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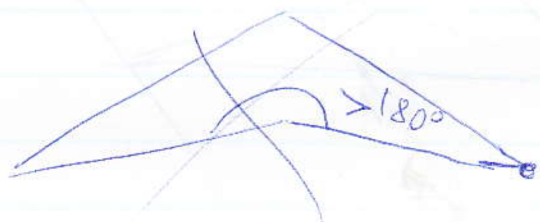
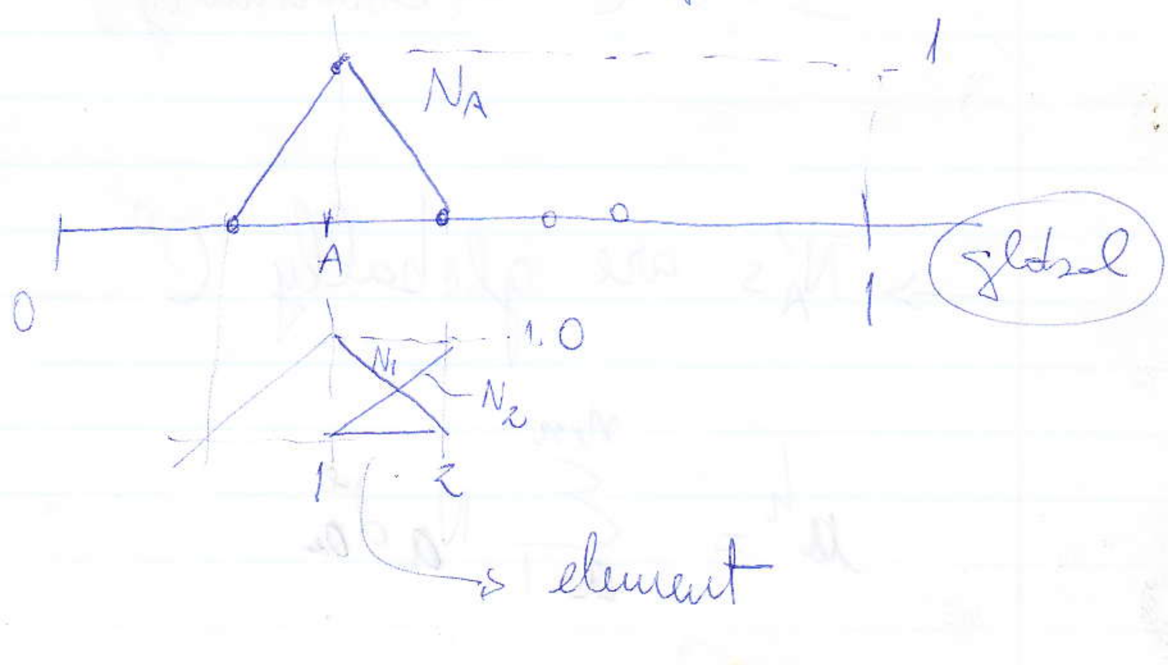
CLASS OF
THURSDAY FEBRUARY 25, 1993

