EFFECT OF P-NORMS ON THE ACCURACY ORDER OF NUMERICAL SOLUTION ERRORS IN CFD

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Abstract. *The objective of this work is to evaluate the use of several p-norm types in the verification of numerical solutions in Computational Fluid Dynamics (CFD). Theoretical aspects of numerical errors and vector norms are discussed, and results of numerical experiments are presented. The one-dimensional advection-diffusion equation is used as example, which is solved by the finite volume method using schemes of first, second and third-order accuracy. The study encompasses seven types of metrics, involving* l_1, l_2 *and* l_∞*-norms; fifteen variables of interest; and fifteen uniform grids. The analytical deductions about the equivalences among norms are corroborated by the numerical results. With regard to the variables of interest studied, it was found that the orders varied according to the type of norm employed. We found permanence, degeneration or elevation of the order of the numerical scheme used here. Among the metrics investigated, the ones that maintained the order of the numerical scheme when applied to the numerical error were: mean* l_1*-norm,* l_2*-norm of mean square nodal errors and the* l_∞*-norm.*

Keywords: *verification, finite volume, numerical error, CFD.*

1. INTRODUCTION

In the current literature it is common to use vector norms in numerical verification procedures in which, basically, the numerical error involved and its order of accuracy are estimated. These norms are used because they characterize metrics that allow for the analysis of the order of accuracy of the error of a given numerical solution. Simonsen and Stern (2003), Falcão et al. (2006), Meyers et al. (2007), Matheou et al. (2008), and Ju et al. (2009) are examples of works that adopt this approach.

Determining the order of accuracy is important, above all, from the following standpoints: use of error estimators for cases of unknown analytical solutions, e.g., GCI (*Grid Convergence Index*) (Roache, 1998) and Richardson (Marchi and Silva, 2002); for confirmation of the theoretical order of accuracy of the numerical model employed; or to determine the practical order of accuracy when the theoretical order is unknown. In the investigation of this order for a fixed number of variables and discretization interval, the choice of the norm to be employed may lead to different results, which in turn may lead to incorrect interpretations. Studies that deal with these effects are currently not available in the literature.

The objective of this work is to evaluate the use of several types of norms in the verification of numerical solutions in CFD. We intend to demonstrate that, in the present context, vector norms are not equivalent, and to identify the metrics that maintain the theoretical order of accuracy of the numerical model adopted. To this end, some theoretical aspects of numerical errors and vector norms are discussed, and the results of numerical experiments are presented.

As the model problem, we consider the one-dimensional advection-diffusion equation solved by the finite volume method with schemes of first, second and third orders of accuracy. The one-dimensional approach is motivated by the possibility of grid refinement up to an order of millions of nodes, which allows for verification of asymptotic behaviors. It is also assumed that the one-dimensional results are applicable to two and three dimensions. In this study we consider seven types of metrics obtained by using l_1, l_2 and l_∞-norms, fifteen variables of interest, and fifteen uniform grids.

2. NUMERICAL VERIFICATION IN CFD

The numerical solution of a problem whose mathematical model is an equation or a set of differential equations can be generalized by (Garbey and Picard, 2008)

$$F\left[\phi(x)\right] = S(x), \quad x \in \Omega \subset R^n, \quad \phi = g \;\; \forall \; x \in \partial\Omega \subset R^n \tag{1}$$

where ϕ represents the dependent variable or set of variables, x is the independent variable or set of variables, Ω is the calculating domain, and $\partial\Omega$ is its boundary. Considering $(V, \|\;\|_V)$ and $(W, \|\;\|_W)$ as two normed vector spaces of finite dimensions, $F_h : V \to W$ corresponds to the operator that represents the application of a numerical method – application of V in W. To exemplify: a process of discretization by finite volumes in grids $M(h)$ parameterized by

