ANALYTICAL EFFECTIVE AND APPARENT ORDER FOR ESTIMATING THE DISCRETIZATION ERROR

Carlos H. Marchi*, Marcio A. V. Pinto*, Luciane Grossi, Giuliano G. La Guardia****

*Department of Mechanics Engineering, Federal University of Paraná, CP 19.011, 81.531-990, Curitiba, PR, Brazil.
e-mail: marchi@ufpr.br; marcio_villela@yahoo.com.br

**Department of Mathematics and Statistics, State University of Ponta Grossa, 84030-900, Ponta Grossa-PR, Brazil.
e-mail: lgbombacini@gmail.com; gguardia@uepg.br

Abstract
Numerical solutions of one equation in fluid dynamics are obtained by the finite difference method with uniform grids and four types of numerical approximations. Analytical effective as well as analytical apparent orders for estimating the discretization and truncation error are established. These analytical orders are investigated for the case in which the size of the grid element is small.

1 - INTRODUCTION

When the error of the numerical solution is caused only by truncated errors (ε), the difference between the analytical exact solution (Φ) of a variable and its numerical solution (ϕ) is called discretization error (E), and is defined by

$$E(\phi) = \Phi - \phi, \tag{1}$$

where the symbols Φ and ϕ represents, respectively, the analytical and numerical solutions of the variables.

Beyond Eq. (1), other method to compute the discretization error is given by (Ferziger and Peric, 1999)

$$E(\phi) = C_1 h^{p_L} + C_2 h^{p_2} + C_3 h^{p_3} + \ldots, \tag{2}$$

where each $C_i \in \mathbf{R}$, $i = 1, 2, 3, \ldots$ are the coefficients. Such coefficients can be positive or negative and also they can depend on the dependent variable (Λ) and their respective derivatives, but they do not depend on the length (h) of the elements of the grid. Equation (2) is called *general equation of the discretization error.*

By means of Eq. (1) or Eq. (2) one can see that the discretization error value can be only computed when the analytical solution of the mathematical model is known. However, in most of cases, the analytical solution is not known, consequently, it is necessary to estimate such analytical solution. Thus, instead of computing the discretization error one computes the estimation of its value. Such estimation is also called uncertainty U of the numerical solution ϕ (Mehta, 1996; Chapra and Canale, 1994):

$$U(\phi) = \phi_\infty - \phi. \tag{3}$$

The Richardson's Estimator $U_{Ri}(\phi)$ (Richardson and Gaunt, 1927; Blottner, 1990) is given by

$$U_{Ri}(\phi)=\phi_\infty-\phi\,, \tag{4}$$

where ϕ represents the numerical solution of the variable of interest and ϕ_∞ is a estimation of the value of the analytical solution; ϕ_∞ is obtained from the generalized Richardson's extrapolation (Roache, 1994) and is given by

$$\phi_\infty=\phi_1+\frac{\phi_1-\phi_2}{q^{p_L}-1}\,, \tag{5}$$

where ϕ_1 and ϕ_2 are the corresponding numerical solution of the fine and coarse grid, respectively, whose length *h* of the elements are h_1 and h_2, p_L is the asymptotic order of the discretization error and *q* is the grid refinement ratio defined by

$$q=\frac{h_2}{h_1}\,. \tag{6}$$

Introducing Eq. (5) into Eq. (4), the Richardson's Estimator becomes

$$U_{Ri}(\phi)=\frac{\phi_1-\phi_2}{q^{p_L}-1}\,. \tag{7}$$

The expression for the Richardson's extrapolation is given by considering that the uncertainty *U* of a numerical solution ϕ that depends on a constant K_U (independent of *h*) and p_L, the asymptotic order of the discretization error.

In the following sections mathematical model, the type of variables, the numerical approximations and their truncation error used in this work are introduced. The expressions are deduced for analytical effective and apparent order. Finally, the results and conclusion of the work are presented.

2. MATHEMATICAL AND NUMERICAL MODELS

The solution of mathematical models of interest is obtained by means of numerical approximation of each one of their terms. For this, we consider that the nodal values utilized in such numerical approximation are obtained by means of analytical solutions, that is, the error in each node is equal to zero.

