EFFECT OF PARAMETERS OF A MULTIGRID METHOD ON THE CPU TIME FOR ONE-DIMENSIONAL PROBLEMS

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Abstract. *In this work the effect on the CPU time of several parameters of a geometric multigrid method is studied. The considered parameters are: coarsening ratio of grids, number of inner iterations, number of grid levels, tolerances and initial estimates. The considered mathematical models involve one-dimensional problems of heat transfer, Poisson and advection-diffusion equations, with Dirichlet boundary conditions. The finite difference method is used to discretizate the differential equations. In the multigrid case, the systems of algebraic equations are solved with the Gauss-Seidel method. Comparisons of singlegrid methods (Gauss-Seidel and TDMA) are made. The used multigrid algorithm is the correction scheme with V-cycle. The restriction is given by injection and the prolongation by linear interpolation. Some literature results are confirmed and some new ones are presented.*

Keywords: *Finite difference; solvers, coarsening ratio, heat transfer, fluid flow, CFD*

1. Introduction

The discretization of mathematical models, which appear in computational fluid dynamics, leads to large-scale systems of algebraic equations of the type

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where **A** is a square matrix, **b** is the independent vector and **x** is the unknown vector. These models appear generally in physical phenomenon that involves fluids in motion with or without heat transfer (Fortuna, 2000; Maliska, 2004). The structure of the matrix **A** depends of the technique used to discretizate the mathematical model.

The finite difference method is very used in computational fluid dynamics (Golub and Ortega, 1992; Tannehill *et al.*, 1997**),** where, in one-dimensional problems, the domain $x \in \Re:\ 0 \le x \le 1$ is partitioned in N_f subintervals, introducing a grid with the points $x_i = (i-1)h$, where $i = 1,...,N_f + 1$ and $h = 1/N_f$ is the length of each interval. This yields a grid of size *h* that is denoted for Ω^h. For each one of the $N_f - 1$ inner points of grid, the differential equation that represents the physical phenomenon is substituted by approximations of finite differences of first and second order (Tannehill *et al.,* 1997; Ferziger and Peric, 1999).

Several numerical techniques have been studied to solve the system (1) with minor computational cost. The solution by direct methods is not recommendable because the matrix of coefficients is very large and the inversion of matrix is an expansive process (Golub and van Loan, 1984). The iterative methods are more appropriate for problems of large-scale (Burden and Faires, 1997). From an initial estimate, it is generated a sequence of solutions that tends to the exact solution of the problem. Methods such as the conjugated gradient (Burden and Faires, 1997) and the pre-conditioned conjugated gradient methods (Dennis and Schnabel, 1983) are more specific for simple geometries and for problems whose matrix of coefficients is badly-conditioned. For more complex geometries, the technique of decomposition of domain (Perng and Street, 1991; Lai and Przekwas, 1996) is also used.

The multigrid method, developed originally by Fedorenko (1964), is presented as an alternative numerical technique to solve systems of equations of the type (1) iteratively. The basic idea is to use a set of grids and to execute iterations in each level of grid and to do approximations of these solutions in a coarser grid alternately. Then, operators that transfer vectors from a finer grid to an immediately coarser grid (restriction) are defined. The same procedure is applied between a coarse grid and an immediately finer grid (prolongation). The linear systems in each grid are solved by an iterative method, which has properties to reduce the oscillatory errors quickly (smoothing properties). Based on this concept, some works were published presenting good numerical results for problems of fluid dynamics. According to Fedorenko (1964), the convergence rate with the use of multigrid techniques is better than singlegrid methods. The objective of the multigrid technique is to speed up the convergence of an iterative scheme (Tannehill *et al*., 1997). The

best performances of the multigrid method are obtained for elliptical problems (Wesseling, 1992), namely, problems dominated by diffusion and the minors for problems dominated by advection (Ferziger and Peric, 1999).