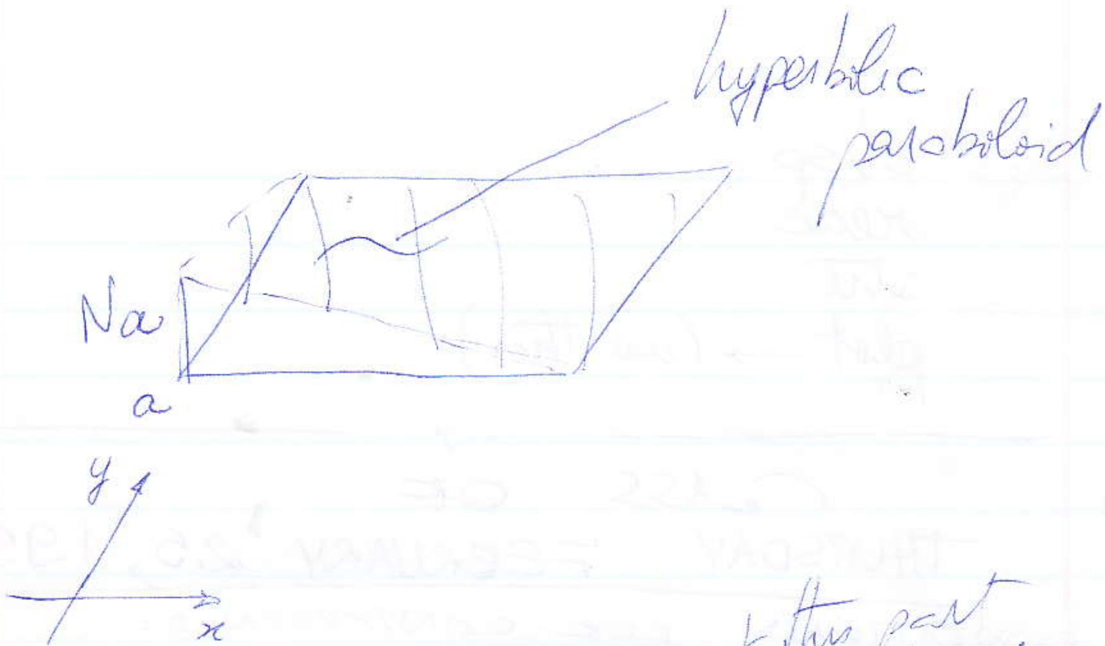
CONDITIONS FOR CONVERGENCE:

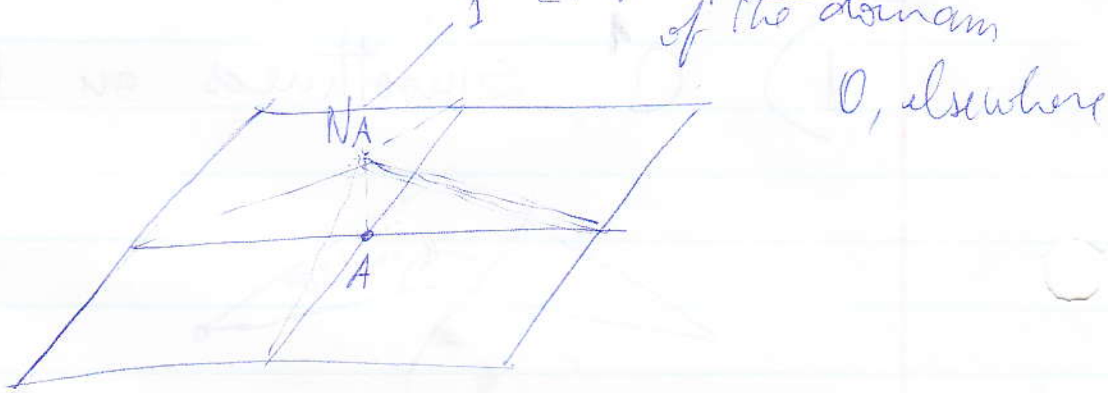
1) C^1 smoothness on Ω^e



2) Global Continuity







$\Longrightarrow C^0$ (continuity)

$\rightarrow$ N_A's are globally C^0

$$u^h = \sum_{a=1}^{n_{en}} N_a d_a^e$$

3. Completeness

$$u^h = \sum_{a=1}^{4} N_a d_a^e$$

$$= \sum_{a=1}^{4} N_a \left(c_0 + c_1 x_a^e + c_2 y_a^e \right)$$

$$= \underbrace{\left(\sum_{a=1}^{4} N_a \right)}_{\text{Want } =1} c_0 + \overbrace{\left(\sum_{a=1}^{4} N_a x_a^e \right)}^{x} c_1 + \underbrace{\left(\sum_{a=1}^{4} N_a y_a^e \right)}_{y} c_2$$

$$u^h = c_0 + c_1 x + c_2 y$$

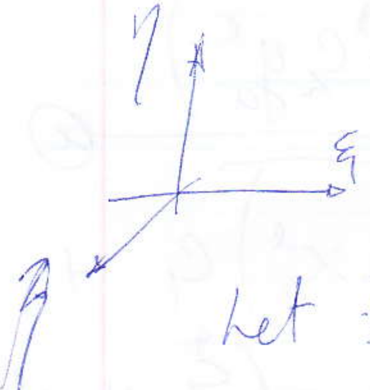
$$\sum_{a=1}^{4} N_a(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1+\xi)(1-\eta) + \frac{1}{4}(1-\xi)(1+\eta)$$

$$= 1$$ //

Q.E.D

Isoparametric Elements

— Parametrize $u^h, \underset{\sim}{x}(\underset{\sim}{\xi})$ using the same shape functions



