PROBLEMS

2A1. Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{jk}S_{kj}$, (d) $a_m a_m$, (e) $S_{mn}a_m a_n$.

2A2. Determine which of these equations have an identical meaning with $a_i = Q_{ij}a_j'$

(a) $a_p = Q_{pm}a_m'$,

(b) $a_p = Q_{qp}a_q'$,

(c) $a_m = a_n' Q_{mn}$.

2A3. Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad [C_{ij}] = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Demonstrate the equivalence of the following subscripted equations and the corresponding matrix equations.

(a) $D_{ji} = B_{ij} \quad [D] = [B]^T$,

(b) $b_i = B_{ij}a_j \quad [b] = [B][a]$,

(c) $c_j = B_{ji}a_i \quad [c] = [B][a]$,

(d) $s = B_{ij}a_i a_j \quad s = [a]^T[B][a]$,

(e) $D_{ik} = B_{ij}C_{jk} \quad [D] = [B][C]$,

(f) $D_{ik} = B_{ij}C_{kj} \quad [D] = [B][C]^T$.

2A4. Given that $T_{ij} = 2\mu E_{ij} + \lambda(E_{kk})\delta_{ij}$, show that

(a)

$$W = \frac{1}{2}T_{ij}E_{ij} = \mu E_{ij}E_{ij} + \frac{\lambda}{2}(E_{kk})^2$$

(b)

$$P = T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + (E_{kk})^2(4\mu\lambda + 3\lambda^2)$$

2A5. Given

$$[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad [S_{ij}] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

(a) Evaluate $[T_{ij}]$ if $T_{ij} = \varepsilon_{ijk}a_k$

(b) Evaluate $[c_i]$ if $c_i = \varepsilon_{ijk}S_{jk}$

(c) Evaluate $[d_i]$ if $d_k = \varepsilon_{ijk}a_ib_j$ and show that this result is the same as $d_k = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_k$

2A6.

(a) If $\varepsilon_{ijk}T_{jk} = 0$,show that $T_{ij} = T_{ji}$

(b) Show that $\delta_{ij}\varepsilon_{ijk} = 0$

2A7. (a)Verify that

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$$

By contracting the result of part (a) show that

(b)$\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}$

(c)$\varepsilon_{ijk}\varepsilon_{ijk} = 6$

2A8. Using the relation of Problem 2A7a, show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

2A9. (a) If $T_{ij} = -T_{ji}$ show that $T_{ij}a_ia_j = 0$

(b) If $T_{ij} = -T_{ji}$ and $S_{ij} = S_{ji}$, show that $T_{kl}S_{kl} = 0$

2A10. Let $T_{ij} = \frac{1}{2}(S_{ij}+S_{ji})$ and $R_{ij} = \frac{1}{2}(S_{ij}-S_{ji})$, show that

$$S_{ij} = T_{ij}+R_{ij},\ T_{ij} = T_{ji},\ \text{and}\ R_{ij} = -R_{ji}$$

2A11. Let $f(x_1,x_2,x_3)$ be a function of x_i and $v_i(x_1,x_2,x_3)$ represent three functions of x_i. By expanding the following equations, show that they correspond to the usual formulas of differential calculus.

(a) $df = \frac{\partial f}{\partial x_i}dx_i$

(b) $dv_i = \frac{\partial v_i}{\partial x_j}dx_j$

2A12. Let $|A_{ij}|$ denote the determinant of the matrix $[A_{ij}]$. Show that $|A_{ij}| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

2B1. A transformation **T** operates on a vector **a** to give $\mathbf{Ta} = \frac{\mathbf{a}}{|\mathbf{a}|}$, where $|\mathbf{a}|$ is the magnitude of **a**. Show that **T** is not a linear transformation.

2B2. (a) A tensor **T** transforms every vector **a** into a vector $\mathbf{Ta} = \mathbf{m}\times\mathbf{a}$, where **m** is a specified vector. Prove that **T** is a linear transformation.