The multigrid methods can be applied to both structured and unstructured grids. In structured grids it is more appropriated the use of geometric multigrid (Wesselind and Oosterlee, 2001) and algebraic multigrid (Stüben, 2001) in unstructured grids. In Briggs *et al.* (2000) the use of geometric and algebraic multigrid for a two-dimensional Poisson problem with Dirichlet boundary conditions is compared. In Wesseling and Oosterlee (2001) many challenges still are seen in the geometric area, as the solution of Navier-Stokes equation, singular perturbation problems, boundary layer problems where strong stretched grids, or same the parallel computing.

Brandt (1977) worked with geometric multigrid for several problems of heat transfer and fluid flow, one and two-dimensional, linear and nonlinear problems. He made comparisons for coarsening ratios $r = 1/2$, $r = 1/3$ and $r = 2/3$, where the coarsening ratio is defined as

$$r = \frac{h_{\Omega^h}}{h_{\Omega^H}} \tag{2}$$

where Ω^h represents a fine grid, Ω^H is an immediately coarser grid and h is the size of each element of the grid for one-dimensional and uniform grids. For the tested problems, the conclusion of Brandt (1977) is that the ratio $r = 1/2$ is the recommendable one, and according to him, this is around the optimal ratio and easier to program. Stüben (1999) developed a study for ratios $r = 1/2$ and $r = 1/4$ in unstructured grids for several problems of heat transfer, two and three-dimensional, linear and nonlinear for flow and electromagnetism problems. In its work the ratio $r = 1/4$ was showed efficient for anisotropic problems (anisotropy by coefficients and it due to highly stretched grids). Briggs *et al.* (2000) worked with $r = 1/2$ affirming to be a universal practical and that $r \neq 1/2$ does not bring advantages. Moro (2004) worked with ratios $r = 1/2$ and $r = 1/4$ in structured grids for problems of diffusion with source term. In his work the ratio $r = 1/4$ was showed sufficiently efficient, with the CPU time less than $r = 1/2$.

In this work one studies the multigrid method in one-dimension to show the principles of the method and to derive some procedures used in general cases. Also, a bigger amount of tests can be made due to rapidity of obtainment of the solutions. This facilitates the study of many parameters. The following parameters are studied: several coarsening ratios ($r = 1/2$, $r = 1/3$, $r = 1/4$, $r = 1/5$ and $r = 1/8$); the number of inner iterations (iterations of the solver to smooth the error components in each grid); and number of grid levels. The goal is to verify the effect of these parameters on the CPU time for a geometric multigrid method. The multigrid algorithm adopted is the Correction Scheme with V-cycle described in Tannehill *et al.* (1997). The results are compared with ones obtained from bibliography. The restriction and prolongation operators for any coarsening ratios in the interval (0,1) are presented. The considered mathematical models in this work involve one-dimensional problems of heat transfer, namely, Poisson and advection-diffusion equations with Dirichlet boundary conditions.

This work is organized in the following form: in Section 2 it is given an overview of multigrid methods, presenting the restriction and prolongation operators, and the iterative Gauss-Seidel method. In Section 3, the mathematical and numerical models are given. In Sections 4 and 5 are described the numerical experiments with their results and the conclusion of the work.

2. Multigrid method

The solution of fluid mechanics and heat transfer problems through numerical methods requires high computational cost and many times impracticable because of many equations to be solved in each iterative step. An alternative method used to improve the convergence rate of these problems is the multigrid method (Briggs *et al.,* 2000), which speeds up the solution of the linear systems involved in the problem. The multigrid methods are iterative methods of solution of linear systems, being therefore, strong dependent of the initial estimate attributed to the unknowns of the problem.

An efficient technique to attenuate the strong oscillations of the residue in each mesh, defined for

$$\mathbf{R} = \mathbf{b} - \mathbf{A}\mathbf{x} \tag{3}$$

it is to smooth the oscillations by a relaxation method. In this work it was opted by the Gauss-Seidel method, once it has good smoothing properties (Briggs *et al.,* 2000).