Let $\square$ denote the parent domain

• Let $\underset{\sim}{x} : \square \rightarrow \Omega^e$ be of the form

$$\rightarrow \underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{a=1}^{n_{en}} N_a(\underset{\sim}{\xi}) \underset{\sim}{x}_a^e$$

We also say:

$$\rightarrow u_i^h(\underset{\sim}{\xi}) = \sum_{a=1}^{n_{en}} N_a(\underset{\sim}{\xi}) d_{ai}^e$$

1) C^1 smoothness in element

Want:

$N_{a,x}$, $N_{a,y}$ to be smooth

$$\langle N_{a,x} \;\; N_{a,y} \rangle = \langle N_{a,\xi} \;\; N_{a,\eta} \rangle \begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix}$$

Mas, pode-se calcular:

$$\begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix} = \left(\underset{\sim}{x}_{,\xi}\right)^{-1}$$

$$J = \det\left(\underset{\sim}{x}_{,\xi}\right) = x_{,\xi}\, y_{,\eta} - x_{,\eta}\, y_{,\xi}$$

$\cancel{\text{Jacobian} \rightarrow J = \det \begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix}}$

THIS IS NOT CORRECT

Implicit function Theorem

We know, $N_{a,\xi}$, $N_{a,\eta}$ are smooth

It turns out that

$N_{a,x}$, $N_{a,y}$ are

smooth if $\boxed{J > 0}$

2) C^0 smoothness
usually considered case-by-case

3) Completeness

$$u^h = \sum_{a=1}^{n_{en}} N_a d_a^e$$

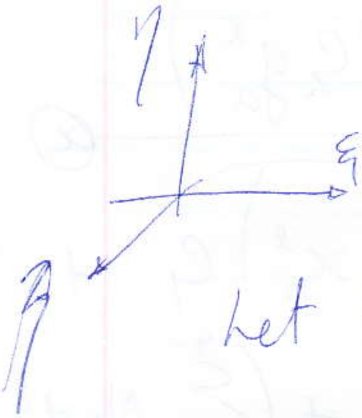
$$= \sum_{a=1}^{n_{en}} N_a (c_0 + c_1 x_a^e + c_2 y_a^e + c_3 z_a^e)$$

$$= c_0 \sum N_a + c_1 \underbrace{\sum N_a x_a^e}_{x} + c_2 \underbrace{\sum N_a y_a^e}_{y} +$$

$$+ c_3 \underbrace{\sum N_a z_a^e}_{z}$$

Isoparametric Elements

— Parametrize $u^h, \underset{\sim}{x}(\underset{\sim}{\xi})$ using the same shape functions



Let $\square$ denote the parent domain

- Let $\underset{\sim}{x} : \square \to \bar{\Omega}^e$ be of the form

$$\rightarrow \underset{\sim}{x}(\underset{\sim}{\xi}) = \sum_{a=1}^{n_{en}} N_a(\underset{\sim}{\xi})\, \underset{\sim}{x}_a^e$$

We also say:

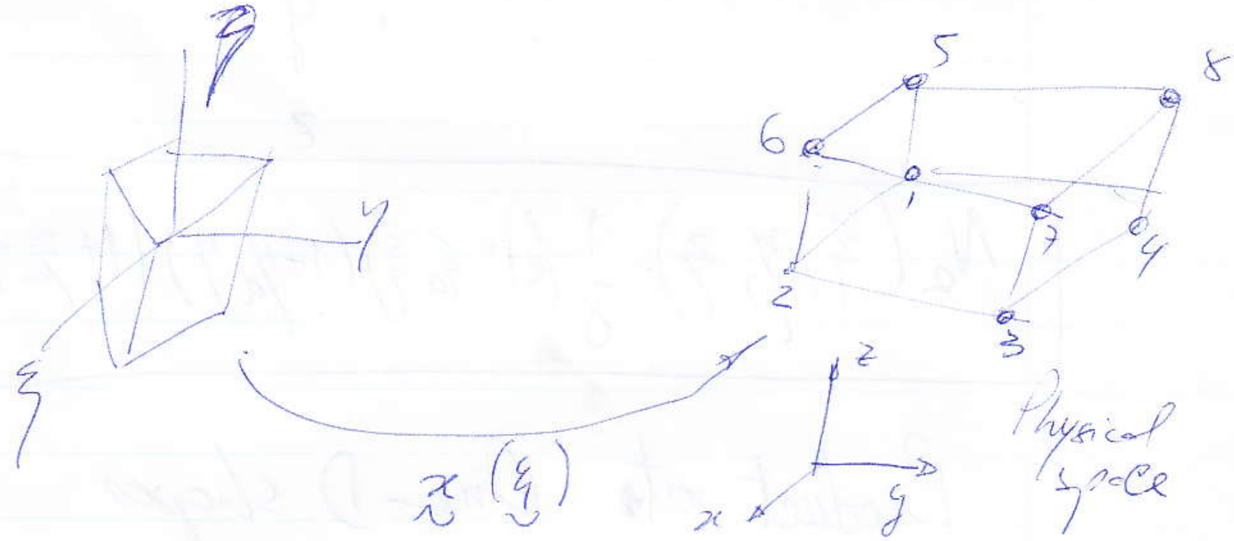
$$\rightarrow u_i^h(\underset{\sim}{\xi}) = \sum_{a=1}^{n_{en}} N_a(\underset{\sim}{\xi})\, d_{ai}^e$$

$$u^h = C_0 \sum N_a + C_1 x + C_2 y + C_3 z$$

Need to prove $= 1$

↳ case-by-case

TRILINEAR BRICK (3D)



$$\underset{\sim}{\xi} = \begin{vmatrix} \xi \\ \eta \\ \zeta \end{vmatrix} \qquad \underset{\sim}{x} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

We assume

$$x\left(\underset{\sim}{\xi}\right) = \alpha_0 + \alpha_1 \xi + \alpha_2 \eta + \alpha_3 \zeta + \\ + \alpha_4 \xi\eta + \alpha_5 \eta\zeta + \alpha_6 \xi\zeta + \alpha_7 \xi\eta\zeta$$

$y(\underset{\sim}{\xi}) = \ldots$

$z(\underset{\sim}{\xi}) = \ldots$

Require that

$$x(\underset{\sim}{\xi}_a) = \sum_{a=1}^{8} N_a(\underset{\sim}{\xi})\, \underset{\sim}{x}_a^e, \quad x(\underset{\sim}{\xi}) = x_a^e$$

$$y$$

$$z$$

$$\boxed{N_a(\xi, \eta, \zeta) = \frac{1}{8}(1+\xi_a \xi)(1+\eta_a \eta)(1+\zeta_a \zeta)}$$

Product of One-D shapes

$$\frac{1}{2}(1+\xi_a \xi)$$

$$\boxed{u^h(\underset{\sim}{\xi}) = \sum_{a=1}^{8} N_a(\underset{\sim}{\xi})\, d_a^e}$$

Triangular, wedge, tetrahedra

→ can be defined via degeneration

(§ 3.4, 3.5)
in the book

collapse

$$\underset{\sim}{\xi} = \begin{vmatrix} \xi \\ \eta \\ \zeta \end{vmatrix}$$

XIII - Lagrange Polynomials and Quadrature

⟹ Two ways to improve an fem answer:

1) decreasing h
"h refinement"

2) (p) refinement
higher polynomial in shape functions

One dimensional Lagrange polynomials

Goal: to write $N_a(\xi)$

Interpolation property $\quad N_a(\xi_b) = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{otherwise} \end{cases}$

Consider the following element

1 — 2 — 3 — ⋯ — $n_{en}-1$ — n_{en}

$a=1$ $\quad a=2$

Lagrange polys. $\underbrace{l_a}_{\text{local node \#}}^{n_{en}-1}$ ← order of polyn.