(b) If $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$, find the matrix of the tensor **T**

2B3. A tensor $\mathbf{T}$ transforms the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ so that

$$\mathbf{T}\mathbf{e}_1 = \mathbf{e}_1+\mathbf{e}_2$$
$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_1-\mathbf{e}_2$$

If $\mathbf{a} = 2\mathbf{e}_1+3\mathbf{e}_2$ and $\mathbf{b} = 3\mathbf{e}_1+2\mathbf{e}_2$, use the linear property of $\mathbf{T}$ to find

(a) $\mathbf{Ta}$ (b) $\mathbf{Tb}$ and (c) $\mathbf{T}(\mathbf{a}+\mathbf{b})$.

2B4. Obtain the matrix for the tensor $\mathbf{T}$ which transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 2\mathbf{e}_1+\mathbf{e}_3$$
$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2+3\mathbf{e}_3$$
$$\mathbf{T}\mathbf{e}_3 = -\mathbf{e}_1+3\mathbf{e}_2$$

2B5. Find the matrix of the tensor $\mathbf{T}$ which transforms any vector $\mathbf{a}$ into a vector $\mathbf{b} = \mathbf{m}(\mathbf{a}\cdot\mathbf{n})$ where

$$\mathbf{m} = \frac{\sqrt{2}}{2}(\mathbf{e}_1+\mathbf{e}_2) \quad \text{and} \quad \mathbf{n} = \frac{\sqrt{2}}{2}(-\mathbf{e}_1+\mathbf{e}_3)$$

2B6. (a) A tensor $\mathbf{T}$ transforms every vector into its mirror image with respect to the plane whose normal is $\mathbf{e}_2$. Find the matrix of $\mathbf{T}$.

b) Do part (a) if the plane has a normal in the $\mathbf{e}_3$ direction instead.

2B7. a) Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_1-axis. Find the matrix of $\mathbf{R}$.

b) Do part (a) if the rotation is about the x_2-axis.

2B8. Consider a plane of reflection which passes through the origin. Let $\mathbf{n}$ be a unit normal vector to the plane and let $\mathbf{r}$ be the position vector for a point in space

(a) Show that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr} = \mathbf{r}-2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{T}$ is the transformation that corresponds to the reflection.

(b) Let $\mathbf{n}=\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$, find the matrix of the linear transformation $\mathbf{T}$ that corresponds to this reflection.

(c) Use this linear transformation to find the mirror image of a vector $\mathbf{a} = \mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$.

2B9. A rigid body undergoes a right hand rotation of angle θ about an axis which is in the direction of the unit vector $\mathbf{m}$. Let the origin of the coordinates be on the axis of rotation and $\mathbf{r}$ be the position vector for a typical point in the body.

(a) Show that the rotated vector of $\mathbf{r}$ is given by $\mathbf{Rr} = (1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta\mathbf{m}\times\mathbf{r}$, where $\mathbf{R}$ is the transformation that corresponds to the rotation.

(b) Let $\mathbf{m}=\frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)$, find the matrix of the linear transformation that corresponds to this rotation.

(c) Use this linear transformation to find the rotated vector of $\mathbf{a} = \mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$.

2B10. (a) Find the matrix of the tensor **S** that transforms every vector into its mirror image in a plane whose normal is $\mathbf{e}_2$ and then by a 45^0 right-hand rotation about the $\mathbf{e}_1$-axis.

(b) Find the matrix of the tensor **T** that transforms every vector by the combination of first the rotation and then the reflection of part (a).

(c) Consider the vector $\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3$, find the transformed vector by using the transformations **S**. Also, find the transformed vector by using the transformation **T**.

2B11. a) Let **R** correspond to a right-hand rotation of angle θ about the x_3-axis.

(a)Find the matrix of $\mathbf{R}^2$.

(b)Show that $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis.

(c)Find the matrix of $\mathbf{R}^n$ for any integer n.