The first iterations of this process, generally, it have fast convergence, characterizing the presence of oscillatory modes of error. However, after some iterations, the process becomes slow, characterizing the predominance of smoothing modes (Brandt, 1977). It is exactly the point where is recommended to transfer the relaxation problem to the coarsest grid. Therefore, the smoothing errors in the fine grid become oscillatory errors in the coarse grid (Wesseling, 1992). For linear problems, the multigrid method is one technique that executes steps of relaxation and approximates solutions of the residual equation in the coarser grid (Correction Scheme) alternately, speeding up the convergence of relaxation scheme (Briggs *et al.,* 2000). The convergence rate of the multigrid is independent of the grid size, that is, it is independent of the number of grid points (Hirsch, 1988; Ferziger and Peric, 1999). It is not very effective only to use

two levels of grid (Roache, 1998). To get a good performance with the multigrid, several grid levels must be used (Tannehill *et al.,* 1999).

In this work the coarsening ratio for uniform grids is given by Eq. (2), where: $r \in (0\,;0.5)$ will be called strong coarsening; $r = 0.5$, standard coarsening; and $r \in (0.5\,;1)$, weak coarsening. One studies the strong and standard coarsening applied to the algorithm found in Tannehill *et al.* (1999). One alternative form of the Eq. (2) is

$$r = \frac{p}{q}, \quad p, q \in Z_+^*, \quad p < q \tag{4}$$

In this case, the coarsening ratio r is called pure ratio if $p = 1$.

The operators of transference of the fine grid to the coarse grid are called restriction operators and they are denoted generically by $\mathbf{I}_{\mathbf{h}}^{\mathbf{H}}\boldsymbol{\varphi}^{\mathbf{h}} = \boldsymbol{\varphi}^{\mathbf{H}}$. Where $\boldsymbol{\varphi}$ assumes the residue $\mathbf{R}$ given by Eq. (3). In this work an operator of injection with its generalized form for any $r \in (0,1)$ was developed. It is given by

$$\boldsymbol{\varphi}_{\mathbf{i}}^{\mathbf{H}} = K_r.\boldsymbol{\varphi}_{\mathbf{cr}}^{\mathbf{h}} + (1-K_r).\boldsymbol{\varphi}_{\mathbf{cr+1}}^{\mathbf{h}}; \quad 2 \le i \le N^H, \text{ with } cr = ceiling\left(\frac{q}{p}(i-1)\right), K_r = cr - \frac{q}{p}(i-1), N^{\mathrm{H}} = N^{\mathrm{h}}.\frac{p}{q} \tag{5}$$

and N^h being the number of intervals of immediately finer grid. The function *ceiling* is defined by

$$ceiling : \Re \to Z \text{, with } x \mapsto ceiling(x) = min\{z \in Z \,/\, z \ge x\} \tag{6}$$

The Eq. (5) is not calculated for $i = 1$ and $i = N^H + 1$, since in this work one adopts the Dirichlet boundary conditions. Therefore, $\mathbf{R} = \mathbf{0}$ at the boundary points.

The operators of transference of the coarse grid to the fine grid are called operators of prolongation, or interpolation, and they are denoted generically by $\mathbf{I}_{\mathbf{H}}^{\mathbf{h}}\boldsymbol{\varphi}^{\mathbf{H}} = \boldsymbol{\varphi}^{\mathbf{h}}$. Where $\boldsymbol{\varphi}$ assumes the approximation of error in the residual equation, namely, correction. As in the case of the restriction operator, in this work an operator of linear interpolation with its generalized form for any $r \in (0,1)$ was developed. It is given by

$$\boldsymbol{\varphi}_{\mathbf{i}}^{\mathbf{h}} = K_p.\boldsymbol{\varphi}_{\mathbf{cp}}^{\mathbf{H}} + (1-K_p).\boldsymbol{\varphi}_{\mathbf{cp+1}}^{\mathbf{H}}; \quad 2 \le i \le N^H, \text{ with } cp = ceiling\left(\frac{p}{q}(i-1)\right), \quad K_p = cp - \frac{p}{q}(i-1) \tag{7}$$

