EXAMPLE 5.1

Consider the isentropic subsonic-supersonic flow through a convergent-divergent nozzle. The reservoir pressure and temperature are 10 atm and 300 K, respectively. There are two locations in the nozzle where $A/A^* = 6$: one in the convergent section and the other in the divergent section. At each location, calculate M, p, T, and u.

■ Solution

In the *convergent* section, the flow is subsonic. From the front of Table A.1, for subsonic flow with $A/A^* = 6$: $\boxed{M = 0.097}$, $p_o/p = 1.006$, and $T_o/T = 1.002$. Hence

$$p = \frac{p}{p_o} p_o = (1.006)^{-1}(10) = \boxed{9.94 \text{ atm}}$$

$$T = \frac{T}{T_o} T_o = (1.002)^{-1}(300) = \boxed{299.4 \text{ K}}$$

$$a = \sqrt{\gamma RT} = \sqrt{(1.4)(287)(299.4)} = 346.8 \text{ m/s}$$

$$u = Ma = (0.097)(346.8) = \boxed{33.6 \text{ m/s}}$$

In the *divergent* section, the flow is supersonic. From the supersonic section of Table A.1, for $A/A^* = 6$: $\boxed{M = 3.368}$, $p_o/p = 63.13$, and $T_o/T = 3.269$. Hence

$$p = \frac{p}{p_o} p_o = (63.13)^{-1}(10) = \boxed{0.1584 \text{ atm}}$$

$$T = \frac{T}{T_o} T_o = (3.269)^{-1}(300) = \boxed{91.77 \text{ K}}$$

$$a = \sqrt{\gamma RT} = \sqrt{(1.4)(287)(91.77)} = 192.0 \text{ m/s}$$

$$u = Ma = (3.368)(192.0) = \boxed{646.7 \text{ m/s}}$$

EXAMPLE 5.2

A supersonic wind tunnel is designed to produce Mach 2.5 flow in the test section with standard sea level conditions. Calculate the exit area ratio and reservoir conditions necessary to achieve these design conditions.

■ Solution

From Table A.1, for $M_e = 2.5$:

$$\boxed{A_e/A^* = 2.637} \qquad p_o/p_e = 17.09 \qquad T_o/T_e = 2.25$$

Also, at standard sea level conditions, $p_e = 1$ atm and $T_e = 288$ K. Hence,

$$p_o = \frac{p_o}{p_e} p_e = (17.09)(1) = \boxed{17.09 \text{ atm}}$$

$$T_o = \frac{T_o}{T_e} T_e = (2.25)(288) = \boxed{648 \text{ K}}$$

EXAMPLE 5.3

Consider a rocket engine burning hydrogen and oxygen; the combustion chamber temperature and pressure are 3517 K and 25 atm, respectively. The molecular weight of the chemically reacting gas in the combustion chamber is 16, and $\gamma = 1.22$. The pressure at the exit of the convergent-divergent rocket nozzle is 1.174×10^{-2} atm. The area of the throat is 0.4 m^2. Assuming a calorically perfect gas and isentropic flow, calculate: (a) the exit Mach number, (b) the exit velocity, (c) the mass flow through the nozzle, and (d) the area of the exit.

■ Solution

Note that for this problem, where $\gamma = 1.22$, the compressible flow tables in the appendix *cannot* be used since the tables are calculated for $\gamma = 1.4$. Thus, to solve this problem, we have to use the governing equations directly.

a. To obtain the exit Mach number, use the isentropic relation given by Eq. (3.30):

$$\frac{p_o}{p_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\gamma/(\gamma-1)}$$

or

$$M_e^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_o}{p_e}\right)^{(\gamma-1)/\gamma} - 1\right] = \frac{2}{0.22}\left[\left(\frac{25}{1.174 \times 10^{-2}}\right)^{0.22/1.22} - 1\right] = 27.116$$

$$\boxed{M_e = 5.21}$$

b. To obtain the exit velocity:

$$\frac{T_e}{T_o} = \left(\frac{p_e}{p_o}\right)^{(\gamma-1)/\gamma} = \left(\frac{1.174 \times 10^{-2}}{25}\right)^{0.180} = 0.2517$$

$$T_e = 0.2517 T_o = 0.2517(3517) = 885.3 \text{ K}$$

From Sec. 1.4, we know that

$$R = \frac{\mathscr{R}}{\mathscr{M}} = \frac{8314}{16} = 519.6 \text{ J/kg} \cdot \text{K}$$