2B12. Rigid body rotations that are small can be described by an orthogonal transformation $\mathbf{R} = \mathbf{I}+\varepsilon\mathbf{R}^*$, where $\varepsilon\to 0$ as the rotation angle approaches zero. Considering two successive rotations $\mathbf{R}_1$ and $\mathbf{R}_2$, show that for small rotations (so that terms containing ε^2 can be neglected) the final result does not depend on the order of the rotations.

2B13. Let **T** and **S** be any two tensors. Show that

(a) $\mathbf{T}^T$is a tensor.

(b) $\mathbf{T}^T+\mathbf{S}^T = (\mathbf{T}+\mathbf{S})^T$

(c) $(\mathbf{TS})^T = \mathbf{S}^T\mathbf{T}^T$.

2B14. Using the form for the reflection in an arbitrary plane of Prob. 2B8, write the reflection tensor in terms of dyadic products.

2B15. For arbitrary tensors **T** and **S**, without relying on the component form, prove that

(a) $(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1}$.

(b) $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$.

2B16. Let **Q** define an orthogonal transformation of coordinates, so that $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$. Consider $\mathbf{e}_i'\cdot\mathbf{e}_j'$ and verify that $Q_{mi}Q_{mj} = \delta_{ij}$.

2B17. The basis $\mathbf{e}_i'$ is obtained by a 30° counterclockwise rotation of the $\mathbf{e}_i$ basis about $\mathbf{e}_3$.

(a) Find the orthogonal transformation **Q** that defines this change of basis, i.e., $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$

(b) By using the vector transformation law, find the components of $\mathbf{a} = \sqrt{3}\,\mathbf{e}_1+\mathbf{e}_2$ in the primed basis (i.e., find a_i')

(c) Do part (b) geometrically.

2B18. Do the previous problem with $\mathbf{e}_i'$ obtained by a 30° clockwise rotation of the $\mathbf{e}_i$-basis about $\mathbf{e}_3$.

2B19. The matrix of a tensor **T** in respect to the basis $\{\mathbf{e}_i\}$ is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find T_{11}', T_{12}' and T_{31}' in respect to a right-hand basis $\mathbf{e}_i'$ where $\mathbf{e}_1'$ is in the direction of $-\mathbf{e}_2+2\mathbf{e}_3$ and $\mathbf{e}_2'$ is in the direction of $\mathbf{e}_1$

2B20. (a) For the tensor of the previous problem, find $[T_{ij}']$ if $\mathbf{e}_i'$ is obtained by a 90° right-hand rotation about the $\mathbf{e}_3$-axis.

(b) Compare both the sum of the diagonal elements and the determinants of $[\mathbf{T}]$ and $[\mathbf{T}]'$.

2B21. The dot product of two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b}_i = b_i\mathbf{e}_i$ is equal to a_ib_i. Show that the dot product is a scalar invariant with respect to an orthogonal transformation of coordinates.

2B22. (a) If T_{ij} are the components of a tensor, show that $T_{ij}T_{ij}$ is a scalar invariant with respect to an orthogonal transformation of coordinates.

(b) Evaluate $T_{ij}T_{ij}$ if in respect to the basis $\mathbf{e}_i$

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{e}_i}$$

(c) Find $[\mathbf{T}]'$ if $\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i$ and

$$[\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{e}_i}$$

(d) Show for this specific $[\mathbf{T}]$ and $[\mathbf{T}]'$ that

$$T_{mn}'T_{mn}' = T_{ij}T_{ij}.$$

2B23. Let $[\mathbf{T}]$ and $[\mathbf{T}]'$ be two matrices of the same tensor **T**, show that

$$\det[\mathbf{T}] = \det[\mathbf{T}]'.$$

2B24. (a) The components of a third-order tensor are R_{ijk}. Show that R_{iik} are components of a vector.

(b) Generalize the result of part (a) by considering the components of a tensor of n^{th} order $R_{ijk\ldots}$ Show that $R_{iik\ldots}$ are components of an $(n\text{-}2)^{th}$ order tensor.

2B25. The components of an arbitrary vector **a** and an arbitrary second-order tensor **T** are related by a triply subscripted quantity R_{ijk} in the manner $a_i = R_{ijk}T_{jk}$ for any rectangular Cartesian basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Prove that R_{ijk} are the components of a third-order tensor.