3. Mathematical and numerical models

The one-dimensional linear problems of diffusion (Poisson equation) and the advection-diffusion equation, with Dirichlet boundary conditions, for steady state and cartesian coordinates can be represented mathematically by (Ferziger and Peric, 1999)

$$u_{xx} = f, \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) = 1 \text{ (diffusion)} \tag{8}$$

$$Pe.u_x = u_{xx}, \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) = 1 \text{ (advection-diffusion)} \tag{9}$$

where u is the unknown, f is a source term that will be given by $f = 1 + 3x + 26x^2$ and $Pe = 20$. In the Eqs. (8) and (9) u represents temperature, u_x and u_{xx} are first and second derivatives, respectively.

The discretization of domain will be developed using uniform grids in N_f subintervals (elements) with grid points given by $x_i = (i-1)h$, where $i = 1,...,N_f + 1$, and $h = 1/N_f$ is the length of each subinterval. The differential equations are given by Eqs. (8) and (9) are discretizated in accordance with the finite difference method. For the Eqs. (8) and (9) the central differencing scheme (CDS) is used for the second order derivatives (diffusion terms), and the upwind differencing scheme (UDS) is used for the first order derivative (advective term). The result equations are

$$\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} = f_i, \quad 2 \le i \le N_f; \quad v_1 = 0, \quad v_{N_f+1} = 1 \text{ (diffusion)} \tag{10}$$

$$Pe.\frac{v_i - v_{i-1}}{h} = \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}, \quad 2 \le i \le N_f; \quad v_1 = 0, \quad v_{N_f+1} = 1 \text{ (advection-diffusion)} \tag{11}$$

where v_i is an approximation (numerical solution) for the analytical exact solution $u(x_i)$ and $f_i = f(x_i)$.

The systems are given by Eqs. (10) and (11) can be represented by a system of algebraic equations of the type

$$\mathbf{Av} = \mathbf{f} \tag{12}$$

where $\mathbf{v} = \left(v_1, ..., v_{N_f+1}\right)^t$ is the unknown vector, $\mathbf{f} = \left(f_1, ..., f_{N_f+1}\right)^t$ is the independent vector and $\mathbf{A}$ is a tridiagonal $\left(N_f + 1\right)\mathbf{x}\left(N_f + 1\right)$, symmetric and positive definite matrix (Briggs *et al.,* 2000; Burden and Faires, 1997).

The Eq. (12) is solved with the TDMA direct method (Tridiagonal Matrix Algorithm) to get information on the performance of the method to problem given by the Eq. (12). The TDMA algorithm is presented in Ferziger and Peric (1999). It is also solved, only in the finest grid and with the Gauss-Seidel method (singlegrid). When applied the multigrid method, the Eq. (12) becomes $\mathbf{Av} = \mathbf{b}$, where $\mathbf{b}$ represents the source term (residue) to each level of grid. In this case, the systems are solved with the Gauss-Seidel method.

4. Numerical results

The algorithms have been implemented in FORTRAN/95 with the Visual Compaq Fortran 6.6. The tests were carried out in a microcomputer with processor Intel Pentium 4 with 2.66 GHz and 1 GB RAM, using double precision arithmetic. The following coarsening ratios were used: standard coarsening, namely, $r = 1/2$; and some strong coarsening using pure rate, as: $r = 1/3$, $r = 1/4$, $r = 1/5$ and $r = 1/8$. Other methods have been used for comparison, for example, the iterative Gauss-Seidel and the direct TDMA method.