$$l_a^{n_{en}-1}(\xi) = \frac{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi - \xi_b)}{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi_a - \xi_b)}$$

$$= \frac{(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{a-1})\overset{\text{skip } a^{\text{th}} \text{ term}}{}(\xi-\xi_{a+1})\cdots(\xi-\xi_{n_{en}})}{(\xi_a-\xi_1)(\xi_a-\xi_2)\cdots(\xi_a-\xi_{a-1})(\xi_a-\xi_{a+1})\cdots(\xi_a-\xi_{n_{en}})}$$

2 cases:

1. suppose $\xi = \xi_a \Rightarrow l_a^{n_{en}-1}(\xi_a) = 1$

2. suppose $\xi = \xi_c,\ c \neq a \Rightarrow l_a^{n_{en}-1}(\xi_c) = 0$

so:

$$l_a^{n_{en}-1}(\xi_b) = \delta_{ab}$$ Q.E.D

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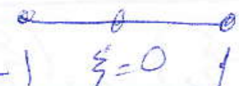
2 - Nodes $(\xi_1 = -1,\ \xi_2 = 1,\ n_{en} = 2)$

$$N_1 = l_1^1(\xi) = \frac{\xi-\xi_2}{(\xi_1-\xi_2)} = \frac{1}{2}(1-\xi)$$

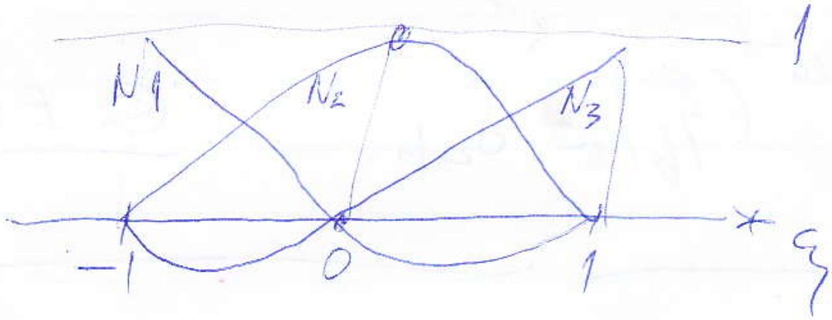
$$N_2 = l_2^1(\xi) = \frac{\xi-\xi_1}{(\xi_2-\xi_1)} = \frac{1}{2}(1+\xi)$$

3-Noded: 

$$\left.\begin{aligned} \xi_1 &= -1 \\ \xi_2 &= 0 \\ \xi_3 &= 1 \end{aligned}\right\} n_{en} = 3$$

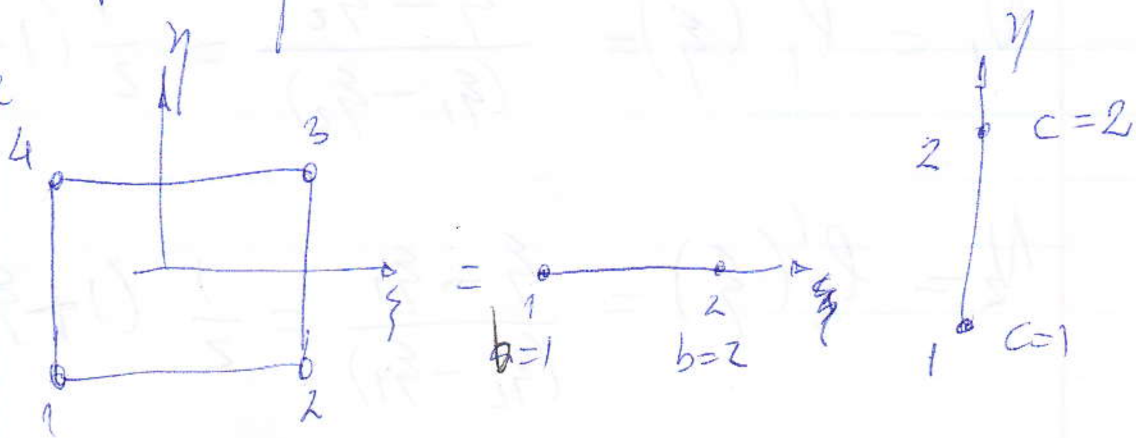
$$N_1 = l_1^2(\xi) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} = \frac{1}{2}\xi(\xi-1)$$