2B26. For any vector **a** and any tensor **T**, show that

(a) $\mathbf{a}\cdot\mathbf{T}^A\mathbf{a} = 0$,

(b) $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^S\mathbf{a}$.

2B27. Any tensor may be decomposed into a symmetric and antisymmetric part. Prove that the decomposition is unique. (Hint: Assume that it is not unique.)

2B28. Given that a tensor **T** has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) find the symmetric and antisymmetric part of **T**.

(b) find the dual vector of the antisymmetric part of **T**.

2B29 From the result of part (a) of Prob. 2B9, for the rotation about an arbitrary axis **m** by an angle θ,

(a) Show that the rotation tensor is given by $\mathbf{R} = (1-\cos\theta)(\mathbf{mm})+\sin\theta\mathbf{E}$, where **E** is the antisymmetric tensor whose dual vector is **m**. [note **mm** denotes the dyadic product of **m** with **m**].

(b) Find $\mathbf{R}^A$, the antisymmetric part of **R**.

(c) Show that the dual vector for $\mathbf{R}^A$ is given by $\sin\theta\mathbf{m}$

2B30. Prove that the only possible real eigenvalues of an orthogonal tensor are $\lambda=\pm 1$.

2B31. Tensors **T**, **R**, and **S** are related by $\mathbf{T} = \mathbf{RS}$. Tensors **R** and **S** have the same eigenvector **n** and corresponding eigenvalues r_1 and s_1. Find an eigenvalue and the corresponding eigenvector of **T**.

2B32. If **n** is a real eigenvector of an antisymmetric tensor **T**, then show that the corresponding eigenvalue vanishes.

2B33. Let **F** be an arbitrary tensor. It can be shown (Polar Decomposition Theorem) that any invertible tensor **F** can be expressed as $\mathbf{F} = \mathbf{VQ} = \mathbf{QU}$, where **Q** is an orthogonal tensor and **U** and **V** are symmetric tensors.

(b) Show that $\mathbf{VV} = \mathbf{FF}^T$ and $\mathbf{UU} = \mathbf{F}^T\mathbf{F}$.

(c) If λ_i and $\mathbf{n}_i$ are the eigenvalues and eigenvectors of **U**, find the eigenvectors and eigenvectors of **V**.

2B34. (a) By inspection find an eigenvector of the dyadic product **ab**

(b) What vector operation does the first scalar invariant of **ab** correspond to?

(c) Show that the second and the third scalar invariants of **ab** vanish. Show that this indicates that zero is a double eigenvalue of **ab**. What are the corresponding eigenvectors?

2B35. A rotation tensor **R** is defined by the relations

$$\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{R}\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{R}\mathbf{e}_3 = \mathbf{e}_1$$

(a) Find the matrix of **R** and verify that $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and $\det|\mathbf{R}| = 1$.

(b) Find the angle of rotation that could have been used to effect this particular rotation.

2B36. For any rotation transformation a basis $\mathbf{e}_i'$ may be chosen so that $\mathbf{e}_3'$ is along the axis of rotation.

(a) Verify that for a right-hand rotation angle θ, the rotation matrix in respect to the $\mathbf{e}_i'$ basis is

$$[\mathbf{R}]' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i'}$$

(b) Find the symmetric and antisymmetric parts of $[\mathbf{R}]'$.

(c) Find the eigenvalues and eigenvectors of $\mathbf{R}^S$.

(d) Find the first scalar invariant of **R**.

(e) Find the dual vector of $\mathbf{R}^A$.

(f) Use the result of (d) and (e) to find the angle of rotation and the axis of rotation for the previous problem.

2B37. (a) If **Q** is an improper orthogonal transformation (corresponding to a reflection), what are the eigenvalues and corresponding eigenvectors of **Q**?

(b) If the matrix **Q** is

$$[\mathbf{Q}] = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

find the normal to the plane of reflection.