$h > 0$. In practice, one has an approximation for the solution ϕ_h in grids $M(h)$ generated by a computational code (numerical algorithm) C applied at points S_h:

$$C : S_h \rightarrow \phi_h \tag{2}$$

The smaller h is, the more refined the discretization of the calculating domain and thus, it is expected that the approximate numerical solution is more accurate. However, this is not always the case, so a structured process of numerical verification is needed. These processes basically consist in obtaining an estimate for the numerical error (E) involved, and its monitoring through the use of a given metric. Among other aspects, the aim is to verify whether

$$\|E\|_V \rightarrow 0 \text{ for } h \rightarrow 0 \tag{3}$$

2.1 Numerical Error and its Order of Accuracy

The numerical error (E) can be defined as the difference between the exact analytical solution (Φ) of a variable of interest and its numerical solution (ϕ), i.e.,

$$E(\phi) = \Phi - \phi \tag{4}$$

where E can be caused by four sources of error (Marchi and Silva, 2002): truncation, iteration, round-off and programming. When the other sources are absent or very minor in relation to truncation errors, E can also be called a discretization error.

By analogy to the general equation of truncation error, the discretization error of a numerical solution is given by (Ferziger and Peric, 2002; Roache, 1998; Marchi and Silva, 2002)

$$E = c_1 h^{P_L} + c_2 h^{P_2} + c_3 h^{P_3} + \ldots \tag{5}$$

where the coefficients c_j, $j = 1, 2, 3...$ are real numbers that are functions of the dependent variable (of the problem) and its derivatives, but are assumed to be independent of the size (h) of the control volumes considered in the discretization process.

By definition, the true orders (P_V) of the error are the exponents of h in Eq. (5). These orders are real numbers that follow the relation: $P_L < P_2 < P_3 < \ldots$. The smallest exponent, P_L, is called the asymptotic order. When $h \rightarrow 0$, the first parcel (Eq. (5)) is the principal component of the discretization error, i.e., it dominates the total value of E (Marchi and Silva, 2002).

P_L is often treated in the literature as error order or accuracy order and is denoted by P. Results and discussions about numerical verification procedures are normally centered on this order. Roy (2005), Falcão et al. (2006) and Matheou et al. (2008) are examples of works that follow this methodology.

There are currently a considerable number of methods to estimate discretization errors. These methods can be classified into a priori and a posteriori methods, but in general both consider the dominant term of the general expression of the discretization error (Eq. (5)). In other words, they consider

$$E \cong c h^P \tag{6}$$

an error of order P for $h \rightarrow 0$. Since it is impossible to adopt this limit in practice, $h \rightarrow 0$ is considered the process of refinement of $M(h)$. Usually, the effective (P_E) and apparent (P_U) orders are admitted, which correspond, respectively, to the local slope for the error curve and its estimate versus h in logarithmic scale graphs (Marchi and Silva, 2002). Therefore, they are employed as approximations for P.

Considering the numerical solutions ϕ_F and ϕ_C, for ϕ in two grids, fine $(M(h_F))$ and coarse $(M(h_C))$, respectively, the algebraic expression for P_E is determined by

$$P_E = \frac{\log\left[E(\phi_C)/E(\phi_F)\right]}{\log(q)} \tag{7}$$

where $E(\phi_F)$ and $E(\phi_C)$ correspond to the errors for ϕ_F and ϕ_C, and $q = h_C / h_F$ is the grid refinement ratio. According to the definition of E, P_E can be obtained only when the analytical solution for ϕ is determined.

Considering $M(h_F)$ (fine), $M(h_C)$ (coarse) and $M(h_{SC})$ (supercoarse) grids, with a constant refinement ratio of $q = h_C / h_F = h_{SC} / h_C$, and their respective numerical solutions ϕ_F, ϕ_C and ϕ_{SC}, the apparent order (P_U) can be obtained by means of (Marchi and Silva, 2002)

$$P_U = \frac{\log\left[(\phi_C - \phi_{SC})/(\phi_F - \phi_C)\right]}{\log(q)} \tag{8}$$

2.2 Global Error by Vector Norm

Each point i of $M(h)$ has a discretization error associated to it – a local error, which is usually treated as a nodal error. However, it is normal to attempt to quantify the error at all the $M(h)$ points to obtain the global error. The estimate of the global discretization error is, among others, an item in which the process of numerical verification can be summarized (Roy, 2005).

For n points, elements or calculation volumes, the global discretization error (E_g) is determined by an expression that involves all the nodal errors, i.e.,

$$E_g = \overset{n}{\underset{i=1}{\Lambda}} c_i h^P \tag{9}$$

where Λ represents the mathematical operator that established this relationship between the nodal values, $i = 1,...,n$.

