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## Repeated Richardson extrapolation to reduce the field discretization error in computational fluid dynamics

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### ABSTRACT

The aim of this study is to extend the use of repeated Richardson extrapolation to one-dimensional (1D) and two-dimensional (2D) fields in computational fluid dynamics (CFD). The following two methods are tested: completed Richardson extrapolation (CRE), a method that has been used previously in the literature, and full Richardson extrapolation (FRE), a new method developed in this study. The Poisson's, advection–diffusion, Laplace's, and Burgers' equations are solved using the finite difference method. The CRE and FRE methods were found to significantly reduce the discretization error of the numerical solutions for all nodes of the grid.

### ARTICLE HISTORY

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### Introduction

The completed Richardson extrapolation (CRE) method [1] can be used to reduce the discretization error in temperature fields and increase the order of accuracy ( $p_0$ ) of numerical solutions without extrapolation. In the CRE method, one Richardson extrapolation [2] is applied to the solutions obtained on two grids, one fine and one coarse, during the post-processing phase. For example, the coarse grid may consist of  $41 \times 41$  nodes, whereas the fine grid consists of  $81 \times 81$  nodes. The numerical solutions for the two grids are initially of order  $p_0 = 2$ ; after applying CRE, the order of accuracy is  $p_f = 4$  for all nodes of the fine grid, where  $p_f$  is the order of accuracy of numerical solution with extrapolation. In Roache and Knupp [1], the finite difference method was augmented with CRE in solving the one-dimensional (1D) Poisson's and advection–diffusion equations and the two-dimensional (2D) Poisson's equation. Using various methods based on one Richardson extrapolation, the order of accuracy of the temperature field obtained using finite differences was increased from  $p_0 = 4$  to  $p_f = 6$  for the 1D and 2D advection–diffusion equations in Sun and Zhang [3], the 2D Poisson's equation in Wang and Zhang [4], and the 3D advection–diffusion equation in Ma and Ge [5] and Zhai et al. [6]. The problems were solved in the steady state in all of these studies. In Richards [7], the CRE method was extended to the 1D transient advection–diffusion equation.

Values of  $p_f$  above 10 have been achieved for the 1D advection–diffusion equation [8] and 2D Laplace's equation [9–11] using repeated Richardson extrapolation (RRE) with the finite difference and finite volume methods on uniform square and triangular grids; in some cases, values as high as  $p_f = 19$  have been reached from initial orders of  $p_0 = 1, 2, \text{ or } 3$ . In these three studies, RRE was applied to global and local variables, such as the average temperature of the entire field, heat transfer rates at boundaries, and temperature at specific nodes of the grid. The RRE method is a post-processing method for solutions obtained on several grids. For cases with a higher number of grids, a larger number of REs can be performed, and a higher  $p_f$  value can, in principle, be achieved.

## Nomenclature

$C$	Richardson correction	$N_g$	total number of nodes in grid $g$
CDS-2	second-order central differencing scheme	$p$	pressure (Pa)
CDS-4	fourth-order central differencing scheme	$P, W, E$	spatial position of the node in the grid
$E$	discretization error in the numerical solution	$p_E$	effective order
$g$	number of a grid	$Pe$	Peclet number
$G$	number of grids	$p_f$	order of accuracy of numerical solution
$h$	distance between two consecutive nodes in each grid	$p_m$	true orders
$k$	weighting factor	$p_0$	theoretical order of accuracy
$L1, L2$ and $Li$	$L1$ -norm, $L2$ -norm, and $L_\infty$ -norm	$r$	grid refinement ratio
$m$	number of Richardson extrapolations	$Re$	Reynolds number
$N$	total number of nodes in grid	$u, v$	dependent variables
		UDS-1	first-order upwind differencing scheme
		$x, y$	spatial coordinates

The primary aim of this study is to apply RRE to fields in CFD, for the first time to our knowledge. The characteristics of the CRE and RRE methods are combined in a new method known as full Richardson extrapolation (FRE). This new algorithm is also a post-processing method for solutions obtained on several grids. The specific aims of this study are the following: (1) to test the FRE method on the 1D Poisson's, advection–diffusion, and Burgers' equations as well as the 2D Laplace's and Burgers' equations; (2) to determine whether the repeated application of CRE can be used to increase  $p_f$  for these equations; and (3) to apply FRE and CRE to numerical solutions of order  $p_0 = 1, 2$  and 4. To the best of our knowledge, these aims have not yet been achieved in the previous literature. The importance of this work lies in obtaining numerical solutions with a high order of accuracy over entire fields during the post-processing phase. Further details on this study are presented in Giacomini [12].