2B38. Show that the second scalar invariant of **T** is

$$I_2 = \frac{T_{ii}T_{jj}}{2} - \frac{T_{ij}T_{ji}}{2}$$

by expanding this equation.

2B39. Using the matrix transformation law for second-order tensors, show that the third scalar invariant is indeed independent of the particular basis.

2B40. A tensor **T** has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the scalar invariants, the principle values and corresponding principal directions of the tensor **T**.

(b) If $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ are the principal directions, write $[\mathbf{T}]_{\mathbf{n}_i}$.

(c) Could the following matrix represent the tensor **T** in respect to some basis?

$$\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2B41. Do the previous Problem for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

2B42. A tensor **T** has a matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the principal values and three mutually orthogonal principal directions.

2B43. The inertia tensor $\bar{\mathbf{I}}_o$ of a rigid body with respect to a point o, is defined by

$$\bar{\mathbf{I}}_o = \int (r^2\mathbf{I} - \mathbf{rr})\rho dV$$

where **r** is the position vector, $r = |\mathbf{r}|, \rho =$ mass density, **I** is the identity tensor, and dV is a differential volume. The moment of inertia, with respect to an axis pass through o, is given by $\bar{\mathbf{I}}_{nn} = \mathbf{n}\cdot\bar{\mathbf{I}}_o\mathbf{n}$, (no sum on n), where **n** is a unit vector in the direction of the axis of interest.

(a) Show that $\bar{\mathbf{I}}_o$ is symmetric.

(b) Letting $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, write out all components of the inertia tensor $\bar{\mathbf{I}}_o$.

(c) The diagonal terms of the inertia matrix are the moments of inertia and the off-diagonal terms the products of inertia. For what axes will the products of inertia be zero? For which axis will the moments of inertia be greatest (or least)?

Let a coordinate frame $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be attached to a rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$. Then, the angular momentum vector $\mathbf{H}_c$, in respect to the mass center, is given by

$$\mathbf{H}_c = \bar{\mathbf{I}}_c\boldsymbol{\omega}$$

and

$$\frac{d\mathbf{e}_i}{dt} = \boldsymbol{\omega}\times\mathbf{e}_i.$$

(d) Let $\boldsymbol{\omega} = \omega_i\mathbf{e}_i$ and demonstrate that

$$\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d\omega_i}{dt}\mathbf{e}_i$$

and that

$$\dot{\mathbf{H}}_c = \frac{d}{dt}\mathbf{H}_c = \bar{\mathbf{I}}_c\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times(\bar{\mathbf{I}}_c\boldsymbol{\omega})$$

2C1. Prove the identities (2C1.2a-e) of Section 2C1.

2C2. Consider the scalar field defined by $\phi = x^2+3xy+2z$.

(a) Find a unit normal to the surface of constant ϕ at the origin (0,0,0).

(b) What is the maximum value of the directional derivative of ϕ at the origin?

(c) Evaluate $d\phi/dr$ at the origin if $d\mathbf{r} = ds(\mathbf{e}_1+\mathbf{e}_3)$.

2C3. Consider the ellipsoid defined by the equation $x^2/a^2+y^2/b^2+z^2/c^2=1$. Find the unit normal vector at a given position (x,y,z).

2C4. Consider a temperature field given by $\theta = 3xy$.

(a) Find the heat flux at the point $A(1,1,1)$ if $\mathbf{q} = -k\nabla\theta$.

(b) Find the heat flux at the same point as part (a) if $\mathbf{q} = -\mathbf{K}\nabla\theta$, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

2C5. Consider an electrostatic potential given by $\phi = \alpha[x\cos\theta+y\sin\theta]$, where α and θ are constants.

(a) Find the electric field $\mathbf{E}$ if $\mathbf{E} = -\nabla\phi$.

(b) Find the electric displacement $\mathbf{D}$ if $\mathbf{D} = \varepsilon\mathbf{E}$, where the matrix of ε is

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

(c) Find the angle θ for which the magnitude of $\mathbf{D}$ is a maximum.