In numerical verification procedures of CFD, Λ is obtained, in most cases, through vector norm. In general, it is common procedure to use l_1, l_2 and l_∞-norms. However, no justifications are available that point to equivalences among these norms, or decision-making criteria with regard to the adoption of a metric characterized by a given norm. As an example, in the numerical verification of a laminar flow problem described by Navier-Stokes equations, Carpenter et al. (2005) consider the use of l_2 and l_∞-norms to determine E_g. At the end, they admit that the results lead to qualitatively similar conclusions and mention the equivalence between the norms. However, they do not describe the orders of accuracy involved. Upon analyzing these results, one finds that the cited orders are distinct.

By definition, from an analytical standpoint, two norms r and s in a vector space $V \subset R^n$, denoted by $\| \, \|_r$ and $\| \, \|_s$, are called equivalent if real constants k_1 and $k_2 \in R$ exist, such that (Golub and Van Loan, 1996)

$$k_1 \|v\|_r \le \|v\|_s \le k_2 \|v\|_r \tag{10}$$

where $v \in V$ is any vector of dimension n. The norms $\| \, \|_1$ (l_1-norm), $\| \, \|_2$ (l_2-norm, or Euclidean norm) and $\| \, \|_\infty$ (l_∞-norm) are examples of analytical equivalence:

$$\|v\|_\infty \le \|v\|_1 \le n\|v\|_\infty \tag{11}$$

$$\|v\|_\infty \le \|v\|_2 \le \sqrt{n}\,\|v\|_\infty \tag{12}$$

$$\frac{1}{n}\|v\|_1 \le \|v\|_2 \le \sqrt{n}\,\|v\|_1 \tag{13}$$

However, in CFD numerical verification procedures, the application of these norms may lead to different interpretations about the accuracy of the numerical results obtained, due to changes in the order of accuracy. It can therefore be stated that, in this context, the equivalence between the orders of l_1, l_2 and l_∞-norms is not verified.

3. EFFECT OF VECTOR NORMS ON THE ACCURACY ORDER

In this section, we investigate the effect resulting from the use of the l_1, l_2 and l_∞-norms on the order of the numerical method utilized. The section begins with a presentation of the problem model adopted, followed by a description of the numerical and analytical results obtained with the use of seven metrics – l_1-norm and its mean, l_2-norm, mean l_2-norm obtained in two distinct ways, l_∞-norm and its mean on the numerical error involved (E) and on the dependent variable (ϕ) of the problem.

3.1 Model Problem

Considering the mathematical model of conservation of thermal energy with steady one-dimensional flow, incompressible fluid, without generation of heat and viscous dissipation, and with constant properties and velocities in a continuous medium, one has the advection-diffusion equation:

$$P_e \frac{d\phi}{dx} = \frac{d^2\phi}{dx^2} \tag{14}$$

where P_e = Peclet number, ϕ is the dependent variable of the problem (temperature) and x is the independent variable (spatial coordinate). The length of the calculating domain (D) considered was the interval $[0,1]$.

The boundary conditions applied (Dirichlet conditions) were: $\phi(0)=0$ e $\phi(1)=1$. Thus, the analytical solution for Eq. (14) is

$$\phi(x) = \frac{e^{P_e x} - 1}{e^{P_e} - 1} \tag{15}$$

The numerical solutions to this problem were obtained using the finite volume method (Versteeg and Malalasekera, 2007), with first, second and third-order numerical approximations. The TDMA method was used to solve the system of equations resulting from the process of discretization (Ferziger and Peric, 2002). The computational code was developed using the Fortran Intel 9.1 application with quadruple precision. The calculations were performed in 15 distinct grids, with a refinement ratio of $q=3$. Among these grids, the coarsest had $n=5$, and the finest grid had $n = 23{,}914{,}845$ calculating volumes, where $h = D/n$.