Hundreds of tests have been carried out with other variants, as: dimension of the problems (of small problems until problems of the order of millions of variables), number of inner iterations, number of grid levels, tolerances and initial estimates. But only some of these tests are presented in this work, since they are the most representative tests, qualitatively enough to show the performance of the multigrid method.

The stopping criterion for outer iterations (needed cycle to smooth the error components) with the iterative methods is given by ratio between the norm L_1 of the residue (Ferziger and Peric, 1999) and the norm of the residue based on the initial estimate, where the residue of each node is calculated through the Eq. (3). In this work it was adopted $\varepsilon = 10^{-7}$ and $\mathbf{v} = \mathbf{0}$ as reference for the stopping criterion and the initial estimate, respectively.

The Fig. 1 shows analytical and numerical solutions for the problems defined by Eqs. (8) and (9). The numerical solutions are compared with the analytical solutions for three distinct methods: the multigrid method using the coarsening ratio $r = 1/2$, the Gauss-Seidel and the TDMA method.

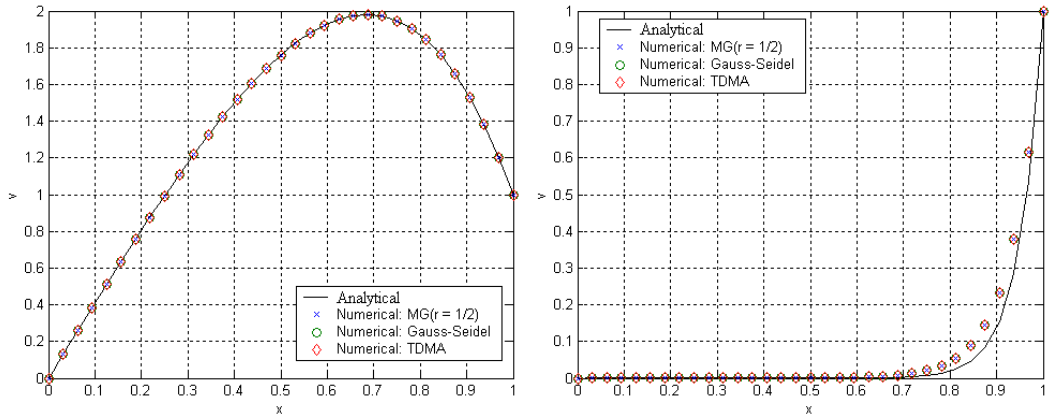

Figure 1. Analytical and numerical solutions for $N_f = 32$ elements:
(left) Poisson and (right) advection-diffusion problems

The focus of study of this work is the minimization of the CPU time. The CPU time is understood for the time expense to carry out the generation of grids, determination of the initial estimate, calculation of the coefficients and solution of the linear system (12). This time is measured using the function TIMEF of the library PORTLIB of the FORTRAN/95. Through of the carried out tests it was verified that the uncertainty of this function is approximately of ±0.05 s.

4.1. Inner iterations (*ITI*)

The Fig. 2 shows the influence of the number of inner iterations on the CPU time and the optimal number for several coarsening ratios and dimensions of problems given by Tab. 1. The Tab. 1 shows, in its first row, the coarsening ratios used in the numerical tests for the Poisson and the advection-diffusion problems and, in its second row, some dimensions of grids. It can be noticed which the number of inner iterations affects the CPU time. For all tested ratios, it was verified that $ITI_{optimal} = 2q - 1$, except the ratio $r = 1/8$, where $ITI_{optimal} = 2q + 2$. This result confirms the one of Tannehill *et al.* (1997) for $r = 1/2$.

