Thermoelastic solutions are developed in Chapter 12, and the current study will now continue under the assumption of isothermal conditions.

Having developed the necessary six constitutive relations, the elasticity field equation system is now complete with fifteen equations (strain-displacement, equilibrium, Hooke's law) for fifteen unknowns (displacements, strains, stresses). Obviously, further simplification is necessary in order to solve specific problems of engineering interest, and these processes are the subject of the next chapter.

References

Chandrasekharaiah DS, and Debnath L: *Continuum Mechanics*, Academic Press, Boston, 1994.

Erdogan F: Fracture mechanics of functionally graded materials, *Composites Engng*, vol 5, pp. 753-770, 1995.

Malvern LE: *Introduction to the Mechanics of a Continuous Medium*, Prentice Hall, Englewood Cliffs, NJ, 1969.

Parameswaran V, and Shukla A: Crack-tip stress fields for dynamic fracture in functionally gradient materials, *Mech. of Materials*, vol 31, pp. 579-596, 1999.

Parameswaran V, and Shukla A: Asymptotic stress fields for stationary cracks along the gradient in functionally graded materials, *J. Appl. Mech.*, vol 69, pp. 240-243, 2002.

Poulos HG, and Davis EH: *Elastic Solutions for Soil and Rock Mechanics*, John Wiley, New York, 1974.

Exercises

4-1. Show that the components of the C_{ij} matrix in equation (4.2.2) are related to the components of C_{ijkl} by the relation

$$C_{ij} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3131} \end{bmatrix}$$

4-2. Explicitly justify the symmetry relations (4.2.4). Note that the first relation follows directly from the symmetry of the stress, while the second condition requires a simple expansion into the form $\sigma_{ij} = \frac{1}{2}(C_{ijkl} + C_{ijlk})e_{lk}$ to arrive at the required conclusion.

4-3. Substituting the general isotropic fourth-order form (4.2.6) into (4.2.3), explicitly develop the stress-strain relation (4.2.7).

4-4. For isotropic materials, show that the fourth-order elasticity tensor can be expressed in the following forms:

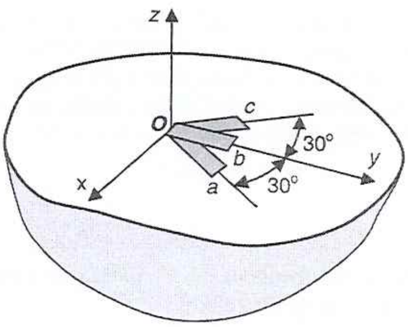
$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$
$$C_{ijkl} = \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + (k - \frac{2}{3}\mu)\delta_{ij}\delta_{kl}$$
$$C_{ijkl} = \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl} + \frac{E}{2(1+\nu)}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

4-5. Following the steps outlined in the text, invert the form of Hooke's law given by (4.2.7) and develop form (4.2.10). Explicitly show that $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ and $\nu = \lambda/[2(\lambda + \mu)]$.

4-6. Using the results of Exercise 4-5, show that $\mu = E/[2(1 + \nu)]$ $\lambda = E\nu/[(1 + \nu)(1 - 2\nu)]$.

4-7. For isotropic materials show that the principal axes of strain coincide with the principal axes of stress. Further, show that the principal stresses can be expressed in terms of the principal strains as $\sigma_i = 2\mu e_i + \lambda e_{kk}$.

4-8. A rosette strain gage (see Exercise 2-7) is mounted on the surface of a stress-free elastic solid at point O as shown in the following figure. The three gage readings give surface extensional strains $e_a = 300 \times 10^{-6}$, $e_b = 400 \times 10^{-6}$, $e_c = 100 \times 10^{-6}$. Assuming that the material is steel with nominal properties given by Table 4-2, determine all stress components at O for the given coordinate system.



4-9. The displacements in an elastic material are given by

$$u = -\frac{M(1 - \nu^2)}{EI}xy, \ v = \frac{M(1 + \nu)\nu}{2EI}y^2 + \frac{M(1 - \nu^2)}{2EI}(x^2 - \frac{l^2}{4}), \ w = 0$$

where M, E, I, and l are constant parameters. Determine the corresponding strain and stress fields and show that this problem represents the pure bending of a rectangular beam in the x,y plane.

4-10. If the elastic constants E, k, and μ are required to be positive, show that Poisson's ratio must satisfy the inequality $-1 < \nu < \frac{1}{2}$. For most real materials it has been found that $0 < \nu < \frac{1}{2}$. Show that this more restrictive inequality in this problem implies that $\lambda > 0$.

