Problems

4.1. The state of stress at certain point of a body is given by

$$[\mathrm{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}_{\mathbf{e}_i} \quad \text{MPa}$$

On each of the coordinate planes (normals $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) (a) what is the normal stress and (b) what is the total shearing stress.

4.2. The state of stress at a certain point of a body is given by

$$[\mathrm{T}] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \quad \text{MPa}$$

(a) Find the stress vector at a point on the plane whose normal is in the direction $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$.

(b) Determine the magnitude of the normal and shearing stresses on this plane.

4.3. Do the previous problem for a plane passing through the point and parallel to the plane $x_1 - 2x_2 + 3x_3 = 4$.

4.4. The stress distribution in a certain body is given by

$$[\mathrm{T}] = \begin{bmatrix} 0 & 100x_1 & -100x_2 \\ 100x_1 & 0 & 0 \\ -100x_2 & 0 & 0 \end{bmatrix}.$$

Find the stress vector acting on a plane which passes through the point $(1/2, \sqrt{3}/2, 3)$ and is tangent to the circular cylindrical surface $x_1^2 + x_2^2 = 1$ at that point.

4.5. Given $T_{11} = 1$ Mpa, $T_{22} = -1$Mpa and all other $T_{ij} = 0$ at a point in a continuum.

(a) Show that the only plane on which the stress vector is zero is the plane with normal in the $\mathbf{e}_3$−direction.

(b) Give three planes on which there is no normal stress acting.

4.6. For the following state of stress

$$[\mathrm{T}] = \begin{bmatrix} 10 & 50 & -50 \\ 50 & 0 & 0 \\ -50 & 0 & 0 \end{bmatrix} \quad \text{MPa}$$

find T_{11}' and T_{13}' where $\mathbf{e}_1'$ is in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{e}_2'$ is in the direction of $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$.

4.7. Consider the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} \alpha x_2 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where α and β are constants.

(a) Determine and sketch the distribution of the stress vector acting on the square in the $x_1 = 0$ plane with vertices located at (0,1,1), (0,-1, 1), (0,1,-1), (0,-1,-1).

(b) Find the total resultant force and moment about the origin of the stress vectors acting on the square of part (a).

4.8. Do the previous problem if the stress distribution is given by

$$T_{11} = \alpha x_2^2$$

and all other $T_{ij} = 0$.

4.9. Do problem 4.7 for the stress distribution

$$T_{11} = \alpha,\ T_{21} = T_{12} = \alpha x_3$$

and all other $T_{ij} = 0$.

4. 10. Consider the following stress distribution for a circular cylindrical bar

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\alpha x_3 & +\alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}$$

(a) What is the distribution of the stress vector on the surfaces defined by $x_2^2 + x_3^2 = 4$, $x_1 = 0$ and $x_1 = l$?

(b) Find the total resultant force and moment on the end face $x_1 = l$.

4.11. An elliptical bar with lateral surface defined by $x_2^2 + 2x_3^2 = 1$ has the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \text{MPa}$$

(a) Show that the stress vector any point(x_1, x_2, x_3) on the lateral surface is zero.

(b) Find the resultant force and resultant moment about the origin O of the stress vector on the left end face $x_1 = 0$.

$$\text{Note: } \int x_2^2 dA = \frac{\pi}{4\sqrt{2}} \text{ and } \int x_3^2 dA = \frac{\pi}{8\sqrt{2}}.$$

4.12. For any stress state **T**., we define the deviatoric stress **S** to be

$$\mathbf{S} = \mathbf{T} - \left(\frac{T_{kk}}{3}\right)\mathbf{I}$$

where T_{kk} is the first invariant of the stress tensor **T**.

(a) Show that the first invariant of the deviatoric stress vanishes.

(b) Given the stress tensor

$$[\mathbf{T}] = 100\begin{bmatrix} 6 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 9 \end{bmatrix} \text{kPa}$$

evaluate **S**

(c) Show that the principal direction of the stress and the deviatoric stress coincide.

(d) Find a relation between the principal values of the stress and the deviatoric stress.

4.13. An octahedral stress plane is defined to make equal angles with each of the principal axes of stress.

(a) How many independent octahedral planes are there at each point?

(b) Show that the normal stress on an octahedral plane is given by one-third the first stress invariant.

(c) Show that the shearing stress on the octahedral plane is given by

$$T_s = \frac{1}{3}[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_1 - T_3)^2]^{1/2},$$

where T_1, T_2, T_3 are the principal values of the stress tensor.

4.14. (a) Let **m** and **n** be two unit vectors that define two planes M and N that pass through a point P. For an arbitrary state of stress defined at the point P, show that the component of the stress vector $\mathbf{t_m}$ in the **n**- direction is equal to the component of the stress vector $\mathbf{t_n}$ in the **m**-direction.

(b) If $\mathbf{m} = \mathbf{e}_1$and $\mathbf{n} = \mathbf{e}_2$, what does the result of part (a) reduce to?

