutting tool.

Schematic:



ASSUMPTIONS: (1) Steady-state, 2-D conduction, (2) Constant properties

Problem 4.76: Analysis of cold plate used to thermally control IBM multi-chip, thermal conduction module.



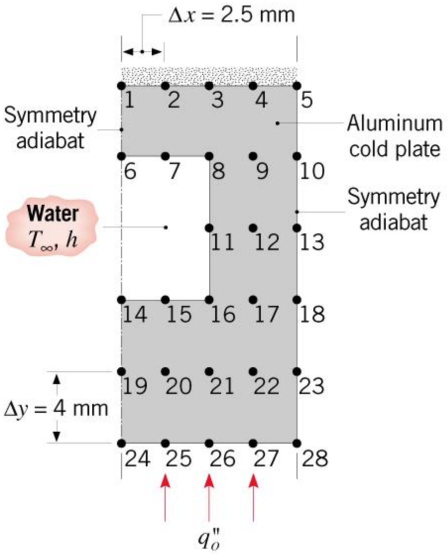
Features:

- Heat dissipated in the chips is transferred by conduction through spring-loaded aluminum pistons to an aluminum cold plate.

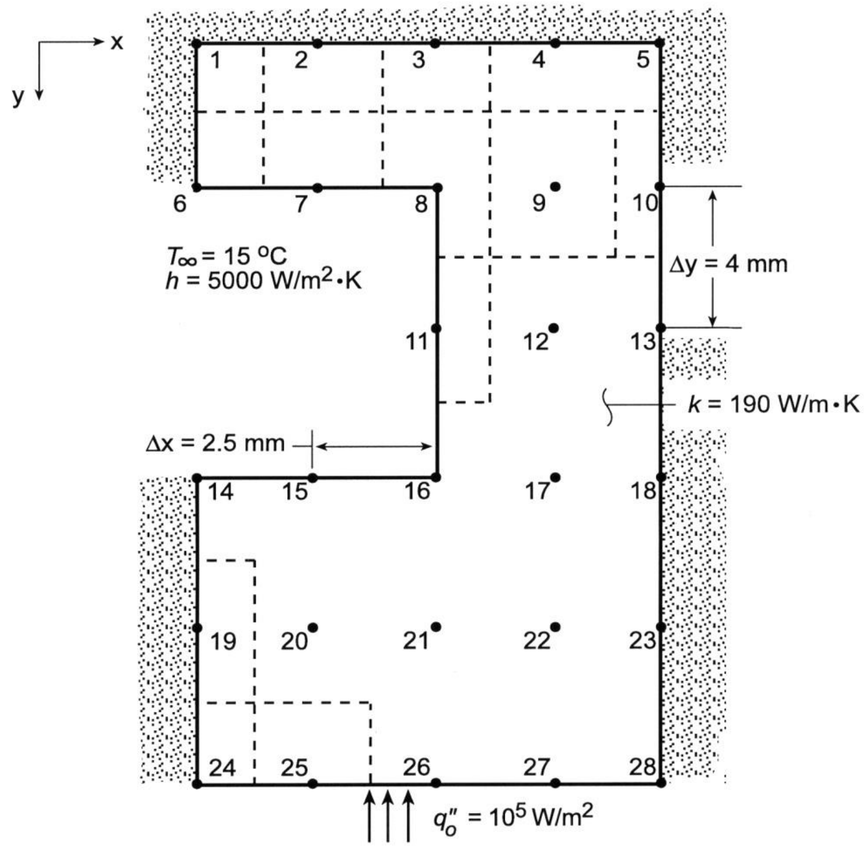
- Nominal operating conditions may be assumed to provide a uniformly distributed heat flux of
  at the base of the cold plate.

- Heat is transferred from the cold plate by water flowing through channels in the cold plate.

Find: (a) Cold plate temperature distribution for the prescribed conditions. (b) Options for operating at larger power levels while remaining within a maximum cold plate temperature of 40°C.



Schematic:



ASSUMPTIONS: (1) Steady-state conditions, (2) Two-dimensional conduction, (3) Constant properties.

ANALYSIS: Finite-difference equations must be obtained for each of the 28 nodes. Applying the energy balance method to regions 1 and 5, which are similar, it follows that

Node 1: $(\Delta y/\Delta x)T_2 + (\Delta x/\Delta y)T_6 - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_1 = 0$

Node 5: $(\Delta y/\Delta x)T_4 + (\Delta x/\Delta y)T_{10} - \left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_5 = 0$

Nodal regions 2, 3 and 4 are similar, and the energy balance method yields a finite-difference equation of the form

Nodes 2,3,4:

$$(\Delta y/\Delta x)\left(T_{m-1,n} + T_{m+1,n}\right) + 2(\Delta x/\Delta y)T_{m,n-1} - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y)\right]T_{m,n} = 0$$

Energy balances applied to the remaining combinations of similar nodes yield the following finite-difference equations.