## Mathematical models

### 1D models

Three 1D mathematical models are used in the tests of the CRE and FRE methods: the Poisson's, advection–diffusion, and Burgers' equations, which can be expressed as follows:

$$\frac{d^2 u}{dx^2} = S_1, \quad Pe \frac{du}{dx} = \frac{d^2 u}{dx^2}, \quad Re u \frac{du}{dx} = \frac{d^2 u}{dx^2} + S_2 \quad (1)$$

Here,  $u$  represents the dependent variable,  $x$  is the spatial coordinate,  $Pe$  is the Peclet number,  $Re$  is the Reynolds number, and  $S_1 = -\pi^2 \sin(\pi x)$  and  $S_2 = e^x [Re(e^x - 1) - e + 1] / (e - 1)^2$  are the source terms. The Dirichlet boundary conditions are  $u(0) = u(1) = 0$  for the Poisson's equation and  $u(0) = 0$  and  $u(1) = 1$  for the advection–diffusion and Burgers' equations. The analytical solutions are  $u = \sin(\pi x)$  for the Poisson's equation and  $u = (e^{xb} - 1) / (e^b - 1)$  for the advection–diffusion and Burgers' equations, where  $b = Pe$  for the advection–diffusion equation and  $b = Re$  for the Burgers' equation.

### 2D models

The first 2D mathematical model used to test the CRE and FRE methods is the Laplace's equation, which can be expressed as follows:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (2)$$

where  $y$  is the second spatial coordinate. The boundary conditions are  $u(x, 1) = \sin(\pi x)$  and  $u(0, y) = u(1, y) = u(x, 0) = 0$ . The analytical solution to the equation is given by  $u = \sin(\pi x) \sinh(\pi y) / \sinh(\pi)$ .

The Burgers' equations comprise the second 2D mathematical model which is used to test the CRE and FRE methods. These equations can be expressed as follows:

$$\begin{cases} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - S(x, y, Re) \end{cases} \quad (3)$$

where  $v$  represents the second dependent variable,  $p$  is the pressure (determined analytically), and  $S$  is the source term. The variables  $p$  and  $S$  are given in Shih et al. [13]. The boundary conditions are as follows:  $u(x, 1) = 16(x^4 - 2x^3 + x^2)$ ,  $u(0, y) = u(1, y) = u(x, 0) = 0$ , and  $v(0, y) = v(1, y) = v(x, 0) = v(x, 1) = 0$ . The analytical solutions [13] are  $u = 8(x^4 - 2x^3 + x^2)(4y^3 - 2y)$  and  $v = -8(4x^3 - 6x^2 + 2x)(y^4 - y^2)$ .

## Numerical models

### Numerical solution without extrapolation

The 1D Poisson's equation was solved using the finite difference method on uniform grids, with a second-order central differencing scheme (CDS-2) [14] or fourth-order standard compact central differencing scheme (CDS-4) [15] that is used to approximate the second-order derivative in Eq. (1). The tridiagonal matrix algorithm [14, 16] was used to solve the system of equations, and the computational program was implemented in the FORTRAN 95 language (version Intel 11.1.065) with quadruple precision. The theoretical order of accuracy ( $p_0$ ) of the numerical solution for  $u$  is 2 with CDS-2 and 4 with CDS-4.

The 1D advection-diffusion equation was solved using the same numerical model as for the 1D Poisson's equation, except that CDS-2 or the first-order upwind differencing scheme (UDS-1) [14] was used to approximate the first-order derivative in Eq. (1), and only CDS-2 was used to approximate the second-order derivative in Eq. (1). The theoretical order of accuracy ( $p_0$ ) of the numerical solution for  $u$  is therefore 1 with UDS-1/CDS-2 and 2 with CDS-2/CDS-2.

The 1D Burgers' equation was solved using the same numerical model as for the 1D Poisson's equation, with the following modifications. First, CDS-2 was used to approximate the first- and second-order derivatives in Eq. (1). In addition, a sufficient number of iterations were performed to reach the machine round-off error, thereby eliminating the iteration error in the solution. The theoretical order of accuracy ( $p_0$ ) of the numerical solution for  $u$  is 2.

The 2D Laplace's equation was also solved using the finite difference method on uniform grids. The CDS-2 method was used to approximate the two second-order derivatives in Eq. (2), and the modified strongly implicit method [17] was used to solve the system of equations associated with the geometric multigrid correction scheme (CS) [18]. The computational program was implemented in the FORTRAN 95 language (version Intel 11.1.065) with quadruple precision. A sufficient number of iterations were performed to reach the machine round-off error, thereby eliminating the iteration error in the solution. The theoretical order of accuracy ( $p_0$ ) of the numerical solution for  $u$  is 2.