The truncation error ε of a variable ϕ is given by

$$\varepsilon(\phi)=\Phi-\phi\,, \tag{8}$$

where Φ is the analytical solution of ϕ, where ϕ is its approximated value. Equation (8) can be represented generically by

$$\varepsilon(\phi)=C_1h^{p_L}+C_2h^{p_2}+C_3h^{p_3}+\ldots\,, \tag{9}$$

where each $C_i\in\mathbf{R}$, $i=1,2,3,\ldots$, are the coefficients. Such coefficients can be positive or negative and they can depend on the dependent variable (Λ) and their respective derivatives, but they do not depend on the length *h*. The Equation (9) is called *general equation of the truncation error.*

When the numerical solution error is caused only by truncation error, the discretization error E (see Eq. (1)), coincides with the truncation error ε (see Eq. (8)). If the exact analytical solution of Φ and its approximated value ϕ are known, the value for the truncation error can be derived by two distinct methods. The first way is by direct application of Eq. (8) and the second one is by replacing in Eq. (8) the exact analytical solution Φ by its Fourier series and also replacing the expression utilized for the numerical approximation ϕ.

Table 1 shows two types of variables for which are presented the types of numerical approximations utilized in this paper as well as the symbols employed to denote the analytical solution (Φ) and the numerical solution (ϕ). The variable considered is the first order derivative of the dependent variable. Its analytical solution is denoted by Λ^i and its numerical approximations are performed of four different manners, denoted by λ^i_{UDS}, λ^i_{CDS}, λ^i_{DDS} and λ^i_{DDS-2}, respectively.

Table 1. Definition of the approximations utilized in this paper.

Type of Variable	Analytical solution (Φ)	Numerical solution (ϕ)	Types of numerical approximations
First order derivative of the dependent variable	Λ^i	λ^i_{UDS}	one-point upstream
		λ^i_{CDS}	central difference
		λ^i_{DDS}	one-point downstream
		λ^i_{DDS-2}	two-point downstream

Numerical approximations and their truncation errors can be obtained from the Taylor series, which is an infinite series defined by (Kreyszig, 2006)

$$\Lambda_x = \sum_{n=0}^{\infty} \Lambda^n_j \frac{(x - x_j)^n}{n!}, \tag{10}$$

where Λ denotes the dependent variable of the mathematical models, Λ_x is the exact analytical value obtained at coordinate x with a Taylor series expansion from the node j, where the exact analytical value of Λ_j and its derivatives ($\Lambda^i_j, \Lambda^{ii}_j, \ldots, \Lambda^n_j$) are known. Equation (10) is valid if Λ is a continuous function of x in the closed interval $[x, x_j]$ and there are continuous derivatives up to the order n in this same interval.

Applying Eq. (10) to the nodes x_{j-1} and x_{j+1} on the uniform grid, one obtains

$$\Lambda_{j-1} = \Lambda_j - \Lambda^i_j h + \Lambda^{ii}_j \frac{h^2}{2} - \Lambda^{iii}_j \frac{h^3}{6} + \ldots \tag{11}$$

$$\Lambda_{j+1} = \Lambda_j + \Lambda^i_j h + \Lambda^{ii}_j \frac{h^2}{2} + \Lambda^{iii}_j \frac{h^3}{6} + \ldots \tag{12}$$

where x_j is a generic node used to perform the numerical approximations; $h = x_j - x_{j-1}$ denotes the grid spacing and Λ_j denotes Λ_{x_j}.

Numerical approximations for the variable Λ^i given in Tab. (1) and others are presented, for example, in Fletcher (1997), Ferziger and Peric (1999), and Tannehill *et al*. (1997). Those use in this work are shown below.

Subtracting Eq. (11) from Eq. (12), one gets an exact analytical expression for the first derivative of the dependent variable at node *j* in the following form:

$$\Lambda_j^i = \frac{\left(\Lambda_{j+1} - \Lambda_{j-1}\right)}{2h} - \Lambda_j^{iii}\frac{h^2}{6} - \Lambda_j^v\frac{h^4}{120} - \Lambda_j^{vii}\frac{h^6}{5{,}040} - \ldots \tag{13}$$

where Λ_j^{iii}, Λ_j^v e Λ_j^{vii} are, respectively, the third, fifth and seventh derivatives of the dependent variable at node *j*. Equation (13) can be rewritten as

$$\Lambda_j^i = \left(\lambda_{CDS}^i\right)_j + \varepsilon\left(\lambda_{CDS}^i\right)_j ,$$

where the first term on the right-hand side of this is the numerical approximation computed by applying central difference for the first derivative, that is,

$$\left(\lambda_{CDS}^i\right)_j = \frac{\left(\Lambda_{j+1} - \Lambda_{j-1}\right)}{2h} , \tag{14}$$

and the remaining terms are the truncation error of λ_{CDS}^i, given by

$$\varepsilon\left(\lambda_{CDS}^i\right)_j = -\Lambda_j^{iii}\frac{h^2}{6} - \Lambda_j^v\frac{h^4}{120} - \Lambda_j^{vii}\frac{h^6}{5{,}040} - \ldots \tag{15}$$