Nodes 6, 14: $(\Delta x/\Delta y)T_1 + (\Delta y/\Delta x)T_7 - \left[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)\right]T_6 = -(h\Delta x/k)T_\infty$

$(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{15} - \left[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta x/k)\right]T_{14} = -(h\Delta x/k)T_\infty$

Nodes 7, 15: $(\Delta y/\Delta x)(T_6 + T_8) + 2(\Delta x/\Delta y)T_2 - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)\right]T_7 = -(2h\Delta x/k)T_\infty$

$(\Delta y/\Delta x)(T_{14} + T_{16}) + 2(\Delta x/\Delta y)T_{20} - 2\left[(\Delta y/\Delta x) + (\Delta x/\Delta y) + (h\Delta x/k)\right]T_{15} = -(2h\Delta x/k)T_\infty$

Nodes 8, 16:

$$(\Delta y/\Delta x)T_7 + 2(\Delta y/\Delta x)T_9 + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_3 - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y) + (h/k)(\Delta x + \Delta y)]T_8 = -(h/k)(\Delta x + \Delta y)T_\infty$$

$$(\Delta y/\Delta x)T_{15} + 2(\Delta y/\Delta x)T_{17} + (\Delta x/\Delta y)T_{11} + 2(\Delta x/\Delta y)T_{21} - [3(\Delta y/\Delta x) + 3(\Delta x/\Delta y) + (h/k)(\Delta x + \Delta y)]T_{16} = -(h/k)(\Delta x + \Delta y)T_\infty$$

Node 11:

$$(\Delta x/\Delta y)T_8 + (\Delta x/\Delta y)T_{16} + 2(\Delta y/\Delta x)T_{12} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x) + (h\Delta y/k)]T_{11} = -(2h\Delta y/k)T_\infty$$

Nodes 9, 12, 17, 20, 21, 22:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + (\Delta x/\Delta y)T_{m,n+1} + (\Delta x/\Delta y)T_{m,n-1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Nodes 10, 13, 18, 23:

$$(\Delta x/\Delta y)T_{n+1,m} + (\Delta x/\Delta y)T_{n-1,m} + 2(\Delta y/\Delta x)T_{m-1,n} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = 0$$

Node 19:

$$(\Delta x/\Delta y)T_{14} + (\Delta x/\Delta y)T_{24} + 2(\Delta y/\Delta x)T_{20} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{19} = 0$$

Nodes 24, 28:

$$(\Delta x/\Delta y)T_{19} + (\Delta y/\Delta x)T_{25} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{24} = -(q_o''\Delta x/k)$$

$$(\Delta x/\Delta y)T_{23} + (\Delta y/\Delta x)T_{27} - [(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{28} = -(q_o''\Delta x/k)$$

Nodes 25, 26, 27:

$$(\Delta y/\Delta x)T_{m-1,n} + (\Delta y/\Delta x)T_{m+1,n} + 2(\Delta x/\Delta y)T_{m,n+1} - 2[(\Delta x/\Delta y) + (\Delta y/\Delta x)]T_{m,n} = -(2q_o''\Delta x/k)$$

Evaluating the coefficients and solving the equations simultaneously, the steady-state temperature distribution (°C), tabulated according to the node locations, is:

23.77	23.91	24.27	24.61	24.74
23.41	23.62	24.31	24.89	25.07
		25.70	26.18	26.33
28.90	28.76	28.26	28.32	28.35
30.72	30.67	30.57	30.53	30.52
32.77	32.74	32.69	32.66	32.65

(b) For the prescribed conditions, the maximum allowable temperature (T_{24} = 40°C) is reached when

$q''_o = 1.407 \times 10^5$ W/m^2 (14.07 W/cm^2).

Options for extending this limit could include use of a copper cold plate (k ≈ 400 W/m·K) and/or increasing the convection coefficient associated with the coolant.

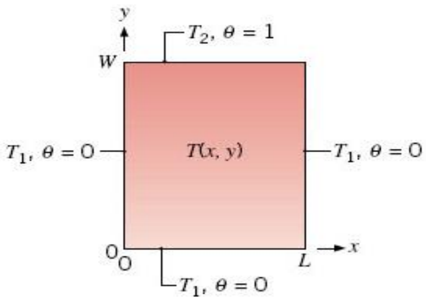
With k = 400 W/m·K, a value of q''_o = 17.37 W/cm^2 may be maintained.