The 2D Burgers' equations were solved using the same numerical model as for the 2D Laplace's equation, with the following modifications. The CDS-2 or UDS-1 method was used to approximate the first-order derivatives in Eq. (3), and a geometric multigrid full-approximation scheme (FAS) [18] was used in place of the CS. The theoretical order of accuracy ( $p_0$ ) of the numerical solutions for  $u$  and  $v$  is 1 with UDS-1/CDS-2 and 2 with CDS-2/CDS-2.

### Numerical solution with multiple extrapolations

For 1D problems, the FRE method computes the extrapolated solution in two steps. The first step considers those nodes that have the same coordinates in the fine and coarse grids, represented by the circles in [Figure 1](#) ( $W$  and  $E$ ). For each node in the fine grid with a corresponding node in the coarse grid, the Richardson correction,  $C$ , was calculated as follows:

$$C_{g,P}^m = \frac{u_{g,P}^{m-1} - u_{g-1,P}^{m-1}}{r^{p_{m-1}} - 1} \quad (4)$$

where  $P$  is the spatial position of the node in the grid ( $W$  or  $E$  in [Figure 1](#));  $p_m$  represents the true orders [9, 11, 19] of the discretization error;  $g = [1, G]$ , where  $g = 1$  corresponds to the coarser grid (with the greatest  $h$ ) and  $g = G$  corresponds to the finer grid (with the smallest  $h$ );  $h$  is the distance between two consecutive nodes in each grid;  $r = h_{g-1}/h_g$  is the grid refinement ratio; and  $m$  is the number of Richardson extrapolations. Here,  $m = 0$  represents no extrapolation (the solution obtained using the methods described in ‘Numerical solution without extrapolation’),  $m = 1$  represents one standard Richardson extrapolation, and  $m > 1$  represents RRE. Equation (4) is valid for  $g = [2, G]$  and  $m = [1, g - 1]$  and was motivated by the works in Roache and Knupp [1] and Marchi et al. [9, 11].

The second step considers the nodes in the fine grid that do not have counterparts in the coarse grid, represented by the diamond in [Figure 1](#) ( $P$ ). For these nodes, the Richardson correction is calculated as follows:

$$C_{g,P}^m = C_{g,W}^m + k_{g,P}^1 (C_{g,E}^m - C_{g,W}^m) \quad (5)$$

where nodes  $W$  and  $E$  lie to the left and the right of node  $P$  ([Figure 1](#)), respectively, and the Richardson corrections for  $W$  and  $E$  are calculated using Eq. (4). The weighting factor  $k$  is given by

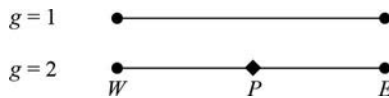
$$k_{g,P}^1 = \frac{u_{g,P}^{m-1} - u_{g,W}^{m-1}}{u_{g,E}^{m-1} - u_{g,W}^{m-1}} \quad (6)$$

Finally, the extrapolated solution is obtained for all of the nodes in the fine grid using the following equation:

$$u_{g,P}^m = u_{g,P}^{m-1} + C_{g,P}^m \quad (7)$$

The FRE algorithm for 1D fields is as follows:

1. Obtain the solution  $u_{g,P}^{m=0}$  without extrapolation for each node  $P$  of each grid  $g$ , that is, for  $P = [1, N_g]$  and  $g = [1, G]$ , where  $N_g$  represents the total number of nodes in grid  $g$ .
2. Set  $m = 1$ .
3. Set  $g = m + 1$ .
4. Using Eq. (4), calculate the Richardson correction for each node of the fine grid ( $g$ ) that coincides with a node in the coarse grid ( $g - 1$ ), represented by the circles in [Figure 1](#).
5. Using Eqs. (5) and (6), calculate the Richardson correction for each node of the fine grid ( $g$ ) that does not coincide with a node in the coarse grid ( $g - 1$ ), represented by the diamond in [Figure 1](#).
6. Using Eq. (7), calculate the solution with  $m$  extrapolations for all of the nodes of the fine grid.
7. Set  $g = g + 1$ .