$$N_3 = l_3^2(\xi) = \frac{1}{2}\xi(\xi+1)$$



Multiple Dimensions

Scheme



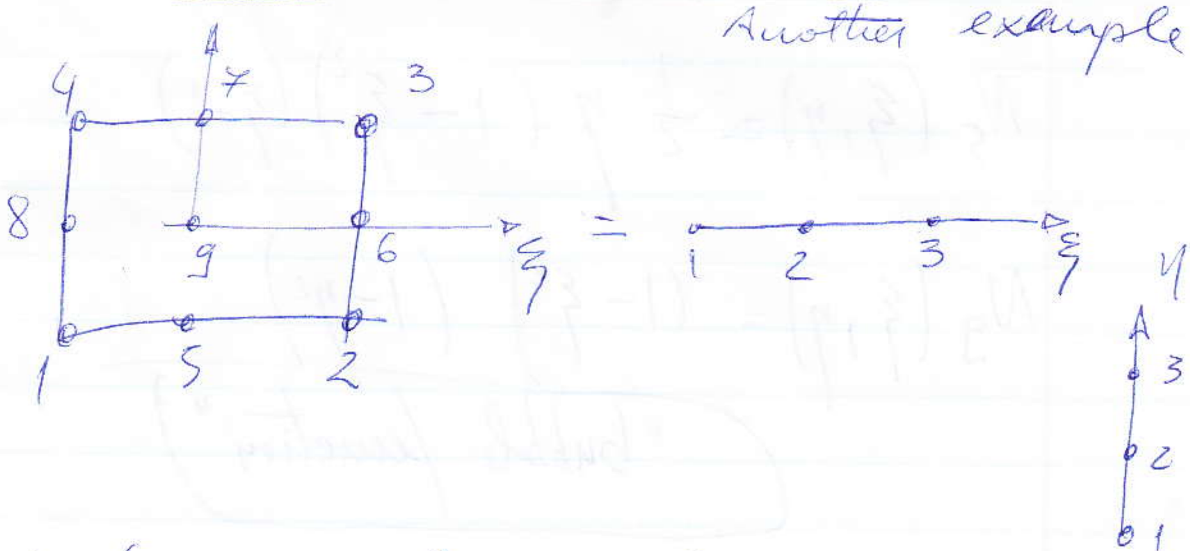
$$N_a(\xi,\eta) = l_b^1(\xi)\, l_c^1(\eta)$$

a	b	c
1	1	1
2	2	1
3	2	2
4	1	2

$$N_3(\xi,\eta) = l_2^1(\xi)\, l_2^1(\eta)$$

$$= \frac{1}{4}(1+\xi)(1+\eta)$$

Another example



$$N_a(\xi,\eta) = l_b^2(\xi)\, l_c^2(\eta)$$

(ROOM 53)

$\bar{a}$	b	c
1	1	1
2	3	1
3	3	3
4	1	3
5	2	1
6	3	2
7	2	3
8	1	2
9	2	2

$$N_1(\xi,\eta) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1)$$

$$N_5(\xi,\eta) = \frac{1}{2}\eta(1-\xi^2)(\eta-1)$$

$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$

"bubble function"

pg 130 of the book

INTEGRAÇÃO NUMÉRICA

1) Regra do Trapézio : 2ª ordem (Precisão)
— integra exatamente pol. de 1º grau

2) Gauss : ordem $2 . n_{int}$

⇒ integram exatamente polinômios de grau

$2 . n_{int} - 1$

Como computar ? (SUBROTINA SHAPE)

$$\underset{\sim}{K}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$$

EL. linear - 2-D

$$\underset{\sim}{B} = [\underset{\sim}{B}_1 , \underset{\sim}{B}_2 , \text{- - - -} , \underset{\sim}{B}_{n_{en}}]$$