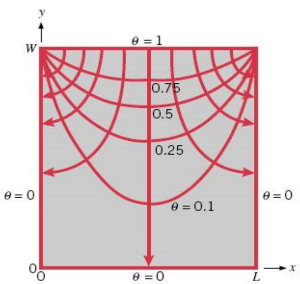
. With k = 400 W/m·K and h = 10,000 W/m^2·K (a practical upper limit), q''_o = 28.65 W/cm^2.

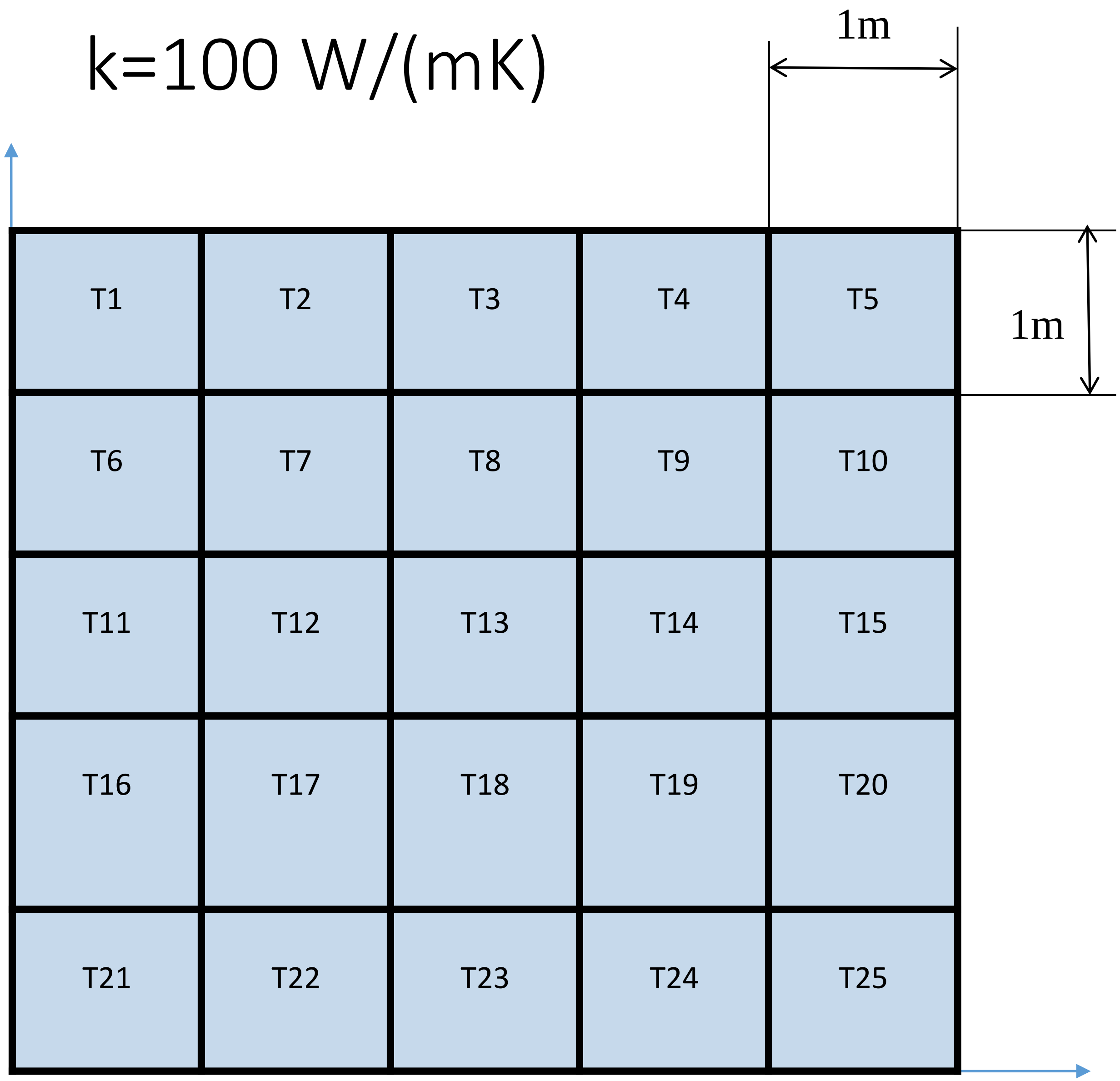
Additional, albeit small, improvements may be realized by relocating the coolant channels closer to the base of the cold plate.