Comparing Eq. (9) with Eq. (15), it can be observed that the true orders of $\varepsilon\left(\lambda_{CDS}^i\right)_j$ are p_V = 2, 4, 6, and so on. Thus, its asymptotic order equals $p_L = 2$. So, one says that the truncation error of λ_{CDS}^i is of second order. Furthermore, $C_1 = -\Lambda_j^{iii}/6$, $C_2 = -\Lambda_j^v/120$, $C_3 = -\Lambda_j^{vii}/5{,}040$, and so on, that is, the coefficients C_i are functions that depends on *x* and also depends on derivatives of the dependent variable.

In a similar way used to deduce Eq. (14), numerical approximations for the first derivative by one-point upstream, one-point downstream and two-point downstream are given, respectively, by

$$\left(\lambda_{UDS}^i\right)_j = \frac{\left(\Lambda_{j+1} - \Lambda_j\right)}{h} ,$$

$$\left(\lambda_{DDS}^i\right)_j = \frac{\left(\Lambda_j - \Lambda_{j-1}\right)}{h} ,$$

$$\left(\lambda_{DDS-2}^i\right)_j = \frac{\left(4\Lambda_{j+1} - 3\Lambda_j - \Lambda_{j-2}\right)}{2h} .$$

Table 2 shows a summary of the true and asymptotic orders of the truncation errors expected for the numerical approximations presented in this work.

Table 2. Expected values for the truncation error orders.

Numerical solution (ϕ)	Types of numerical approximation	True orders (p_V)	Asymptotic order (p_L)
λ^i_{UDS}	one-point upstream	1, 2, 3, ...	1
λ^i_{CDS}	central difference	2, 4, 6, ...	2
λ^i_{DDS}	one-point downstream	1, 2, 3, ...	1
λ^i_{DDS-2}	two-point downstream	2, 3, 4, ...	2

3.2 Apparent order

According to Eq. (19) and Eq. (20), it is necessary to know the exact analytical solution Φ to compute the effective order p_E. However, in practical cases, when the analytical solution is not known, the asymptotic order p_L is verified by means of the *apparent order* p_U, given in the following.

The apparent order p_U is defined as the local inclination of the uncertainty curve *U* of the numerical solution ϕ *versus* the length *h* of the elements of grid in logarithmic graphic. More formally, the apparent order is given by

$$U(\phi) = K_U h^{p_U} , \tag{25}$$

where K_U is a coefficient that does not depend on *h*. Since $U(\phi) = \phi_\infty - \phi$ then

$$K_U h^{p_U} = \phi_\infty - \phi . \tag{26}$$

By applying Eq. (26) to three different solutions ϕ_1, ϕ_2 and ϕ_3 with lengths h_1, h_2 and h_3, respectively, one obtains

$$K_U h_1^{p_U} = \phi_\infty - \phi_1 , \tag{27}$$

$$K_U h_2^{p_U} = \phi_\infty - \phi_2 , \tag{28}$$

$$K_U h_3^{p_U} = \phi_\infty - \phi_3 . \tag{29}$$

Solving such system of equations it follows that

$$p_U = \frac{\log(\psi_U)}{\log(q)} , \tag{30}$$

where

$$q = \frac{h_3}{h_2} = \frac{h_2}{h_1} , \tag{31}$$

and

$$\psi_U = \frac{\phi_2 - \phi_3}{\phi_1 - \phi_2}. \tag{32}$$

Replacing Eq. (1) in Eq. (32) it follows that (to simplify the notation we consider that $E_1 = E(\phi_1)$, $E_2 = E(\phi_2)$ e $E_3 = E(\phi_3)$)

$$\psi_U = \frac{[(\Phi - E_2) - (\Phi - E_3)]}{[(\Phi - E_1) - (\Phi - E_2)]} = \frac{E_3 - E_2}{E_2 - E_1} \tag{33}$$

and thus we one has

$$\psi_U = \frac{\left(\frac{E_3}{E_2} - 1\right)}{\left(1 - \frac{E_1}{E_2}\right)}. \tag{34}$$