$$a_e = \sqrt{\gamma R T_e} = \sqrt{(1.22)(519.6)(885.3)} = 749.1 \text{ m/s}$$

$$V_e = M_e a_e = (5.21)(749.1) = \boxed{3903 \text{ m/s}}$$

c. Since we are given $A^* = 0.4 \text{ m}^2$, let us calculate the mass flow at the throat. First, obtain ρ_o from the equation of state:

$$\rho_o = \frac{p_o}{RT_o} = \frac{(25)(1.01 \times 10^5)}{(519.6)(3517)} = 1.382 \text{ kg/m}^3$$

From Eq. (3.36)

$$\frac{\rho^*}{\rho_o} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)} = \left(\frac{2}{2.22}\right)^{4.545} = 0.622$$

$$\rho^* = 0.622\rho_o = (0.622)(1.382) = 0.860 \text{ kg/m}^3$$

From Eq. (3.34)

$$\frac{T^*}{T_o} = \frac{2}{\gamma + 1} = \frac{2}{2.22} = 0.9$$

$$T^* = 0.9T_o = (0.9)(3517) = 3168 \text{ K}$$

$$a^* = \sqrt{\gamma R T^*} = \sqrt{(1.22)(519.6)(3168)} = 1417 \text{ m/s}$$

$$\dot{m} = \rho A V = \rho^* A^* a^* = (0.860)(0.4)(1417) = \boxed{487.4 \text{ kg/s}}$$

d. At the exit, since $\dot{m} = \text{const}$,

$$\dot{m} = \rho_e A_e V_e = 487.4 \text{ kg/s}$$

$$\rho_e = \frac{p_e}{RT_e} = \frac{(1.174 \times 10^{-2})(1.01 \times 10^5)}{(519.6)(885.3)} = 0.00258 \text{ kg/m}^3$$

$$A_e = \frac{\dot{m}}{\rho_e V_e} = \frac{487.4}{(0.00258)(3903)} = \boxed{48.4 \text{ m}^2}$$

EXAMPLE 5.4

Consider the flow through a convergent-divergent duct with an exit-to-throat area ratio of 2. The reservoir pressure is 1 atm, and the exit pressure is 0.95 atm. Calculate the Mach numbers at the throat and at the exit.

■ **Solution**

First, let us analyze this problem. If the flow were supersonic in the divergent portion, then from Table A.1, for an area ratio of $A_e/A^* = 2$, $p_o/p_e = 10.69$; thus p_e would have to be $p_e = p_o/10.69 = (1\text{ atm})/10.69 = 0.0935\text{ atm}$. This is considerably less than the given $p_e = 0.95\text{ atm}$. Therefore, we do not have a subsonic-supersonic isentropic flow as was the case in Examples 5.1 through 5.3. *Question:* Is the flow completely subsonic? If this were the case, the throat area A_t is *not* equal to A^*, and $A_t > A^*$. Let us examine A_t and A^*. From Table A.1, for $p_o/p_e = 1/0.95 = 1.053$, $A_e/A^* = 2.17$ (nearest entry). However, for the given problem, $A_e/A_t = 2$. Thus, $A_t > A^*$, and the flow is completely subsonic. From Table A.1, since $p_o/p_e = 1.053$, we have

$$\boxed{M_e = 0.28}$$

At the throat,

$$\frac{A_t}{A^*} = \frac{A_t}{A_e}\frac{A_e}{A^*} = \tfrac{1}{2}(2.17) = 1.085$$

From Table A.1, for $A_t/A^* = 1.085$, we have

$$\boxed{M_t = 0.72}$$

EXAMPLE 5.5

Consider a convergent-divergent duct with an exit-to-throat area ratio of 1.6. Calculate the exit-to-reservoir pressure ratio required to achieve sonic flow at the throat, but subsonic flow everywhere else.