Exercício 1

k=100 W/(mK)







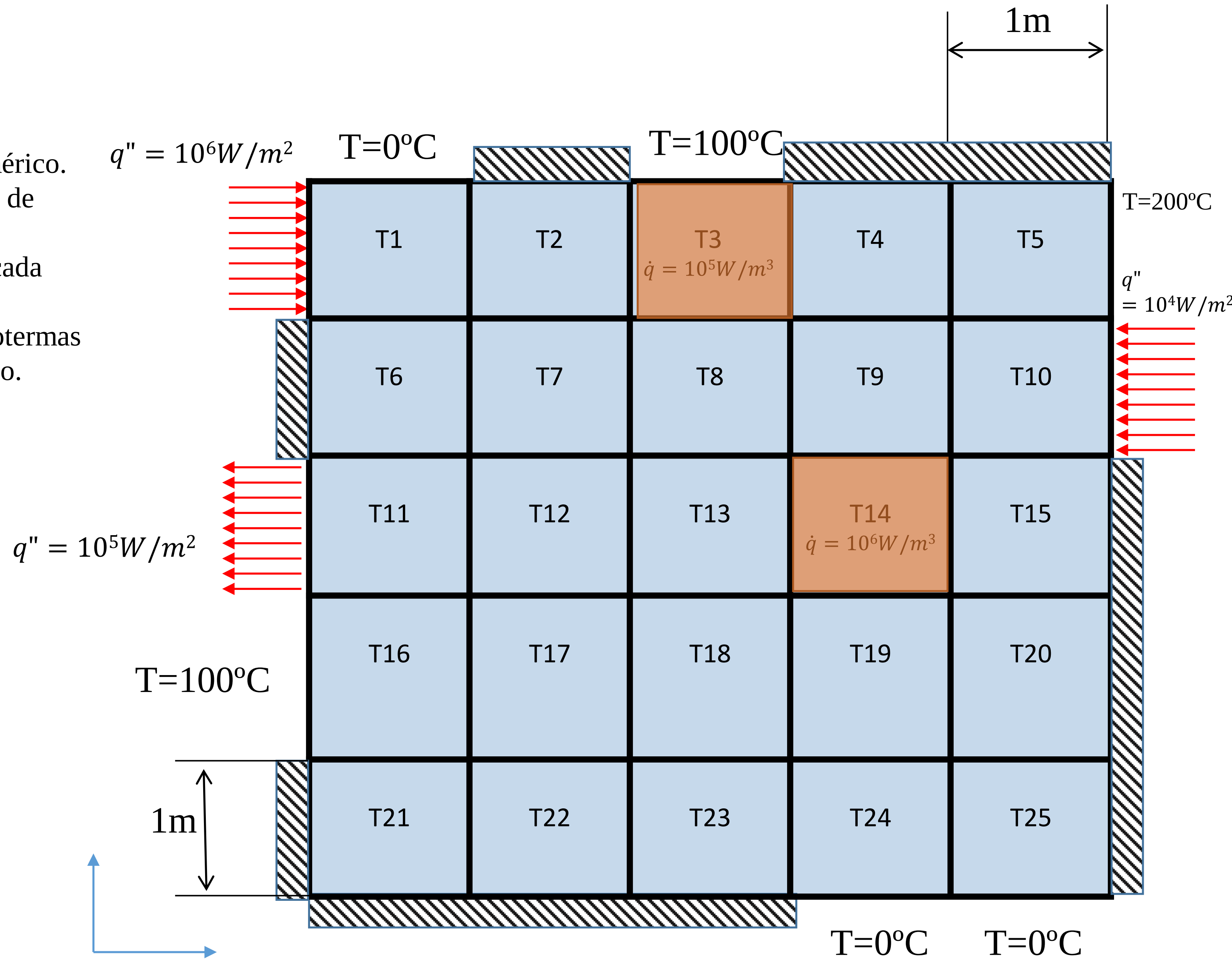
Calcular:

1) T1,....,T25. Analítico e numérico.
2) Comparar resultados
3) Calcular fluxos de calor nas paredes e no centro.
4) Desenhar as isotermas e linhas de fluxo.

Exercício 2 k=100 W/(mK), w=1m

Calcular:

1) T1,....,T6. Numérico.
2) Calcular fluxos de calor vertical e horizontal em cada volume.
3) Desenhar as isotermas e linhas de fluxo.



Exercício 3 k=100 W/(mK), w=1m
NÃO COBRADO

Calcular:

1) T1,....,T6. Numérico.
2) Calcular fluxos de calor vertical e horizontal em cada volume.
3) Desenhar as isotermas e linhas de fluxo.