For the Richardson's Estimator U_{Ri} (see Eq. (4)), the uncertainty of the numerical solution ϕ_1 obtained in the fine grid of length h_1 is given by

$$U_{Ri}(\phi) = \frac{\phi_1 - \phi_2}{q^{p_U} - 1}. \tag{35}$$

Applying the general equation of the error discretization E (see Eq. (2)), in three grids of different lengths h_1, h_2 and h_3 (fine, coarse and super-coarse) one has

$$E_1 = C_1 h_1^{p_L} + C_2 h_1^{p_2} + C_3 h_1^{p_3} + \ldots\ , \tag{36}$$

$$E_2 = C_1 h_2^{p_L} + C_2 h_2^{p_2} + C_3 h_2^{p_3} + \ldots, \tag{37}$$

$$E_3 = C_1 h_3^{p_L} + C_2 h_3^{p_2} + C_3 h_3^{p_3} + \ldots \tag{38}$$

Since the equalities $h_3 = q h_2$ and $h_1 = \frac{h_2}{q}$ hold (see Eq. (31)), replacing such equations in Eq. (36) and Eq. (38), respectively, it implies that

$$E_1 = \frac{h_2^{p_L}}{q^{p_L}}\left[C_1 + C_2\left(\frac{h_2}{q}\right)^{p_2 - p_L} + C_3\left(\frac{h_2}{q}\right)^{p_3 - p_L} + \ldots\right], \tag{39}$$

$$E_3 = (q h_2)^{p_L}\left[C_1 + C_2 (q h_2)^{p_2 - p_L} + C_3 (q h_2)^{p_3 - p_L} + \ldots\right]. \tag{40}$$

From Eq. (37) and Eq. (39) one obtains

$$\frac{E_1}{E_2} = \frac{1}{q^{p_L}} \frac{\left[C_1 + C_2\left(h_2 / q\right)^{p_2-p_L} + C_3\left(h_2 / q\right)^{p_3-p_L} + \ldots\right]}{\left[C_1 + C_2 h_2^{\,p_2-p_L} + C_3 h_2^{\,p_3-p_L} + \ldots\right]} = \frac{1}{q^{p_L}} \frac{A}{B} \qquad (41)$$

and from Eq. (37) and Eq. (40) one obtains

$$\frac{E_3}{E_2} = q^{p_L} \frac{\left[C_1 + C_2(qh_2)^{p_2-p_L} + C_3(qh_2)^{p_3-p_L} + \ldots\right]}{\left[C_1 + C_2 h_2^{\,p_2-p_L} + C_3 h_2^{\,p_3-p_L} + \ldots\right]} = q^{p_L} \frac{C}{B}, \qquad (42)$$

where

$$A = \left[C_1 + C_2\left(h_2 / q\right)^{p_2-p_L} + C_3\left(h_2 / q\right)^{p_3-p_L} + \ldots\right],$$

$$B = \left[C_1 + C_2 h_2^{\,p_2-p_L} + C_3 h_2^{\,p_3-p_L} + \ldots\right],$$

$$C = \left[C_1 + C_2(qh_2)^{p_2-p_L} + C_3(qh_2)^{p_3-p_L} + \ldots\right].$$

Replacing Eq. (41) and Eq. (42) in Eq. (34) one obtains

$$\psi_U = q^{p_L} \frac{(q^{p_L} C - B)}{(q^{p_L} B - A)}. \qquad (43)$$

Replacing Eq. (43) in Eq. (30) it follows that

$$p_U = p_L + \Delta p_U, \qquad (44)$$

where $\Delta p_U = \log(K) / \log(q)$ and K is given by $K = (q^{p_L} C - B) / (q^{p_L} B - A)$.

When $q \to 1$ it follows that $K \to 1$, resulting in indetermination of Δp_U. Eliminating this indetermination one obtains

$$\Delta p_U = 2 \frac{(C_2 p_2 (p_2 - p_L) h^{p_2-p_L} + C_3 p_3 (p_3 - p_L) h^{p_3-p_L} + \ldots)}{(C_1 p_L + C_2 p_2 h^{p_2-p_L} + C_3 p_3 h^{p_3-p_L} + \ldots)}, \qquad (45)$$

where $h = h_2$ is the length of the grid. Equation (44) and Eq. (45) are used for the computation of the apparent order.

Based on the analysis of Section 3.1.1, we conclude that the possible values for the apparent order are the same of the effective order.

Remark: Note that the effective order p_U equals the asymptotic order p_L (for each h) if the general equation of the error discretization E consists of only one term. However, when $h \to 0$ it follows that $p_U \to p_L$.