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}$$

$$\underset{\sim}{\xi} = \begin{vmatrix} \xi \\ \eta \end{vmatrix}$$

$$\underset{\sim}{K}^e = \int_{-1}^{1} \int_{-1}^{1} \underset{\sim}{B}^T(\underset{\sim}{\xi}) \, \underset{\sim}{D}(\underset{\sim}{\xi}) \, \underset{\sim}{B}(\underset{\sim}{\xi}) \, j(\underset{\sim}{\xi}) \, d\xi \, d\eta$$

$$j(\underset{\sim}{\xi}) = \det [\underset{\sim}{x}_{,\underset{\sim}{\xi}}]$$

CONCLUSÃO: Dadas as coordenadas

$$\tilde{\underset{\sim}{\xi}}_l = (\tilde{\xi}_l, \tilde{\eta}_l) \rightarrow \text{pontos de gauss}$$

precisa-se de uma subrotina para calcular

$$N_{a,x}(\tilde{\underset{\sim}{\xi}}_l), \quad N_{a,y}(\tilde{\underset{\sim}{\xi}}_l), \quad j(\tilde{\underset{\sim}{\xi}}_l)$$

pois, pela quadratura gaussiana, tem-se:

$$\boxed{k^e \cong \sum_{l=1}^{n_{int}} w_l \, \underset{\sim}{B}^T(\tilde{\underset{\sim}{\xi}}_l) \, \underset{\sim}{D}(\tilde{\underset{\sim}{\xi}}_l) \, \underset{\sim}{B}(\tilde{\underset{\sim}{\xi}}_l) \, j(\tilde{\underset{\sim}{\xi}}_l)}$$

REGRA DA CADEIA:

$$\overbrace{\langle N_{a,x} \quad N_{a,y} \rangle}^{\text{Queremos ?}} = \underbrace{\langle N_{a,\xi} \quad N_{a,\eta} \rangle}_{\text{facilmente calculadas}} \underbrace{\begin{vmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{vmatrix}}_{\text{NÃO É DIRETO } (*)}$$

$$x = \sum_{a=1}^{n_{en}} \underbrace{N_a}_{\text{funções de } \xi,\ \eta} x_a^e$$

obtemos

$$\begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}^{-1} = \frac{1}{j} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

Dados $(\tilde{\xi}_\ell, \tilde{\eta}_\ell)$ → pontos de gauss e

coordenadas nodais (x_a^e, y_a^e)

1) calcula-se:

$$x_{,\xi} \;,\; x_{,\eta} \;,\; y_{,\xi} \;,\; y_{,\eta}$$

2) jacobiano:

$$j(\xi,\eta) = x_{,\xi}\, y_{,\eta} - x_{,\eta}\, y_{,\xi}$$

3) derivadas espaciais:

$$\langle N_{a,x} \;\; N_{a,y} \rangle = \frac{1}{j} \langle N_{a,\xi} \;\; N_{a,\eta} \rangle \begin{vmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{vmatrix}$$

$$\forall a = 1, 2 \ldots n_{en}$$

CÁLCULO DE $\underset{\sim}{k}^e$

1) Subrotina pgauss nos dá $\underset{\sim}{\xi}$ e $\underset{\sim}{w}$

2) LOOP p/ INTEGRAÇÃO $(\underset{\sim}{k}^e)$

$$\underset{\sim}{k}^e = \int_{\square} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, j \, d\square \simeq \sum_{\ell=1}^{n_{int}} \left(\underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, j \right) \Big|_{\underset{\sim}{\xi}_\ell} w_\ell$$