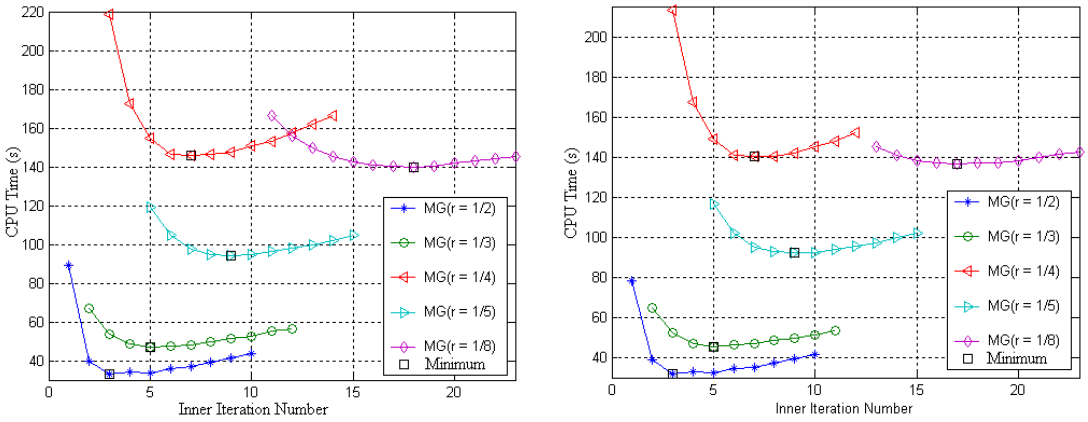

Figure 2. CPU time versus inner iterations for:
(left) Poisson and (right) advection-diffusion problems

Table 1. Coarsening ratios and some grids.

Ratio	1/2	1/3	1/4	1/5	1/8
N_f	1,048,576	1,062,882	2,097,152	781,250	524,288

One notices that, for the most tested ratios and for both problems, the Fig. 2 illustrates the strong slope until reaching minimums in the graphs and, after these points, the growth rate is lower. Other numerical tests have shown that, relatively to the minimum point, a sensible reduction of the number of inner iterations increases drastically the CPU time, reaching even the order of hours for some coarsening ratios. On the other hand, a sensible increase in inner iterations increases small increment of the CPU time. Therefore, it is recommended to use $ITI = 2q$ for the pure ratios of the form $r = 1/q$; for example, for $r = 1/2$, it is recommended to use $ITI = 4$. For $ITI = 2q$, the CPU time changes a short variation to the optimal number of inner iterations: about 3.5% for the Poisson problem and about 3.4% for the advection-diffusion problem.

4.2. Grid levels (*L*)

The Fig. 3 shows the influence of the number of grid levels on the CPU time and the optimal number for several coarsening ratios and dimensions of problems given by Tab. 1. One notices that the number of grid levels can affect significantly the CPU time, and $L_{optimal} \approx L_{maximum}$ for all tested ratios to both problems, where $L_{maximum}$ is the possible maximum number of levels. It has similar results to ones of Tannehill *et al.* (1997) and Mesquita and De-Lemos (2004), both for cases where $r = 1/2$.

One notices that the Fig. 3 illustrates the strong slope until reaching minimums in the graphs and, after these points, the growth rate is lower until reaching $L_{maximum}$. Other numerical tests have shown that, relatively to the minimum point, a sensible reduction of the number of grid levels increases drastically the CPU time, reaching even the order of hours for some coarsening ratios. On the other hand, a sensible increase in grid levels increases small increment of the CPU time. Therefore, it is recommended to use $L = L_{maximum}$ for the pure ratios of the form $r = 1/q$; for instance, for $r = 1/2$ and $N_f = 2^{20} = 1{,}048{,}576$, it is recommended to use $L = 20$, that is exactly $L_{maximum}$. This type of procedure has a lot of computational advantages, because it does not need extra looping to take the decision of the exact number of grid levels. It is dependent of the dimension of the problem and the coarsening ratio. When adopting $L = L_{maximum}$, the

variation of the CPU time is sufficiently small: about 0.90% for the Poisson problem and about 0.58% for the advection-diffusion problem.