2C6. Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be scalar fields, and let $\mathbf{v}(x,y,z)$ and $\mathbf{w}(x,y,z)$ be vector fields. By writing the subscripted component form, verify the following identities:

(a) $\nabla(\phi+\psi) = \nabla\phi+\nabla\psi$

Sample solution:

$$[\nabla(\phi+\psi)]_i = \frac{\partial}{\partial x_i}(\phi+\psi) = \frac{\partial\phi}{\partial x_i}+\frac{\partial\psi}{\partial x_i} = (\nabla\phi)_i+(\nabla\psi)_i$$

(b) $\mathrm{div}(\mathbf{v}+\mathbf{w}) = \mathrm{div}\mathbf{v}+\mathrm{div}\mathbf{w}$,

(c) $\mathrm{div}(\phi\mathbf{v}) = (\nabla\phi)\cdot\mathbf{v}+\phi(\mathrm{div}\mathbf{v})$,

(d) $\mathrm{curl}(\nabla\phi) = 0$,

(e) $\mathrm{div}(\mathrm{curl}\mathbf{v}) = 0$.

2C7. Consider the vector field $\mathbf{v} = x^2\mathbf{e}_1 + z^2\mathbf{e}_2 + y^2\mathbf{e}_3$. For the point (1, 1, 0):

(a) Find the matrix of $\nabla\mathbf{v}$.

(b) Find the vector $(\nabla\mathbf{v})\mathbf{v}$.

(c) Find div $\mathbf{v}$ and curl $\mathbf{v}$.

(d) if $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, find the differential $d\mathbf{v}$.

2D1. Obtain Eq. (2D1.15)

2D2. Calculate div $\mathbf{u}$ for the following vector field in cylindrical coordinates:

(a) $u_r = u_\theta = 0$, $u_z = A + Br^2$,

(b) $u_r = \frac{\sin\theta}{r}$, $u_\theta = 0$, $u_z = 0$,

(c) $u_r = \frac{1}{2}\sin\theta r^2$, $u_\theta = \frac{1}{2}\cos\theta r^2$, $,u_z = 0$,

(d) $u_r = \frac{\sin\theta}{r^2}$, $u_\theta = -\frac{\cos\theta}{r^2}$, $u_z = 0$.

2D3. Calculate div $\mathbf{u}$ for the following vector field in spherical coordinates:

$$u_r = Ar + \frac{B}{r^2},\quad u_\theta = u_\phi = 0$$

2D4. Calculate $\nabla\mathbf{u}$ for the following vector field in cylindrical coordinate

$$u_r = \frac{A}{r},\quad u_\theta = Br,\quad v_z = 0$$

2D5. Calculate $\nabla\mathbf{u}$ for the following vector field in spherical coordinate

$$u_r = Ar + \frac{B}{r^2},\quad u_\theta = u_\phi = 0$$

2D6. Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = \frac{Az}{R^3} - \frac{3r^2z}{R^5}, \quad T_{\theta\theta} = \frac{Az}{R^3}, \quad T_{zz} = -\left[\frac{Az}{R^3} + \frac{3z^3}{R^5}\right], \quad T_{rz} = -\left[\frac{Ar}{R^3} + \frac{3rz^2}{R^5}\right]$$

$$T_{z\theta} = T_{r\theta} = 0, \text{ where } R^2 = r^2 + z^2$$

2D7. Calculate div **T** for the following tensor field in cylindrical coordinates:

$$T_{rr} = A + \frac{B}{r^2}, \quad T_{\theta\theta} = A - \frac{B}{r^2}, \quad T_{zz} = \text{constant}, \quad T_{r\theta} = T_{rz} = T_{\theta z} = 0$$

2D8. Calculate div **T** for the following tensor field in spherical coordinates:

$$T_{rr} = A - \frac{2B}{r^3}, \quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3}$$

$$T_{\theta r} = T_{\phi r} = T_{\phi\phi} = 0$$

2D9. Derive Eq. (2D3.24b) and Eq. (2D3.24c).