4.15. Let **m** be a unit vector that defines a plane M passing through a point P. Show that the stress vector on any plane that contains the stress traction $\mathbf{t}_m$ lies in the M-plane.

4.16. Let $\mathbf{t_m}$ and $\mathbf{t_n}$ be stress vectors on planes defined by the unit vectors **m** and **n** and pass through the point P. Show that if **k** is a unit vector that determines a plane that contains $\mathbf{t_m}$ and $\mathbf{t_n}$, then $\mathbf{t_m}$ is perpendicular to **m** and **n**.

4. 17. True or false

(i) Symmetry of stress tensor is not valid if the body has an angular acceleration.

(ii) On the plane of maximum normal stress, the shearing stress is always zero.

4.18. True or false

(i) On the plane of maximum shearing stress, the normal stress is zero.

(ii) A plane with its normal in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$ has a stress vector $\mathbf{t} = 50\mathbf{e}_1 + 100\mathbf{e}_2 - 100\mathbf{e}_3$ MPa. It is a principal plane.

4.19. Why can the following two matrices not represent the same stress tensor?

$$\begin{bmatrix} 100 & 200 & 40 \\ 200 & 0 & -30 \\ 40 & -30 & -50 \end{bmatrix} \text{MPa} \qquad \begin{bmatrix} 40 & 100 & 60 \\ 100 & 100 & 0 \\ 60 & 0 & 20 \end{bmatrix} \text{MPa.}$$

4.20. Given a

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{Mpa}$$

(a) Find the magnitude of shearing stress on the plane whose normal is in the direction of $\mathbf{e}_1 + \mathbf{e}_2$.

(b) Find the maximum and minimum normal stresses and the planes on which they act.

(c) Find the maximum shearing stress and the plane on which it acts.

4.21. The stress components at a point are given by

$$T_{11} = 100\text{MPa},\ T_{22} = 300\text{ MPa},\ T_{33} = 400\text{ MPa},\ T_{12} = T_{13} = T_{23} = 0$$

(a) Find the maximum shearing stress and the planes on which it acts.

(b) Find the normal stress on these planes.

(c) Are there any plane/planes on which the normal stress is 500 MPa?

4.22. The principal values of a stress tensor $\mathbf{T}$ are: $T_1 = 10$ MPa , $T_2 = -10$ MPa and $T_3 = 30$ MPa. If the matrix of the stress is given by

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{bmatrix} \times 10\text{Mpa}$$

find the value of T_{11} and T_{33}.

4.23. If the state of stress at a point is

$$[\mathbf{T}] = \begin{bmatrix} 300 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & 400 \end{bmatrix} \text{kPa}$$

find (a) the magnitude of the shearing stress on the plane whose normal is in the direction of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$, and (b)the maximum shearing stress.

4.24. Given

$$[\mathbf{T}] = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{MPa}.$$

(a) Find the stress vector on the plane whose normal is in the direction $\mathbf{e}_1 + \mathbf{e}_2$.

(b) Find the normal stress on the same plane.

(c) Find the magnitude of the shearing stress on the same plane.

(d) Find the maximum shearing stress and the planes on which this maximum shearing stress acts.

4.25. The stress state in which the only non-vanishing stress components are a single pair of shearing stresses is called simple shear. Take $T_{12} = T_{21} = \tau$ and all other $T_{ij} = 0$.

(a) Find the principal values and principal directions of this stress state.

(b) Find the maximum shearing stress and the plane on which it acts.

4.26. The stress state in which only the three normal stress components do not vanish is called tri-axial stress state. Take $T_{11} = \sigma_1$, $T_{22} = \sigma_2$, $T_{33} = \sigma_3$ with $\sigma_1 > \sigma_2 > \sigma_3$ and all other $T_{ij} = 0$. Find the maximum shearing stress and the plane on which it acts.

4.27. Show that the symmetry of the stress tensor is not valid if there are body moments per unit volume, as in the case of a polarized anisotropic dielectric solid.

4.28. Given the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} x_1 + x_2 & T_{12}(x_1,x_2) & 0 \\ T_{12}(x_1,x_2) & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$$

find T_{12} so that the stress distribution is in equilibrium with zero body force and so that the stress vector on $x_1 = 1$ is given by $\mathbf{t} = (1 + x_2)\mathbf{e}_1 + (5 - x_2)\mathbf{e}_2$.

4.29. Suppose the body force vector is $\mathbf{B} = -g\mathbf{e}_3$, where g is a constant. Consider the following stress tensor

$$[\mathbf{T}] = \alpha \begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T_{33} \end{bmatrix}$$

and find an expression for T_{33} such that $\mathbf{T}$ satisfies the equations of equilibrium.

4.30. In the absence of body forces, the equilibrium stress distribution for a certain body is

$$[\mathbf{T}] = \begin{bmatrix} Ax_2 & x_1 & 0 \\ x_1 & Bx_1 + Cx_2 & 0 \\ 0 & 0 & \frac{1}{2}(T_{11} + T_{22}) \end{bmatrix}$$