**Figure 1.** Types of nodes for extrapolation in 1D problems.

8. Return to step 4 if  $g \leq G$ .
9. Set  $m = m + 1$ .
10. Return to step 3 if  $m \leq G - 1$ .

The FRE algorithm for 2D fields is as follows:

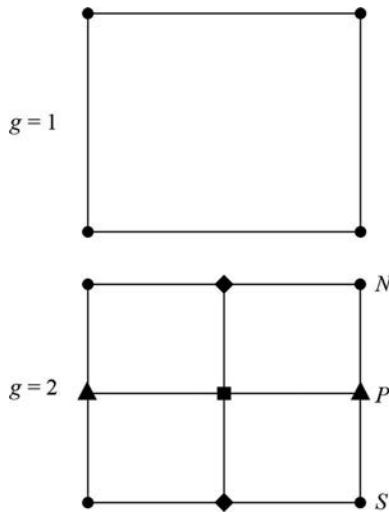
1. Obtain the solution  $u_{g,P}^{m=0}$  without extrapolation for each node  $P$  of each grid  $g$ , that is, for  $P = [1, N_g]$  and  $g = [1, G]$ .
2. Set  $m = 1$ .
3. Set  $g = m + 1$ .
4. Using Eq. (4), calculate the Richardson correction for each node of the fine grid ( $g$ ) that coincides with a node in the coarse grid ( $g - 1$ ) (such that the nodes in the fine and coarse grids have the same coordinates), represented by the circles in [Figure 2](#).
5. For the nodes of the fine grid ( $g$ ) whose vertical coordinates correspond to a row of the coarse grid ( $g - 1$ ) but whose horizontal coordinates do not coincide with a node in the coarse grid, represented by the diamonds in [Figure 2](#), calculate the Richardson correction using Eqs. (5) and (6), as in the 1D case.
6. For the nodes of the fine grid ( $g$ ) whose horizontal coordinates correspond to a column of the coarse grid ( $g - 1$ ) but whose vertical coordinates do not coincide with a node in the coarse grid, represented by the triangles in [Figure 2](#), calculate the Richardson correction as follows:

$$C_{g,P}^m = C_{g,S}^m + k_{g,P}^2 (C_{g,N}^m - C_{g,S}^m) \tag{8}$$

where

$$k_{g,P}^2 = \frac{u_{g,P}^{m-1} - u_{g,S}^{m-1}}{u_{g,N}^{m-1} - u_{g,S}^{m-1}} \tag{9}$$

7. For the nodes in the fine grid ( $g$ ) that do not coincide with the coarse grid ( $g - 1$ ) in either the horizontal or vertical direction, represented by the square in [Figure 2](#), calculate the Richardson correction using Eqs. (5) and (6), where the square node in [Figure 2](#) becomes node  $P$  in Eqs. (5) and (6).
8. Using Eq. (7), calculate the solution with  $m$  extrapolations for all nodes of the fine grid ( $g$ ).



**Figure 2.** Types of nodes for extrapolation in 2D problems.

9. Set  $g = g + 1$ .
10. Return to step 4 if  $g \leq G$ .
11. Set  $m = m + 1$ .
12. Return to step 3 if  $m \leq G - 1$ .

The CRE method, developed by Roache and Knupp [1], was applied to both 1D and 2D problems using the two algorithms mentioned above, but only for  $m = 1$  with weighting factors  $k = \frac{1}{2}$  for the 1D case and  $k = \frac{1}{2}$  and  $\frac{1}{4}$  for the 2D case. That is, the last two steps of each algorithm were not executed. In this study, the RRE method is combined with the CRE method, that is, the above two algorithms are executed in their entirety.

### Performance analysis of the CRE and FRE methods

Three types of criteria [20, 21] were used to analyze the performance of CRE and FRE in reducing the discretization error for all of the nodes in the grid:

1. The average discretization error:

$$L1 = \frac{\sum_{i=1}^N |E_i|}{N} \quad (10)$$

2. The root-mean-square discretization error:

$$L2 = \left[ \frac{\sum_{i=1}^N (E_i)^2}{N} \right]^{1/2} \quad (11)$$

3. The maximum discretization error:

$$Li = \max_{1 \leq i \leq N} |E_i| \quad (12)$$

where max refers to the maximum value of  $E_i$  among the  $N$  values.

These three criteria were chosen because, among the metrics investigated in Marchi and Martins [21], they maintain the order of accuracy of the numerical solution. In Eqs. (10)–(12),  $N$  represents all of the nodes of the grid except the boundary nodes, and the discretization error in the numerical solution for each node  $i$  is defined as follows:

$$E_i = u_i^{\text{analytical}} - u_i^{\text{numerical}} \quad (13)$$

The concept of the effective order of RRE [9, 11] was used to quantify the efficiency of the CRE and FRE methods in reducing the discretization error for all of the nodes in the grid:

$$(p_E)_{g,m} = \frac{\log\left(\frac{L1_{g-1,m-1}}{L1_{g,m}}\right)}{\log(r)} \quad (14)$$

where  $g = [2, G]$  and  $m = [1, g - 1]$ . Equation (14) can also be applied when  $m = 0$  and  $g = [2, G]$  if the subindex  $m - 1$  is taken to be equal to  $m$  in this case. A higher value of  $p_E$  corresponds to a higher efficiency of the RRE method and a more accurate numerical solution.