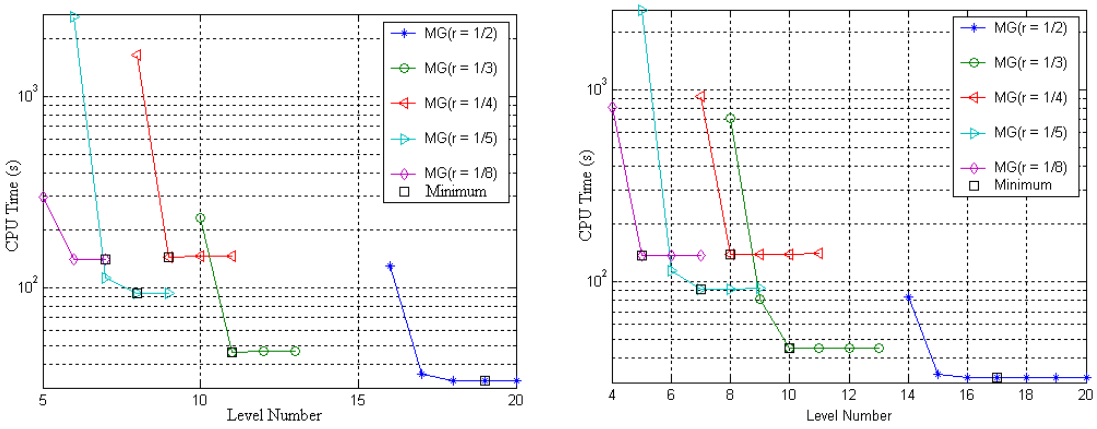

Figure 3. CPU time versus grid levels for:
(left) Poisson and (right) advection-diffusion problems

4.3. CPU time versus coarsening ratios

The Fig. 4 shows the various coarsening ratios for the multigrid method. The comparison among Multigrid, Gauss-Seidel and TDMA singlegrid methods versus the number of elements N_f of grid are shown in the Fig 4 as well. It can be noticed that the TDMA method is the most efficient among tested methods for both problems. However, the goal here is to verify the effect of the multigrid parameters on the CPU time for a geometric multigrid method in one-dimensional problems to obtain information to the future research.

With the discrete least square approximation for the data in Fig. 4 and using the geometric curve $t = bN_f^a$, one obtains $a \approx 1.00$ for TDMA method, $a \approx 1.05$ for multigrid methods and $a \approx 3.00$ for Gauss-Seidel method.

For the same ε, $\mathbf{v}$ and N_f, amongst the coarsening ratios to the multigrid methods, for the ratios $r_1 = 1/q_1$ and $r_2 = 1/q_2$, and the respective CPU times t_1 and t_2, as for the Poisson as for the advection-diffusion problem, it can be noticed that $q_1 < q_2 \Rightarrow t_1 < t_2$. Therefore, in the multigrid case, it is recommended to use the ratio $r = 1/2$.

It was verified that the number of outer iterations (*ITE*) in order to reach the convergence increases with N_f until the point in that this number stabilizes and becomes independent of N_f. This process of stabilization is given by a asymptotic form, such that, $q_1 < q_2 \Rightarrow (ITE)_1 < (ITE)_2$ and $q_1 < q_2 \Rightarrow (N_f)_1 < (N_f)_2$, where $(ITE)_i$ is the number of outer iterations in order to reach the convergence, referring to the reason $r_i = 1/q_i$. The notation for $(N_f)_i$ is analogous.