4. EXAMPLES

In the practical cases of CFD, that is, when it is desirable to obtain the numerical solution for a specific problem, the analytical solution is unknown. However, for the examples shown in this section, we consider that the analytical solution of the dependent variable (Λ) and their respective derivatives are known. Here we give some examples of the adopted procedure to compute truncation and discretization errors by applying Taylor series.

The function used is

$$\Lambda = x^4 ,$$

whose its respective derivatives are given by:

$$\Lambda^i = 4x^3 ,$$

$$\Lambda^{ii} = 12x^2 ,$$

$$\Lambda^{iii} = 24x ,$$

$$\Lambda^{iv} = 24 ,$$

$$\Lambda^v = \Lambda^{vi} = \ldots = 0 .$$

We present some examples of applications to compute the effective p_E, Eq. (23), and the apparent order p_U, Eq. (44), both based on only one numerical solution. The computations of p_E and p_U are applied to four numerical approximations (λ^i_{UDS}, λ^i_{CDS}, λ^i_{DDS} and λ^i_{DDS-2}) and the results are exhibited in Tabs. 3, 5, 6, 8 and 10 and in Figs. 1 to 4. The differences $|p_E - p_L|$ and $|p_U - p_L|$ are shown in Tabs. 4, 7, 9 and 11. Examples 1 to 4, we consider only the node $x_j = 8$ and $h = 4$, 2, 1, $1/2$, $1/4$, ..., $1/256$. Example 5, we consider only the node $x_j = 0$ and $h = 1$, $1/2$, $1/3$, $1/4$, $1/5$, ..., $1/15$, $1/16$, ..., $1/50$.

Example 1

For the numerical approximation λ^i_{UDS}, one has:

$$\varepsilon(\lambda^i_{UDS}) = 6x^2 h - 4xh^2 + 1h^3 , \tag{47}$$

where $C_1 = 6x^2$ and $p_L = 1$, $C_2 = -4x$ and $p_2 = 2$ and $C_3 = 1$ and $p_3 = 3$.Therefore,

$$p_E(\lambda^i_{UDS}) = 1 + \Delta p \text{, where } \Delta p = \frac{2h(h-16)}{(384 - 32h + h^2)} ,$$

$$p_U(\lambda^i_{UDS}) = 1 + \Delta p \text{, where } \Delta p = \frac{4h(3h-32)}{(384 - 64h + 3h^2)} .$$

Table 3. Effective (p_E) and apparent order (p_U) of the uncertainty for the numerical approximation of the first order derivative with one-point upstream (λ^i_{UDS}). Asymptotic order $p_L = 1$.

H	p_E	p_U
4.000000000E+00	0.64705882352941	-0.81818181818182
2.000000000E+00	0.82716049382716	0.22388059701493
1.000000000E+00	0.91501416430595	0.64086687306502
5.000000000E-01	0.95790902919212	0.82707299787385
2.500000000E-01	0.97905933189297	0.91512476659311
1.250000000E-01	0.98955635047901	0.95795072090414
6.250000000E-02	0.99478490280490	0.97907136909840
3.125000000E-02	0.99739414012999	0.98955955323886
1.561500000E-02	0.99869749308799	0.99478572717114
7.812500000E-03	0.99934885240405	0.99739434914989
3.906250000E-03	0.99967445268003	0.99869754570708