The results are presented only for  $L1$  as the results for  $L2$  and  $Li$  are qualitatively similar. The results are provided for all three criteria in Giacomini [12].

**Results**

For each problem, the numerical solution was obtained on 10 grids with 3, 5, 9, 17, 33, 65, 129, 257, 513, and 1,025 nodes in each coordinate direction; therefore,  $r = 2$  and  $G = 10$ .

**1D Poisson's equation**

Figure 3 shows the results for the 1D Poisson's equation. Figure 3(a) shows the  $L_1$  results for  $m = 0$ , the numerical solution obtained without extrapolation using the CDS-2 method;  $m = 1$ , the numerical solution obtained with one extrapolation; and  $m = g - 1$ , the numerical solution obtained with the maximum possible number of extrapolations for each grid  $g$ . Each symbol along the curves in Figure 3(a) represents a grid of different size  $h$ . For a given grid size, the smallest error is obtained with  $m = 1$  for the CRE method, while the error is smallest with  $m = g - 1$  (limited to 7) for FRE. There is a small increase in the error for  $m > 1$  with the CRE method. In the case of the finest grid (1,025 nodes and  $h = 9.765625 \times 10^{-4}$ ), the errors were reduced by factors of  $4.25 \times 10^5$  and  $4.83 \times 10^{24}$  compared to  $m = 0$  (no extrapolation) using the CRE and FRE methods, respectively; therefore, the FRE method reduces the error by a much greater factor compared to the CRE method.

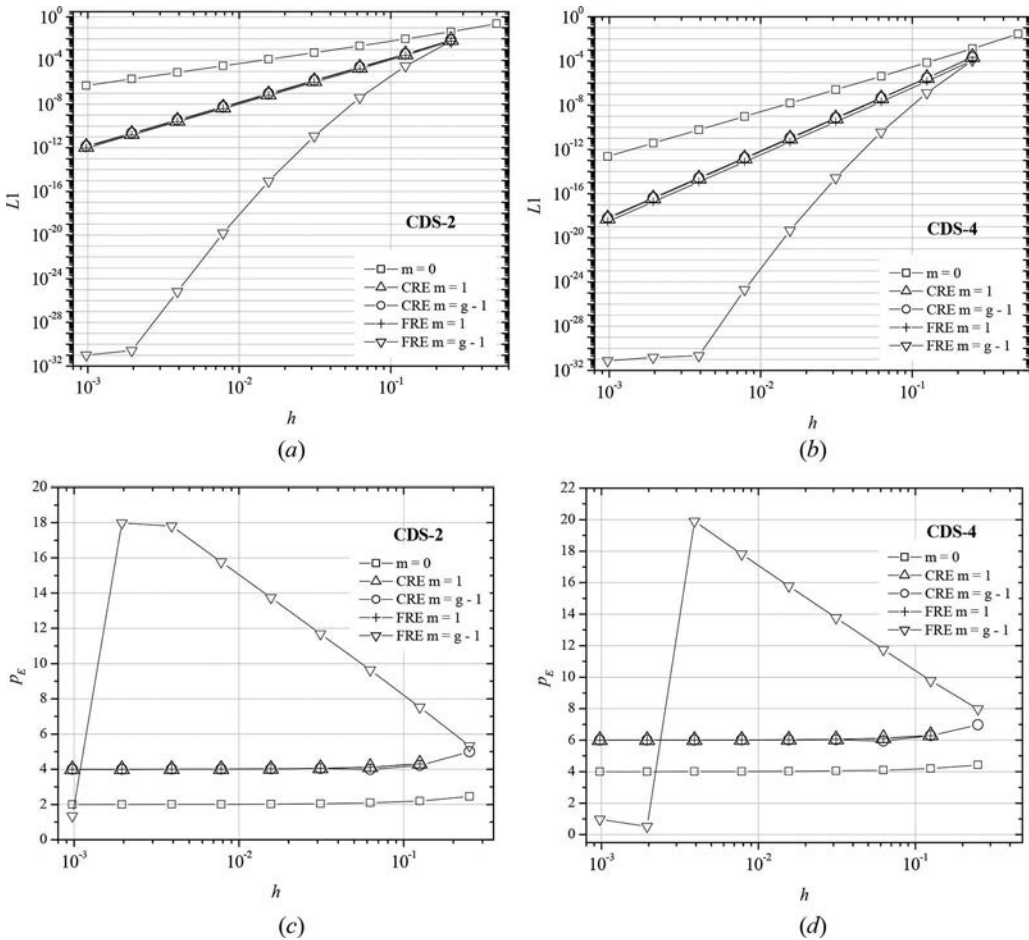


Figure 3. Results for the 1D Poisson equation.