■ **Solution**

Since $M = 1$ at the throat, $A_t = A^*$. Thus

$$\frac{A_e}{A_t} = \frac{A_e}{A^*} = 1.6$$

From Table A.1, the subsonic entry that corresponds to $A_e/A^* = 1.6$ is $p_o/p_e = 1.1117$. Hence

$$\frac{p_e}{p_o} = \frac{1}{1.1117} = \boxed{0.9}$$

For this area ratio of $A_e/A_t = 1.6$, if the exit-to-reservoir pressure ratio is greater than 0.9, the flow through the duct is completely subsonic. If this pressure ratio is less than 0.9, then the flow will expand to supersonic speed downstream of the throat. However, unless $p_e/p_o = 1/7.128 = 0.1403$, which corresponds to an isentropic expansion to the exit, there will be shock waves either at the lip of the nozzle (overexpanded case) or a normal shock somewhere inside the duct. Which of these cases hold depends upon the prescribed value of p_e/p_o.

EXAMPLE 5.6

Consider a convergent-divergent nozzle with an exit-to-throat area ratio of 3. A normal shock wave is inside the divergent portion at a location where the local area ratio is $A/A_t = 2$. Calculate the exit-to-reservoir pressure ratio.

■ **Solution**

For this case, we have an isentropic subsonic-supersonic expansion through the part of the nozzle upstream of the normal shock. Let the subscripts 1 and 2 denote conditions immediately upstream and downstream of the shock, respectively. The local Mach number M_1 just ahead of the shock is obtained from Table A.1 for $A_1/A_1^* = 2$, namely $M_1 = 2.2$. From Table A.2, for $M_1 = 2.2$, $M_2 = 0.5471$ and $p_{o_2}/p_{o_1} = 0.6281$. From Table A.1, for $M_2 = 0.5471$, we have $A_2/A_2^* = 1.27$. Note an important fact at this stage of our calculation. The normal shock is assumed to be infinitely thin, hence $A_1 = A_2$. However, we have previously shown that $A_1/A_1^* = 2$ and $A_2/A_2^* = 1.27$. *Clearly, the value of A^* changes across the shock wave.* This is due to the entropy increase across the shock. A_1^* is the flow area necessary to achieve Mach 1 isentropically in the flow upstream of the shock, and A_2^* is the flow area necessary to achieve Mach 1 isentropically in the flow downstream of the shock. Since the entropy is different for these two flows, then A^* is different for the two flows. Proceeding with the calculation,

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_2}\frac{A_2}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_2}\frac{A_2}{A_2^*} = (3)\left(\tfrac{1}{2}\right)(1.27) = 1.905$$

The flow is *subsonic* behind the normal shock wave, and hence is subsonic throughout the remainder of the divergent portion downstream of the shock. Thus, from the subsonic entries in Table A.1, we have for $A_e/A_2^* = 1.905$, $M_e = 0.32$ and $p_{o_e}/p_e = 1.074$. Thus, since $p_o = p_{o_1}$ and $p_{o_e} = p_{o_2}$, we have

$$\frac{p_e}{p_o} = \frac{p_e}{p_{o_e}}\frac{p_{o_e}}{p_{o_2}}\frac{p_{o_2}}{p_{o_1}}\frac{p_{o_1}}{p_o} = \left(\frac{1}{1.074}\right)(1)(0.6281)(1) = \boxed{0.585}$$

EXAMPLE 5.7

Consider a convergent-divergent nozzle with an exit-to-throat area ratio of 3. The inlet reservoir pressure is 1 atm and the exit static pressure is 0.5 atm. For this pressure ratio, a normal shock will stand somewhere inside the divergent portion of the nozzle. Calculate the location of the shock wave using (a) a trial-and-error solution and (b) the direct solution. Compare the results.

■ Solution

a. Assume $A/A_t = A/A_1^* = 2.3$. From Table A.1, $M_1 = 2.35$. From Table A.2, $M_2 = 0.5286$ and $p_{o_2}/p_{o_1} = 0.5615$. From Table A.1, for $M_2 = 0.5286$, $A/A_2^* = 1.303$. (Recall that we are using nearest entries in the table.) Hence,

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_1^*}\frac{A_1^*}{A}\frac{A}{A_2^*} = (3)\left(\frac{1}{2.3}\right)(1.303) = 1.7$$

For $A_e/A_2^* = 1.7$, from Table A.1, $M_e = 0.36$, and $p_{o_e}/p_e = 1.094$. Hence,

$$p_e = \frac{p_e}{p_{o_e}}\frac{p_{o_2}}{p_{o_1}}p_{o_1} = \frac{1}{1.094}(0.5615)(1) = 0.513\,\text{atm}$$

