References

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Exercises

6-1. The uniaxial deformation case as shown in Figure 6-1 was used to determine the strain energy under uniform stress with zero body force. Determine this strain energy for the case in which the stress varies continuously as a function of x and also include the effect of a body force F_x. Neglecting higher-order terms, show that the result is the same as previously given by (6.1.4).

6-2. Since the strain energy has physical meaning that is independent of the choice of coordinate axes, it must be invariant to all coordinate transformations. Because U is a *quadratic form* in the strains or stresses, it cannot depend on the third invariants III_e or I_3, and so it must depend only on the first two invariants of the strain or stress tensors. Show that the strain energy can be written in the following forms

$$
\begin{aligned}
U &= \left(\frac{1}{2}\lambda + \mu\right) I_e^2 - 2\mu II_e \\
&= \frac{1}{2E}\left(I_1^2 - 2(1+\nu)I_2\right)
\end{aligned}
$$

6-3. Starting with the general expression (6.1.7), explicitly develop forms (6.1.9) and (6.1.10) for the strain energy density.

6-4. Differentiate the general three-dimensional strain energy form (6.1.9) to show that

$$\sigma_{ij} = \frac{\partial U(e)}{\partial e_{ij}}$$

6-5. Using equations (6.1.12), develop the symmetry relations (6.1.13), and use these to prove the symmetry in the elasticity tensor $C_{ijkl} = C_{klij}$.

6-6. Verify the decomposition of the strain energy into volumetric and deviatoric parts as given by equations (6.1.16) and (6.1.17).

6-7. Starting with relations (6.1.16) and (6.1.17), show that the volumetric and distortional strain energies can be expressed in terms of the invariants of the stress matrix as

$$U_v = \frac{1-2\nu}{6E} I_1^2$$
$$U_d = -\frac{1}{4\mu}\left(I_1^2 + 2I_2\right)$$

Results from Exercise 6-6 may be helpful.

6-8. Show that the distortional strain energy given by (6.1.17) is related to the octahedral shear stress $(3.5.4)_2$ by the relation

$$U_d = \frac{3}{2}\frac{1+\nu}{E}\tau_{oct}^2 = \frac{3}{4\mu}\tau_{oct}^2$$

Results from Exercise 3-5 may be helpful.

6-9. A two-dimensional state of *plane stress* in the x,y-plane is defined by the stress matrix

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determine the strain energy density for this case in terms of these nonzero stress components.

6-10. The stress field for a beam of length $2l$ and depth $2c$ under end bending moments M (see Figure 8-2) is given by

$$\sigma_x = -\frac{3M}{2c^3}\,y\,, \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Determine the strain energy density and show that the total strain energy in the beam is given by

$$U_T = \frac{3M^2 l}{2Ec^3} = \frac{M^2 l}{EI}$$

where I is the section moment of inertia. Assume unit thickness in the z-direction.

6-11. The stress field for the torsion of a rod of circular cross-section is given by

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0,\ \tau_{xz} = -\mu\alpha y,\ \tau_{yz} = \mu\alpha x$$

where α is a constant and the z-axis coincides with the axis of the rod. Evaluate the strain energy density for this case, and determine the total strain energy in a rod with section radius R and length L.

6-12. Using the reciprocal theorem, choose the first state as $u_i^{(1)} = Ax_i,\ F_i^{(1)} = 0,\ T_i^{(1)} = 3kAn_i$, and take the second state as $u_i,\ F_i,\ T_i$ to show that the change in volume of the body is given by

$$\Delta V = \int_V e_{ii}dV = \frac{1}{3k}\left\{\int_V F_i x_i dV + \int_S T_i x_i dS\right\}$$

where A is an arbitrary constant and k is the bulk modulus.

6-13. Rework Example 6-2 using the trigonometric Ritz approximation $w_j = \sin \frac{j\pi x}{l}$. Develop a two-term approximate solution, and compare it with the displacement solution developed in the text. Also compare each of these approximations with the exact solution (6.7.9) at midspan $x = l/2$.

6-14. Using the bending formulae (6.6.9), compare the maximum bending stresses from the cases presented in Example 6-2 and Exercise 6-13. Numerically compare these results with the exact solution; see (6.7.9) at midspan $x = l/2$.