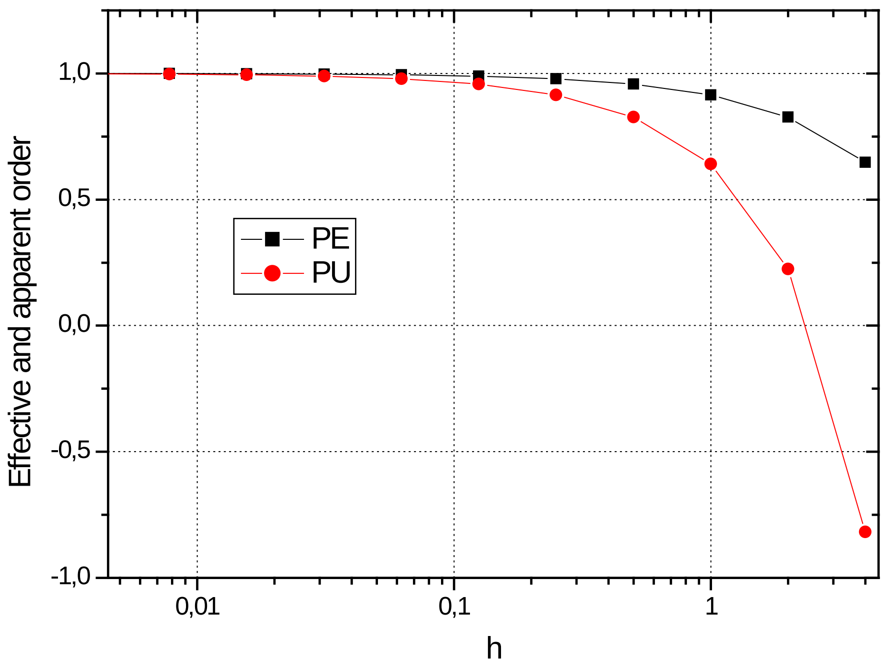

Figure 1. Effective (p_E) and apparent error order (p_U) of the uncertainty of $\varepsilon(\lambda^i_{UDS})$.

Example 2

For the numerical approximation λ^i_{CDS}, one has:

$$\varepsilon(\lambda^i_{CDS}) = -4xh^2, \tag{48}$$

where $C_1 = -4x$ and $p_L = 2$. Thus,

$$p_E(\lambda^i_{CDS}) = 2 + \Delta p \text{, where } \Delta p = 0,$$

$$p_U(\lambda^i_{CDS}) = 2 + \Delta p \text{, where } \Delta p = 0 .$$

According to Tab. 5, for the numerical approximation λ^i_{CDS} we have $p_E = p_U = 2$. Therefore, they do not depend on *h*.

Table 5. Effective (p_E) and apparent error order (p_U) of the uncertainty for the numerical approximation of the first order derivative with central difference (λ^i_{CDS}). Asymptotic order $p_L = 2$.

h	p_E	p_U
4.000000000E+00	2.000000000E+00	2.000000000E+00
2.000000000E+00	2.000000000E+00	2.000000000E+00
1.000000000E+00	2.000000000E+00	2.000000000E+00
5.000000000E-01	2.000000000E+00	2.000000000E+00
2.500000000E-01	2.000000000E+00	2.000000000E+00
1.250000000E-01	2.000000000E+00	2.000000000E+00
6.250000000E-02	2.000000000E+00	2.000000000E+00
3.125000000E-02	2.000000000E+00	2.000000000E+00
1.561500000E-02	2.000000000E+00	2.000000000E+00
7.812500000E-03	2.000000000E+00	2.000000000E+00
3.906250000E-03	2.000000000E+00	2.000000000E+00

Example 3

For the numerical approximation λ^i_{DDS}, one has:

$$\varepsilon(\lambda^i_{DDS}) = -6x^2h - 4xh^2 - 1h^3 , \tag{49}$$

where $C_1 = -6x^2$, $p_L = 1$, $C_2 = -4x$, $p_2 = 2$, $C_3 = -1$ and $p_3 = 3$. Thus,

$$p_E(\lambda^i_{DDS}) = 1 + \Delta p \text{, where } \Delta p = \frac{2h(h+16)}{(384 + 32h + h^2)} ,$$

$$p_U(\lambda^i_{DDS}) = 1 + \Delta p \text{, where } \Delta p = \frac{4h(3h+32)}{(384 + 64h + 3h^2)} .$$

Table 6. Effective (p_E) and apparent error order (p_U) of the uncertainty for the numerical approximation of the first order derivative with one-point downstream (λ^i_{DDS}). Asymptotic order $p_L = 1$.

h	p_E	p_U
4.000000000E+00	1.30303030303030	2.02325581395349
2.000000000E+00	1.15929203539823	1.58015267175573
1.000000000E+00	1.08153477218225	1.31042128603104
5.000000000E-01	1.04122423485322	1.16076784643071
2.500000000E-01	1.02072373664913	1.08183663907543
1.250000000E-01	1.01038940120002	1.04128970547208
6.250000000E-02	1.00520153414898	1.02073874218522
3.125000000E-02	1.00260246904812	1.01039297487521
1.561500000E-02	1.00130165920276	1.00520240487570
7.812500000E-03	1.00065093566840	1.00260268386296
3.906250000E-03	1.00032549433807	1.00130171254621