The variables of interest, i.e., the variables for which the solutions were obtained and the orders analyzed, were:

(a) $E[\phi(1/2)]$, the nodal error for the numerical solution of ϕ at the central point of the grid;

(b) $\|E\|_1$, the global error, determined using l_1-norm, of the values of the nodal errors $E_i,\ i=1,\ldots,n$;

(c) $\overline{\|E\|}_1$, the mean l_1-norm of E – the ratio of $\|E\|_1$ to the number of calculating volumes n;

(d) $\|\phi\|_1$, the l_1-norm of ϕ – use of l_1-norm on the nodal values $\phi_i,\ i=1,\ldots,n$;

(e) $\overline{\|\phi\|}_1$, the mean l_1-norm of ϕ – the ratio of $\|\phi\|_1$ to the number of calculating volumes n;

(f) $\|E\|_2$, the global error, determined by the use of l_2-norm, of the values of the nodal errors $E_i,\ i=1,\ldots,n$;

(g) $\overline{\|E\|}_2$, the mean l_2-norm of E – the ratio of $\|E\|_2$ to the number of calculating volumes n;

(h) $\|E/n\|_2$, the l_2-norm of the mean square nodal errors ($E_i,\ i=1,\ldots,n$), i.e.,

$$\|E/n\|_2 = \sqrt{\frac{\sum_{i=1}^{n} E_i^{\,2}}{n}} \tag{16}$$

(i) $\|\phi\|_2$, the application of l_2-norm on the nodal values $\phi_i,\ i=1,\ldots,n$;

(j) $\overline{\|\phi\|}_2$, the mean l_2-norm of ϕ – the ratio of $\|\phi\|_2$ to the number of calculating volumes n;

(j) $\|\phi/n\|_2$, the l_2-norm of the mean square nodal values ($\phi_i,\ i=1,\ldots,n$), i.e.,

$$\|\phi/n\|_2 = \sqrt{\frac{\sum_{i=1}^{n} \phi_i^{\,2}}{n}} \tag{17}$$

(k) $\|E\|_\infty$, the global error determined by application of the l_∞-norm on the nodal errors E_i, $i = 1,...,n$;

(l) $\overline{\|E\|}_\infty$, the mean l_∞-norm of E – the ratio of $\|E\|_\infty$ to the number of calculating volumes n;

(m) $\|\phi\|_\infty$, the application of the l_∞-norm on the nodal values ϕ_i, $i = 1,...,n$;

(n) $\overline{\|\phi\|}_\infty$, the mean l_∞-norm of ϕ – the ratio of $\|\phi\|_\infty$ to the number of calculating volumes n.

3.2 Results

The results presented below consist of the determination of the practical and theoretical order of accuracy for the nodal and global errors, considering the use of the above described metrics. The practical orders of accuracy were obtained with P_E and P_U (Eqs. (7) and (8)). The theoretical orders of accuracy were determined considering the definitions of accuracy order, local error, global error and analytical equivalence between vector norms (Eqs. (4), (5), (6), (9) (10), (11), (12) and (13)). Based on the calculation of P_E and P_U, an investigation was also made of the practical orders of convergence generated by the application of the seven metrics on ϕ. These results were confirmed considering the algebraic development – application of the metrics – on Eq. (15).

3.2.1 Accuracy order (*P*) of the nodal error

In the first, second and third-order numerical schemes it was found that, for the variable $E[\phi(½)]$, $P_E, P_U \approx P$. These results confirm the theoretical order of accuracy (*P*) of the nodal error.

3.2.2 Accuracy order of $\|E\|_1$ and $\overline{\|E\|}_1$

Analytically, considering Eqs. (6) and (9), it is possible to identify the effect caused by $\|E\|_1$ on the order of accuracy of the nodal error. In other words, the application of l_1-norm to obtain the global error results in

$$\|E\|_1 = D\,\bar{c}\,h^{P-1} \tag{18}$$

where $\bar{c} = \frac{1}{n}\sum_{i=i}^{n}|c_i|$, $i = 1,...,n$ represents the number of volumes and $D = n.h$ the size of the calculating domain.

Upon investigating $\|E\|_1$ for the first-order numerical scheme, we found that $\|E\|_1 \to 0.414...$ when $h \to 0$. This behavior can be verified in Eq. (18), where $P = 1$ leads to: $\|E\|_1 = D\,\bar{c}$. In this case, it was found that: $P_E = 0 \neq P_U = 1$. $P_E = 0$ is due to the fact that the global error does not approach zero with the refinement of $M(h)$ ($\|E\|_1 \neq 0,\ h \to 0$). Hence, the numerator of Eq. (7) becomes null. On the other hand, $P_U = 1$ is justified by the convergent behavior of $\|E\|_1$. That is, for the first order scheme, the numerator of Eq. (8) approaches the denominator, with $h \to 0$.