4-11. Under the condition that E is positive and bounded, determine the elastic moduli λ, μ, and k for the special cases of Poisson's ratio: $\nu = 0, \frac{1}{4}, \frac{1}{2}$. Discuss the special circumstances for the case with $\nu = \frac{1}{2}$.

4-12. Consider the three deformation cases of simple tension, pure shear, and hydrostatic compression as discussed in Section 4.3. Using the nominal values from Table 4-2, calculate the resulting strains in each of these cases for

(a) Aluminum: with loadings ($\sigma = 150\ MPa$, $\tau = 75\ MPa$, $p = 500\ MPa$)

(b) Steel: with loadings ($\sigma = 300\ MPa$, $\tau = 150\ MPa$, $p = 500\ MPa$)

(c) Rubber: with loadings ($\sigma = 15\ MPa$, $\tau = 7\ MPa$, $p = 500\ MPa$)

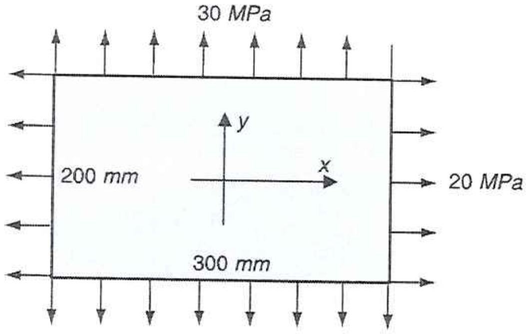
Note that for aluminum and steel, these tensile and shear loadings are close to the yield values of the material.

4-13. Show that Hooke's law for an isotropic material may be expressed in terms of spherical and deviatoric tensors by the two relations

$$\tilde{\sigma}_{ij} = 3k\tilde{e}_{ij}, \;\; \hat{\sigma}_{ij} = 2\mu\hat{e}_{ij}$$

4-14. A sample is subjected to a test under *plane stress* conditions (specified by $\sigma_z = \tau_{zx} = \tau_{zy} = 0$) using a special loading frame that maintains an in-plane loading constraint $\sigma_x = 2\sigma_y$. Determine the slope of the stress-strain response σ_x vs. e_x for this sample.

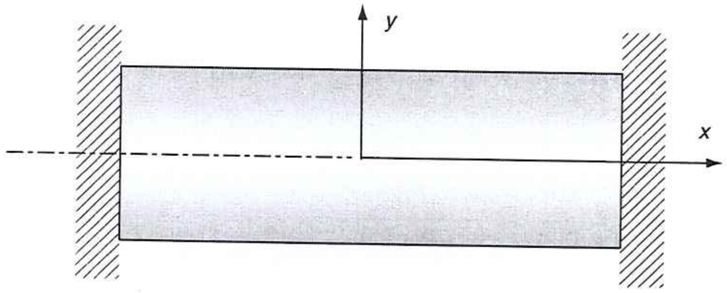
4-15. A rectangular steel plate (thickness 4 mm) is subjected to a uniform biaxial stress field as shown in the following figure. Assuming all fields are uniform, determine changes in the dimensions of the plate under this loading.



4-16. Redo Exercise 4-15 for the case where the vertical loading is 50 MPa in tension and the horizontal loading is 50 MPa in compression.

4-17. Consider the one-dimensional thermoelastic problem of a uniform bar constrained in the axial x direction but allowed to expand freely in the y and z directions, as shown in the following figure. Taking the reference temperature to be zero, show that the only nonzero stress and strain components are given by

$$\sigma_x = -E\alpha T$$
$$e_y = e_z = \alpha(1+\nu)T$$



4-18. Verify that Hooke's law for isotropic thermoelastic materials can be expressed in the form

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_x + \nu(e_y + e_z)] - \frac{E}{1-2\nu}\alpha(T - T_o)$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_y + \nu(e_z + e_x)] - \frac{E}{1-2\nu}\alpha(T - T_o)$$
$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_z + \nu(e_x + e_y)] - \frac{E}{1-2\nu}\alpha(T - T_o)$$
$$\tau_{xy} = \frac{E}{1+\nu}e_{xy}, \quad \tau_{yz} = \frac{E}{1+\nu}e_{yz}, \quad \tau_{zx} = \frac{E}{1+\nu}e_{zx}$$