Since p_e should be 0.5 atm, assume a new A/A_1^* (closer to the exit), and start over again. Assume $A/A_1^* = 2.4$. For this, $M_1 = 2.4$, $M_2 = 0.5231$, $p_{o_2}/p_{o_1} = 0.5401$, and $A/A_2^* = 1.303$. (Again, recall that we are using nearest entries.) Hence, $A_e/A_2^* = (3)(1/2.4)\,(1.303) = 1.629$. With this, $M_e = 0.39$ and $p_{o_e}/p_e = 1.111$. Hence,

$$p_e = \frac{p_e}{p_{o_e}}\frac{p_{o_2}}{p_{o_1}}p_{o_1} = \frac{1}{1.111}(0.5401)(1) = 0.486\,\text{atm}$$

Since p_e should be 0.5 atm, the value of 0.486 atm is too low by about the same amount as the first iteration is too high. Splitting the difference, the correct location of the normal shock wave is approximately $\boxed{A/A_t = 2.35}$.

b. Using the direct method, from the specified conditions

$$\left(\frac{p_e}{p_{o_1}}\right)\left(\frac{A_e}{A_1^*}\right) = \left(\frac{0.5}{1.0}\right)(3) = 1.5$$

From Eq. (5.28)

$$M_e^2 = -\frac{1}{\gamma - 1} + \sqrt{\frac{1}{(\gamma - 1)^2} + \left(\frac{2}{\gamma - 1}\right)\left(\frac{2}{\gamma + 1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{p_{o_e} A_e^*}{p_e A_e}\right)^2}$$

$$= -2.5 + \sqrt{(2.5)^2 + (5)(0.8333)^6\left(\frac{1}{1.5}\right)^2}$$

$$= -2.5 + \sqrt{6.994} = 0.1447$$

Hence,

$$M_e = 0.38$$

From Table A.1 for $M_e = 0.38$, $p_{o_e}/p_e = 1.094$. From Eq. (5.29),

$$\frac{p_{o2}}{p_{o_1}} = \frac{p_{o_e}}{p_e}\frac{p_e}{p_{o_1}} = (1.094)\left(\frac{0.5}{1}\right) = 0.547$$

From Table A.2, for $p_{o2}/p_{o_1} = 0.547$, $M_1 = 2.38$. From Table A.1, for $M_1 = 2.38$, $A/A_1^* = A/A_t = \boxed{2.36}$. This direct answer compares to that obtained with the iteration in part (a) to within 0.4 percent.

EXAMPLE 5.8

Consider the wind tunnel described in Example 5.2. Estimate the ratio of diffuser throat area to nozzle throat area required to allow the tunnel to start. Also, assuming that the diffuser efficiency is 1.2 after the tunnel has started, calculate the pressure ratio across the tunnel necessary for running, i.e., calculate the ratio of total pressure at the diffuser exit to the reservoir pressure.

■ Solution

From Table A.2, for $M = 2.5$: $p_{o_2}/p_{o_1} = 0.499$. From Eq. (5.36)

$$\frac{A_{t2}}{A_{t_1}} = \frac{p_{o_1}}{p_{o_2}} = \frac{1}{0.499} = \boxed{2.00}$$

From Eq. (5.30)

$$\left(\frac{p_{d_o}}{p_o}\right)_{\text{actual}} = \eta_D\left(\frac{p_{o_2}}{p_{o_1}}\right)_{\text{normal shock}} = (1.2)(0.499) = \boxed{0.599}$$

Note: In Example 5.2, standard sea level conditions were stipulated in the test section. For this case, the pressure at the diffuser exit is far above atmospheric pressure. Specifically, from Example 5.2, $p_o = 17.09$ atm; hence $p_{d_o} = (0.599)(17.09) = 10.23$ atm. If the diffuser exhausted directly to the atmosphere, the flow would rapidly expand to supersonic velocity in the free jet downstream of the tunnel exit, with accompanying tremendous losses. Therefore, for this particular wind tunnel, a *closed circuit* design is by far the best. That is, the low subsonic flow at the exit of the diffuser is ducted right back to the entrance of the nozzle. The tunnel forms a closed loop, and the pressure loss in passing through the tunnel and the return loop is made up by a fan with a motor drive. Since the gas is also heated by the addition of power from this motor drive, a cooler must also be inserted in the return loop. See Chap. 5 of Ref. 9 for a more detailed discussion of the design of a closed-loop (or closed-return) supersonic wind tunnel.