For the second and third order schemes, the values obtained for P_E and P_U confirmed the degeneration of one unit on the value of P, resulting from the application of $\|E\|_1$ ($P_E, P_U \to P-1,\ h \to 0$).

Analytically, the use of the mean l_1-norm on the values of the nodal errors leads to

$$\overline{\|E\|}_1 = \bar{c}\,h^{P} \tag{19}$$

This behavior was also found in the three numerical schemes (Fig.1). In other words, for $\overline{\|E\|}_1$ one has: $P_E, P_U \to P$, with $h \to 0$.

3.2.3 Accuracy order of $\|E\|_\infty$ and $\overline{\|E\|}_\infty$

Considering the use of the l_∞-norm on Eq. (9), one has

$$\|E\|_\infty = c^* h^P \quad (20)$$

where $c^* = \max\{|c_1|,...,|c_n|\}$. For this metric, in the three numerical schemes adopted, we found that the value of P was maintained. In other words, $P_E, P_U \to P$ for $h \to 0$.

With regard to the metric $\|\overline{E}\|_\infty$ in the three numerical schemes, the order of accuracy of the local discretization error was found to increase by one unit. That is, for $\|\overline{E}\|_\infty$ $P_E, P_U \to P+1$ with $h \to 0$. This behavior was also found in the algebraic expression obtained for $\|\overline{E}\|_\infty$, using Eq. (20):

$$\|\overline{E}\|_\infty = \frac{c^*}{D} h^{P+1} \quad (21)$$

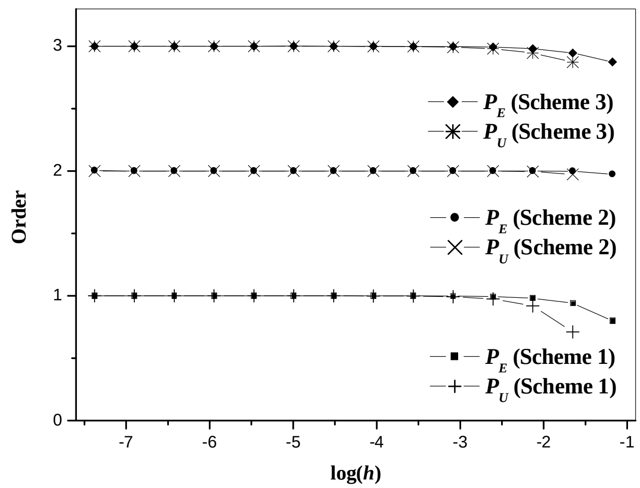

Figure 1. P_E and P_U for $\|\overline{E}\|_1$

3.2.4 Accuracy order of $\|E\|_2$, $\|\overline{E}\|_2$ and $\|E/n\|_2$

We considered here the global error determined by $\|E\|_2$, $\|\overline{E}\|_2$ and the metric $\|E/n\|_2$ determined by Eq. (16). Analytically, considering Eqs. (12) and (20), one finds that

$$c^* h^P \le \|E\|_2 \le \sqrt{D}\, c^* h^{P-1/2} \quad (22)$$

In other words, the order of accuracy resulting from $\|E\|_2$ belongs to the interval $[P-1/2, P]$.

The calculation of the practical orders of $\|E\|_2$ on the numerical schemes with $P = 1$, 2 and 3, respectively, corroborates this result. In other words, numerically, $\|E\|_2$ presented $P_E, P_U \to P-1/2$ with $h \to 0$, for the three numerical schemes adopted (Fig. 2).

Analogously to the obtainment of Eq. (22), for $\|\overline{E}\|_2$ one has the following relationship:

$$\frac{c^*}{D} h^{P+1} \le \|\overline{E}\|_2 \le \frac{c^*}{\sqrt{D}}\, h^{P+1/2} \quad (23)$$

where is can be seen that the order of accuracy resulting from $\|\bar{E}\|_2$ belongs to the interval $[P+1/2, P+1]$. In the calculation of the practical orders of $\|\bar{E}\|_2$, for the three numerical schemes, it was found that $P_E, P_U \rightarrow P+1/2$ with $h \rightarrow 0$.