Figure 3(c) shows the effective order ( $p_E$ ) of each error curve in Figure 3(a). For  $m = 0$  (no extrapolation), the values of  $p_E$  tend toward 2, which is the theoretical value expected for  $p_0$  in the numerical solution for  $u$  with CDS-2. In the CRE method, the values of  $p_E$  tend toward 4 for  $m \geq 1$ . In the FRE method, the values of  $p_E$  increase with  $m$  for  $m = g - 1$ , reaching a maximum value of approximately 18 for  $m = 7$  and 8; for  $m = 9$ , the value of  $p_E$  is lower than 2 due to rounding errors in  $L1$ .

Figure 3(b) shows the  $L1$  results obtained with CDS-4 for the 1D Poisson's equation. These results are qualitatively similar to those in Figure 3(a) with CDS-2. However, comparing Figure 3(a) and 3(b), the errors are smaller for CDS-4 compared to CDS-2 for  $m = 0$  with a given  $h$  and for CRE and FRE with the same  $m$ , as expected.

Figure 3(d) shows the  $p_E$  values for each error curve in Figure 3(b). As expected, the values of  $p_E$  for  $m = 0$  tend toward 4, which is the theoretical value of  $p_0$  for the numerical solution of  $u$  with CDS-4. In the CRE method, the values of  $p_E$  tend toward 6 for  $m \geq 1$ . In the FRE method, the values of  $p_E$  increase with  $m$  for  $m = g - 1$ , reaching a maximum value of 20 for  $m = 7$ ; for  $m = 8$  and 9, the value of  $p_E$  is lower than 4 due to rounding errors in  $L1$ .

For both CDS-2 and CDS-4, the error decreases continuously as  $m$  increases between  $m = 1$  and  $m = g - 1$  (limited to 7) for the FRE method. That is, for each grid refinement, when  $m$  is increased,  $p_E$  is also increased, and the error is significantly reduced. This is the ideal theoretical behavior of RRE, which was verified in [9, 11] for CDS-2, but only for global and local variables rather than for entire fields as in this study. Because of this behavior, the FRE method exhibits much better performance compared to the CRE method; for the two (CDS-2) or three (CDS-4) finest grids, the  $L1$  values reach the machine round-off error of the quadruple precision that was used in the calculations.

For both CDS-2 and CDS-4, the smallest values of  $L2$  and  $Li$  (compared to  $m = 0$ ) are obtained for  $m = 1$  and  $m = g - 1$  (limited to 7) for CRE and FRE, respectively, for a given grid size  $h$ .

### 1D advection–diffusion equation

Figure 4 shows the results for the 1D advection–diffusion equation with  $Pe = 10$ . The  $L1$  results for UDS-1/CDS-2 are shown in Figure 4(a). For a given grid size  $h$ , the most significant error reduction (compared to  $m = 0$ ) is achieved with  $m = 2$  for both CRE and FRE, and the smallest error occurs when  $m = 3$ . For  $m > 3$ , there is a small increase in the error. For the finest grid with  $m = 3$ , the error is reduced by factors of 11,981 and 15,155 (compared to  $m = 0$ ) using the CRE and FRE methods, respectively; therefore, the FRE method reduces the error by a greater factor compared to the CRE method. For the finest grid, the  $L1$  ratio between  $m = 2$  and  $m = 3$  is 1.24 and 1.59 for CRE and FRE, respectively. For a given grid size  $h$ , the most significant reduction in  $L2$  and  $Li$  (compared to  $m = 0$ ) is achieved with  $m = 2$  for both CRE and FRE methods.

Figure 4(c) shows the  $p_E$  values corresponding to each error curve in Figure 4(a). For  $m = 0$ , the values of  $p_E$  tend toward 1, which is the theoretically expected value for  $p_0$  in the numerical solution of  $u$  with UDS-1/CDS-2. In the CRE and FRE methods, the values of  $p_E$  tend toward 2 for  $m = 1$  and toward 3 for  $m > 1$ .