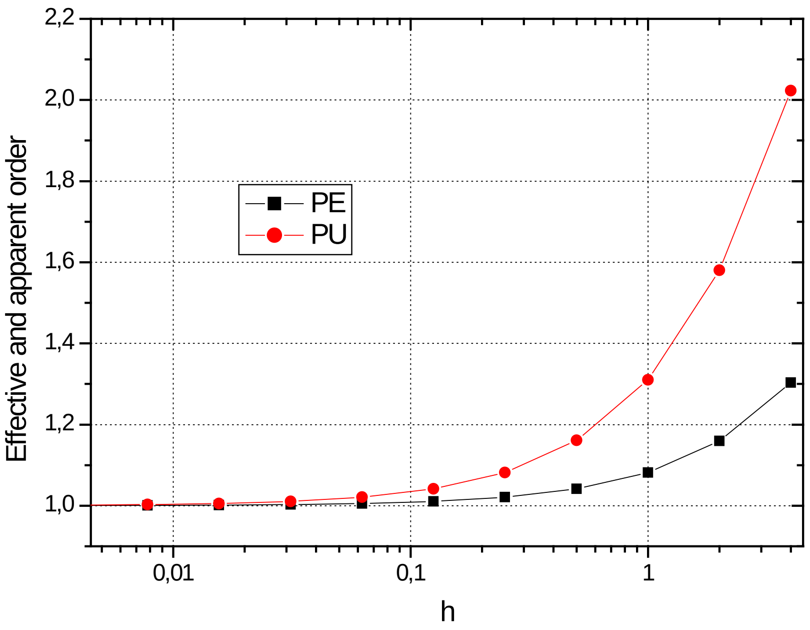

Figure 2. Effective (p_E) and apparent error order (p_U) of the uncertainty of $\varepsilon(\lambda^i_{DDS})$.

Example 4

For the numerical approximation λ^i_{DDS-2}, one has:

$$\varepsilon(\lambda^i_{DDS-2}) = 8xh^2 + 6h^3, \tag{50}$$

where $C_1 = 8x$ e $p_L = 2$ e $C_2 = 6$ e $p_2 = 3$. Thus,

$$p_E(\lambda^i_{DDS-2}) = 1 + \Delta p \text{, where } \Delta p = \frac{6h}{(64+6h)},$$

$$p_U(\lambda^i_{DDS-2}) = 1 + \Delta p \text{, where } \Delta p = \frac{18h}{(64+9h)}.$$

Table 8. Effective (p_E) and apparent error order (p_U) of the uncertainty for the numerical approximation of the first order derivative two-point downstream (λ^i_{DDS-2}). Asymptotic order $p_L = 2$.

h	p_E	p_U
4.000000000E+00	2.17647058823529	2.72000000000000
2.000000000E+00	2.12000000000000	2.43902439024390
1.000000000E+00	2.07317073170732	2.24657534246575
5.000000000E-01	2.04109589041096	2.13138686131387
2.500000000E-01	2.02189781021898	2.06792452830189
1.250000000E-01	2.01132075471698	2.03454894433781
6.250000000E-02	2.00575815738964	2.01742497579864
3.125000000E-02	2.00290416263311	2.00875060768109
1.561500000E-02	2.00145843461351	2.00438489646772
7.812500000E-03	2.00073081607795	2.00219485428606
3.906250000E-03	2.00036580904768	2.00109802964680

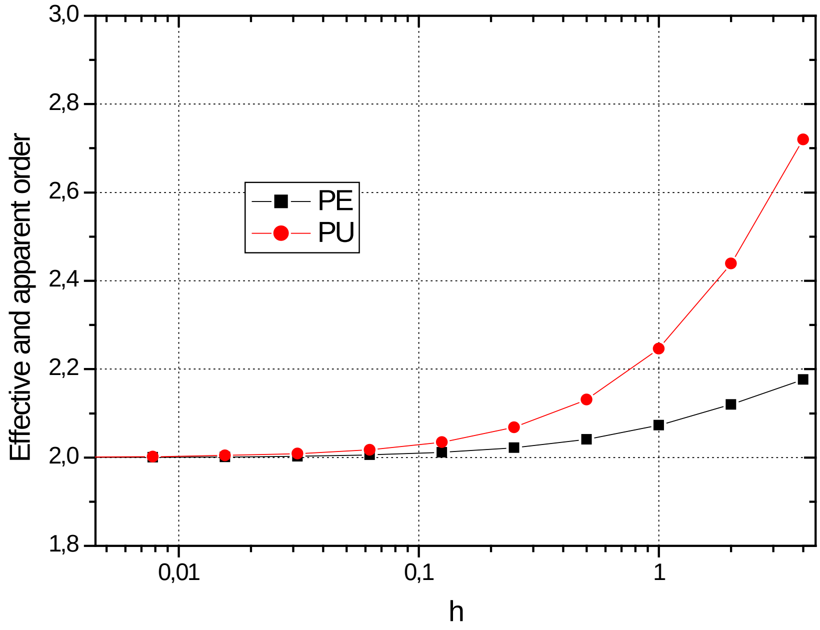

Figure 3. Error effective order (p_E) and uncertainty apparent order (p_U) of $\varepsilon(\lambda^i_{DDS-2})$.