Analytically, the metric $\|E/n\|_2$ can be analyzed considering Eqs. (6), (9) and (16),

$$\|E/n\|_2 = \frac{\|c\|_2}{\sqrt{n}} h^P \tag{24}$$

where $c = (c_1, ..., c_n)$. Considering Eqs. (12), (13), (18), (19) and (20), one finds the following relationship:

$$\bar{c}\, h^P \leq \|E/n\|_2 \leq c^* h^P \tag{25}$$

The numerical results obtained for $\|E/n\|_2$ indicated the permanence of the order of accuracy of the nodal error in all the numerical schemes adopted $(P_E, P_U \rightarrow P,\ h \rightarrow 0)$. It can therefore be stated that $\|E/n\|_2$ maintains the order of accuracy of the numerical scheme adopted. In the works of Falcão et al. (2006) and Roy (2005), this metric is used for purposes of numerical verification.

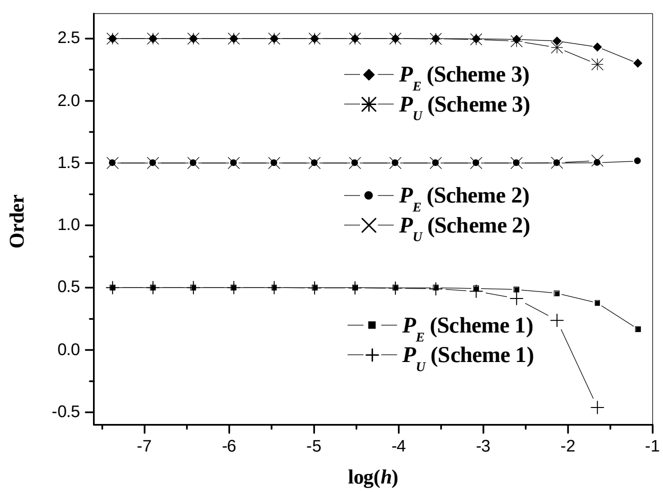

Figure 2. P_E and P_U for $\|E\|_2$

3.2.5 Convergence order of $\|\phi\|_1$ and $\|\bar{\phi}\|_1$

Upon investigating the behavior presented by $\|\phi\|_1$, in the three numerical schemes adopted, P_U was found to be constant $(P_U = -1)$. In this case, P_E could not be determined because $n \rightarrow \infty,\ h \rightarrow 0\ \Rightarrow \|\phi\|_1 \rightarrow \infty$ (unlimited analytical solution).

The analytical expression for $\|\phi\|_1$, considering Eq. (15), is determined by

$$\|\phi\|_1 = \left(\frac{1}{e^{P_e} - 1}\right) \cdot \left[e^{\frac{1}{2} P_e h} \cdot \left(\frac{e^{P_e} - 1}{e^{P_e h} - 1}\right) - n \right] \tag{26}$$

Using linear approximation based on power series (Kreyszig, 1999) for the exponential term $(e^{P_e h})$ with $h \rightarrow 0$, one has

$$\|\phi\|_1 = \frac{1}{2} + \left(\frac{1}{P_e} - \frac{1}{e^{P_e} - 1} \right) h^{-1} \tag{27}$$

By analogy to the accuracy order of the numerical error, it can be stated that $\|\phi\|_1$ has an order of convergence -1.

The analytical expression obtained for $\|\overline{\phi}\|_1$, on Eq. (15), results in

$$\|\overline{\phi}\|_1 = \frac{(e^{P_e} - P_e - 1)}{P_e(e^{P_e} - 1)} \tag{28}$$

In the three numerical schemes, it was found that $\|\overline{\phi}\|_1 \to 0.193...$ for $h \to 0$, which confirms the result of Eq. (28). As for the practical order of accuracy, it was found that $P_E, P_U \to P$ with $h \to 0$ in the first and second-order schemes. In the third-order numerical scheme, we observed: $P_E, P_U \to P - 1$ with $h \to 0$. In other words, for $P = 3$, there was degeneration of one unit in the accuracy order of the numerical scheme in response to the application of $\|\overline{\phi}\|_1$.