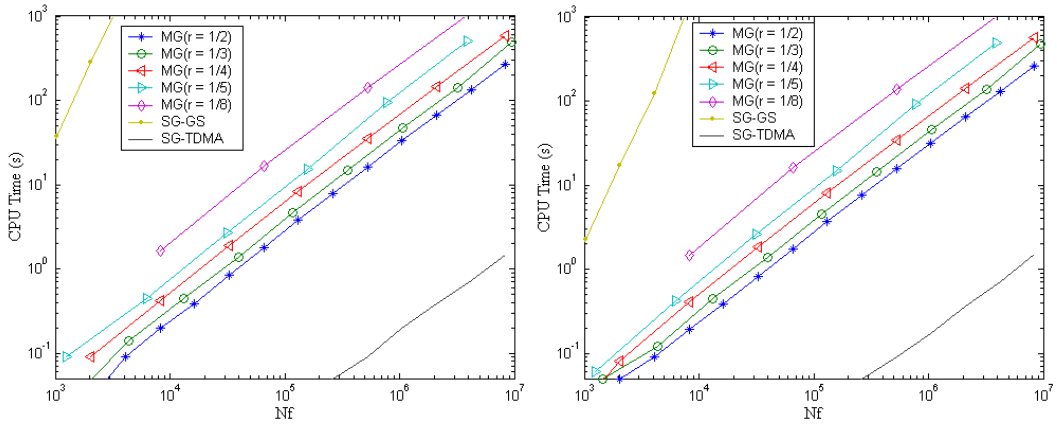

Figure 4. CPU time versus N_f elements for several methods:
(left) Poisson and (right) advection-diffusion problems

4.4. Tolerance and initial estimate

Other cases with the variations of tolerance ε as of the initial estimate $\mathbf{v}$ have also been studied. For the tolerance, one studied the cases where $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-10}$, beyond $\varepsilon = 10^{-7}$ used in the previous results. The results of the studies of the number of optimal inner iterations and of grid levels for the Poisson problem has resulted small variation in the CPU time: about 2.4% if adopting $ITI = 2q$ and 0.8% if $L = L_{maximum}$. Similar results have been obtained for the advection-diffusion problem: small variation in CPU time, about 3.4% if adopting $ITI = 2q$ and 0.61% if $L = L_{maximum}$.

For the variation of initial estimate, one studied the cases where $\mathbf{v} = \mathbf{1}/\mathbf{2}$ and $\mathbf{v} = \mathbf{1}$, beyond $\mathbf{v} = \mathbf{0}$ used in the previous results. The results obtained for the number of optimal inner iterations for the Poisson problem has resulted a substantial variation in the CPU time: about 28.7% if adopting $ITI = 2q$. The results for the optimal level number have resulted small variation: approximately 0.90% if $L = L_{maximum}$. Similar results have been obtained for the advection-diffusion problem: the variation in the CPU time is about 22.0% if adopting $ITI = 2q$; and small variation, approximately 0.57% if $L = L_{maximum}$. This shows that the minimization problem of the CPU time versus inner iterations and the grid level number is weakly dependent of the tolerance and strong dependent of the initial estimate, at least for the tested Poisson and the advection-diffusion problems.

5. Conclusion

In this work it was studied the influence of coarsening ratio of grids on the CPU time in the geometric multigrid method (correction scheme with V-cycle). The considered mathematical models involve one-dimensional problems of heat transfer with Dirichlet boundary conditions. The equations were discretizated with the finite difference method. The comparisons were made with the Gauss-Seidel and the TDMA singlegrid methods.

Based on the results of this work, one verified that:

1) The numbers of inner iterations and the grid levels affect the CPU time significantly. It is recommended to use $ITI = 2q$ and $L = L_{maximum}$ for pure ratios $r = 1/q$ and any N_f.
2) The optimal numbers of the inner iterations and the grid levels are weakly dependent of the tolerance demanded for the problem, but they are strongly dependent of the initial estimate.
3) For the same ε, $\mathbf{v}$ and N_f, amongst the coarsening ratios to multigrid methods, it was verified that $t_{cpu}(r = 1/2) < t_{cpu}(r = 1/3) < t_{cpu}(r = 1/4) < t_{cpu}(r = 1/5) < t_{cpu}(r = 1/8)$. Therefore, it is recommended to use the ratio $r = 1/2$.

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