Figure 4(b) shows the  $L1$  results for CDS-2/CDS-2. For a given grid size  $h$ , the most significant error reduction (compared to  $m = 0$ ) is achieved with  $m = 1$  for the CRE and FRE methods, and the smallest error occurs when  $m = 2$  for both CRE and FRE methods. For  $m > 2$ , there is a small increase in the error. For the finest grid with  $m = 2$ , the error is reduced by factors of 32,873 and 41,748 (compared to  $m = 0$ ) using the CRE and FRE methods, respectively. The FRE method therefore reduces the error by a greater factor compared to the CRE method. For the finest grid, the  $L1$  ratio between  $m = 1$  and  $m = 2$  is 1.30 and 1.11 for the CRE and FRE methods, respectively. For a given grid size  $h$ , the most significant reduction in  $L2$  and  $Li$  (compared to  $m = 0$ ) is achieved with  $m = 1$  for both CRE and FRE methods.

Figure 4(d) shows the  $p_E$  values corresponding to each error curve in Figure 4(b). For  $m = 0$ , the values of  $p_E$  tend toward 2, which is the theoretically expected value for  $p_0$  in the numerical solution for  $u$  with CDS-2/CDS-2. For both CRE and FRE, the values of  $p_E$  tend toward 4 for  $m \geq 1$ .

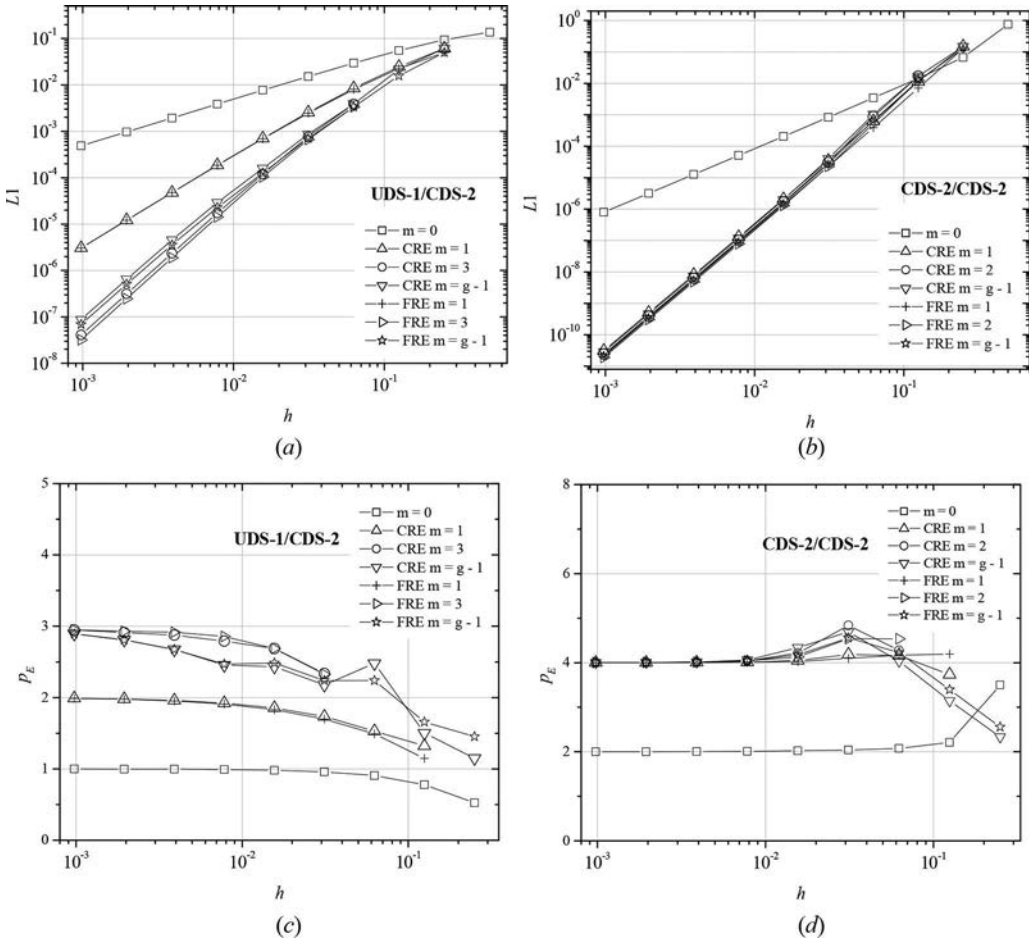


Figure 4. Results for the 1D advection-diffusion equation.