3.2.6 Convergence order of $\|\phi\|_2$, $\|\overline{\phi}\|_2$ and $\|\phi / n\|_2$

In the three numerical schemes, the metric $\|\phi\|_2$ presented constant P_U $(P_U = -1/2)$. Similarly to the previous case ($\|\phi\|_1$), P_E could not be calculated. The analytical expression for $\|\phi\|_2$, considering Eq. (15), is determined by

$$\|\phi\|_2 = \frac{Pe}{e^{Pe} - 1} h \sqrt{\frac{1}{3}h^{-3} + \frac{1}{2}h^{-2} - \frac{1}{3}h^{-1} - \frac{1}{2}h^0 + \frac{1}{4}h} \tag{29}$$

and, by analogy with the definition of asymptotic order (Eq. 5), one has

$$\|\phi\|_2 \cong \frac{Pe}{e^{Pe} - 1} h^{-1/2} \tag{30}$$

This expression corroborates the numerical results obtained for $\|\phi\|_2$ and confirms the negative order of convergence ($-1/2$), i.e., divergence $(h \to 0 \Rightarrow \|\phi\|_2 \to \infty)$.

Distinct behaviors were identified for $\|\overline{\phi}\|_2$ and for the metric $\|\phi / n\|_2$ determined by Eq. (17). In the calculation of $\|\overline{\phi}\|_2$, it was observed that $\|\overline{\phi}\|_2 \to 0$ and $P_E, P_U \to 1/2$ with $h \to 0$, in the three numerical schemes. Analytically, this result is confirmed by dividing Eq. (30) by $n = D/h$.

In the three numerical schemes, we found that $\|\phi / n\|_2 \to 0.314...$ for $h \to 0$. This result is confirmed by the analytical expression obtained for $\|\phi / n\|_2$, considering Eqs. (15) and (17), i.e.,

$$\|\phi / n\|_2 = \frac{1}{e^{P_e} - 1} \sqrt{\frac{e^{2P_e} - 4e^{P_e} + 2P_e + 3}{2P_e}} \tag{31}$$

The practical order of convergence (P_E and P_U) of $\|\phi / n\|_2$ presented a behavior similar to $\|\overline{\phi}\|_1$. That is, it preserved the accuracy order of the first and second-order schemes, and degenerated by one unit the value of the accuracy order of the third-order scheme.

3.2.7 Convergence order of $\|\phi\|_\infty$ and $\|\overline{\phi}\|_\infty$

In the three numerical schemes employed here, the calculations of $\|\phi\|_\infty$ and $\|\overline{\phi}\|_\infty$ led to: $\|\phi\|_\infty \to 1$ and $\|\overline{\phi}\|_\infty \to 0$ for $h \to 0$. These values are confirmed by the condition imposed on the right boundary of the problem model (Section 3.1). In this case, the refinement of $M(h)$ led to $P_E, P_U \to 1$. Analytically, this result can be evaluated considering the use of the l_∞-norm in Eq. (15) and linear approximation based on power series (Kreyszig, 1999).

4. CONCLUSIONS

An analysis was made of the effect caused by the use of vector p-norms to obtain the global discretization error, where the discretization error was defined as the difference between the exact analytical solution and the numerical solution for a given variable of interest. With the definitions of the effective order, apparent order and asymptotic order of the error, and with the convergence of these orders in the numerical experiments, the parameter order of accuracy (*P*) was adopted. The investigation then focused on the behavior of this order in the local and global errors.

According to the definition of the error, the variation of ϕ is equal to the variation of its error (*E*), with the refinement process of $M(h)$, i.e., $\Delta\phi = \Delta E$. However, since the vector norms addressed in this work (represented by $\| \ \|_k$) do not characterize linear operators, one has: $\Delta\|\phi\|_k \neq \Delta\|E\|_k$. As a result, the respective orders of accuracy (for $\|E\|_k$) and of convergence (for $\|\phi\|_k$) did not present a direct relationship.

By means of algebraic development and numerical experimentation, we found that *P* can be degenerated, elevated or maintained using vector p-norms. Based on the results obtained in the calculation of the global error, we found that the metrics: $\|\overline{E}\|_1$, $\|E/n\|_2$ and $\|E\|_\infty$ maintain the order *P*.

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