Example 5

In this example we deal with the discretization error (*E*) of first derivative of dependent variable with two-point downstream (λ^i_{DDS-2}). For numerical approximation, one has (Marchi; 2002):

$$E(\lambda^i_{DDS-2}) = -h^2 + 6h^3, \tag{51}$$

where $C_1 = -1$ e $p_L = 2$, $C_2 = 6$ e $p_2 = 3$. Therefore,

$$p_E(\lambda^i_{UDS}) = 2 + \Delta p \text{, where } \Delta p = \frac{6h}{(6h-1)},$$

$$p_U(\lambda^i_{UDS}) = 2 + \Delta p \text{, where } \Delta p = \frac{12h}{(9h-1)}.$$

Table 10. Effective (p_E) and apparent order (p_U) of the uncertainty for the numerical approximation of the first order derivative with two-point downstream (λ^i_{DDS-2}). Asymptotic order $p_L = 2$.

h	p_E	p_U
1.00000000000000	3.20000000000000	3.50000000000000
0.50000000000000	3.50000000000000	3.71428571428571
0.33333333333333	4.00000000000000	4.00000000000000
0.25000000000000	5.00000000000000	4.40000000000000
0.20000000000000	8.00000000000000	5.00000000000000
0.16666666666667	∞	6.00000000000000
0.14285714285714	-4.00000000000000	8.00000000000000
0.12500000000000	-1.00000000000000	14.00000000000000
0.11111111111111	0.00000000000000	∞
0.10000000000000	0.50000000000000	-10.00000000000001
0.09090909090909	0.80000000000000	-4.00000000000000
0.08333333333333	1.00000000000000	-2.00000000000000
0.07692307692308	1.14285714285714	-1.00000000000000
0.07142857142857	1.25000000000000	-0.40000000000000
0.06666666666667	1.33333333333333	0
…	…	…
0.02040816326531	1.86046511627907	1.70000000000000
0.02000000000000	1.86363636363636	1.70731707317073

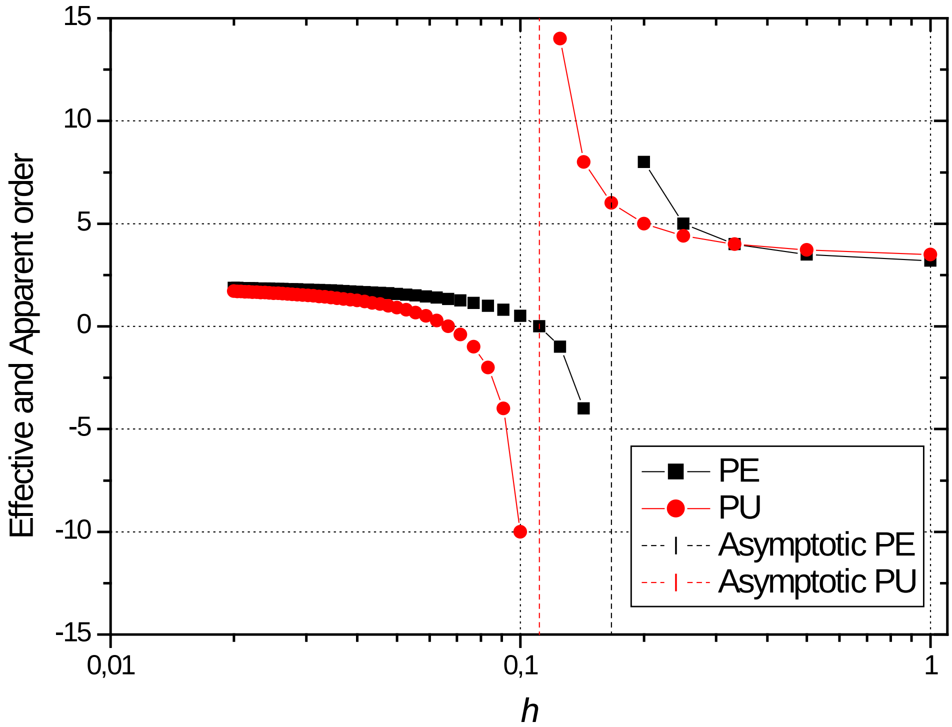

Figure 4. Effective (p_E) and apparent error order (p_U) of the uncertainty of $E(\lambda^{i}_{DDS-2})$.