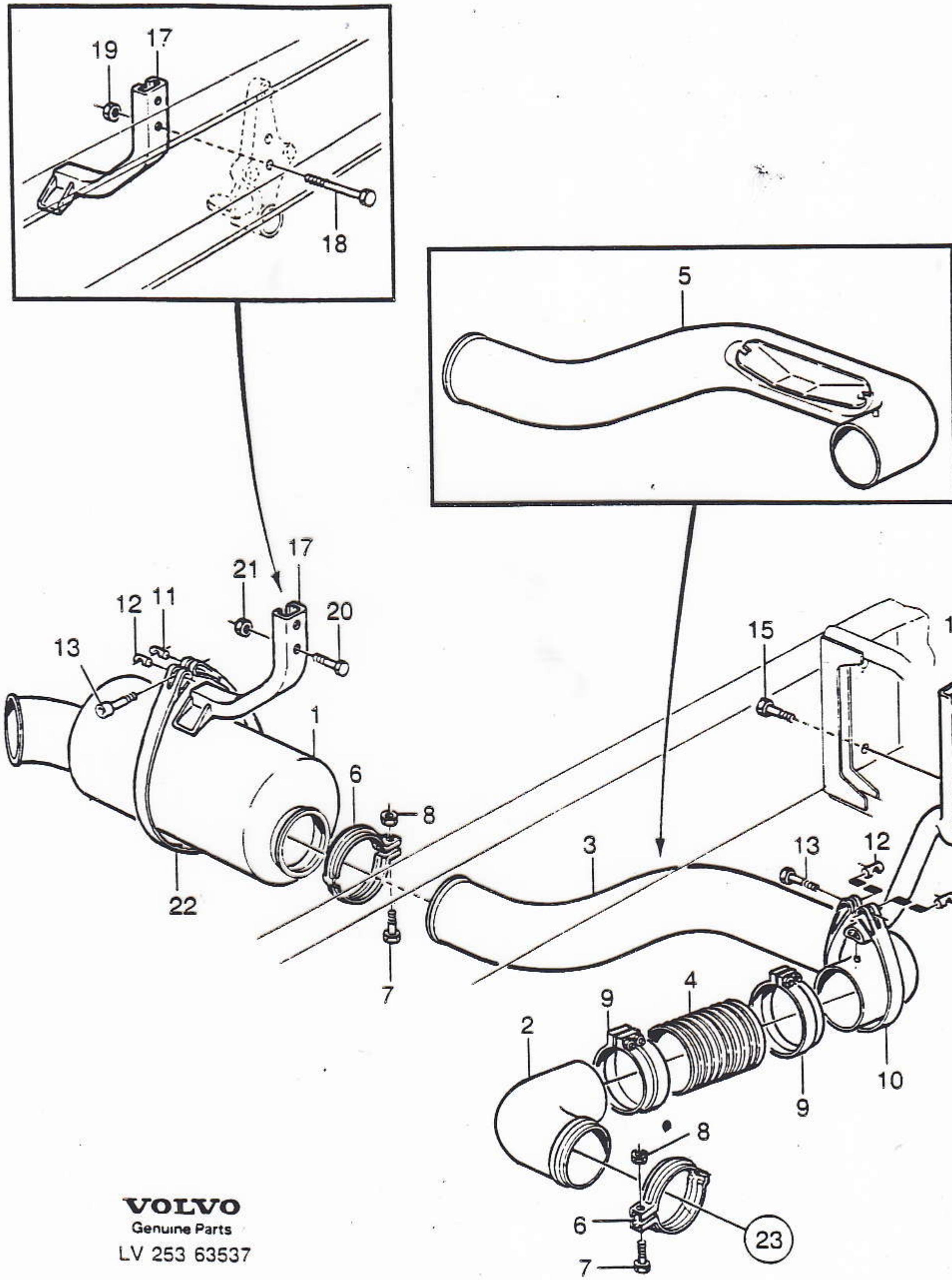
Seja
$$\tilde{\underset{\sim}{D}} = j(\underset{\sim}{\xi}_l)\, w_l\, \underset{\sim}{D}(\underset{\sim}{\xi}_l)$$

então:
$$\underset{\sim}{K}^e \cong \sum_{l=1}^{n_{int}} \left(\underset{\sim}{B}^T \tilde{\underset{\sim}{D}}\, \underset{\sim}{B}\right)_l$$

For $l = 1, \ldots, n_{int}$

- use "shape" para obter
  $j(\underset{\sim}{\xi}_l),\ N_{a,x}(\underset{\sim}{\xi}_l),\ N_{a,y}(\underset{\sim}{\xi}_l)$ $(a = 1, \ldots, n_{en})$
- Monte $\underset{\sim}{B}$
- Calcule $\tilde{\underset{\sim}{D}} = j\, w_l\, \underset{\sim}{D}$
- Calcule $\underset{\sim}{B}^T \tilde{\underset{\sim}{D}}\, \underset{\sim}{B}$, adiciona dentro da expressão para $\underset{\sim}{K}^e$
- Analogamente, calcule o vetor força

$$\underset{\sim}{f}^e = \int_{\square} \underset{\sim}{N} f\, d\square = \sum_{l=1}^{n_{int}} N_a(\underset{\sim}{\xi}_l)\, j(\underset{\sim}{\xi}_l)\, w_l \cdot f(\underset{\sim}{\xi}_l)$$



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