Comparison of Figure 4(a) and 4(b) reveals that for a given grid size  $h$ , the  $L_1$  values are lower with CDS-2/CDS-2 compared to UDS-1/CDS-2 for  $m = 0$  and for CRE and FRE methods with any  $m$ , as expected.

### 1D Burgers' equation

Figure 5 shows the results for the 1D Burgers' equation with  $Re = 1$ . The  $L_1$  results for CDS-2/CDS-2 are shown in Figure 5(a). For a given grid size  $h$ , the most significant error reduction (compared to  $m = 0$ ) is achieved with  $m = 1$  for both CRE and FRE methods. The smallest error occurs with  $m = 2$  for both CRE and FRE methods, and there is a small increase in the error for  $m > 2$ . For the finest grid with  $m = 2$ , the error is reduced by factors of approximately 105,000 and 102,000 (compared to  $m = 0$ ) using the CRE and FRE methods, respectively. The CRE method therefore yields a slightly larger error reduction compared to the FRE method. For the finest grid, the  $L_1$  ratio between  $m = 1$  and  $m = 2$  is 1.51 and 1.55 for the CRE and FRE methods, respectively. For a given grid size  $h$ , the most significant reduction in  $L_2$  and  $L_i$  (compared to  $m = 0$ ) is achieved with  $m = 1$  for both CRE and FRE methods.

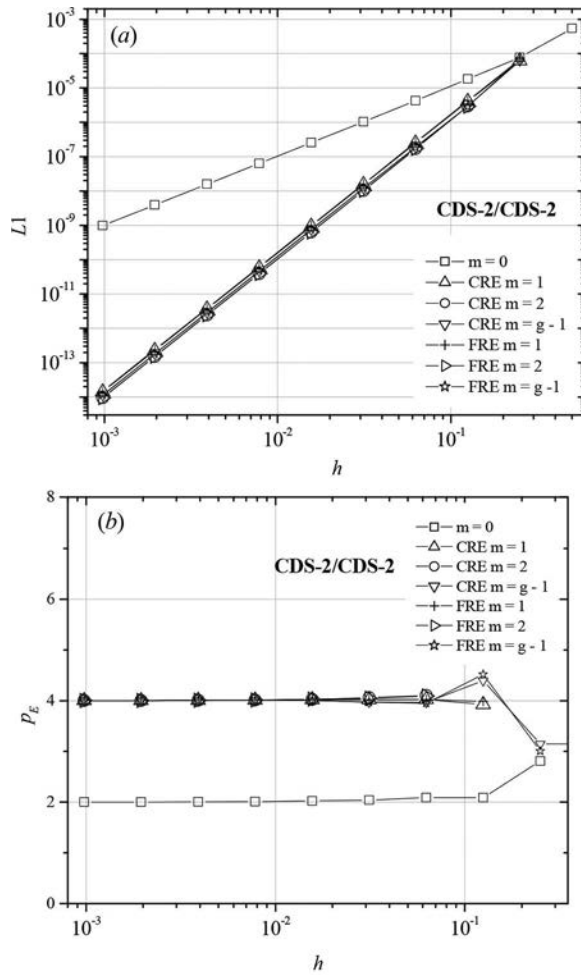


Figure 5. Results for the 1D Burgers' equation.

Figure 5(b) shows the  $p_E$  values corresponding to each error curve in Figure 5(a). For  $m = 0$ , the values of  $p_E$  tend toward 2, which is the theoretically expected value for  $p_0$  in the numerical solution of  $u$  with CDS-2/CDS-2. In the CRE and FRE methods, the values of  $p_E$  tend toward 4 for  $m \geq 1$ .

The CRE and FRE results for the 1D Burgers' equation are qualitatively similar to those for the 1D advection–diffusion equation discretized using CDS-2/CDS-2.

### 2D Laplace's equation

Figure 6 shows the results for the 2D Laplace's equation. The  $L_1$  results for CDS-2 are shown in Figure 6(a). For a given grid size  $h$ , the most significant error reduction compared to  $m = 0$  is achieved with  $m = 1$  for both CRE and FRE methods, and the smallest error occurs with  $m = 2$  for both methods. For  $m > 2$ , there is a small increase in the error. For the finest grid with  $m = 2$ , the error is reduced by factors of  $1.81 \times 10^5$  and  $2.72 \times 10^5$  (compared to  $m = 0$ ) using the CRE and FRE methods, respectively. The FRE method therefore reduces the error by a greater factor compared to the CRE method. For the finest grid, the  $L_1$  ratio between  $m = 1$  and  $m = 2$  is 1.76 and 1.88 for

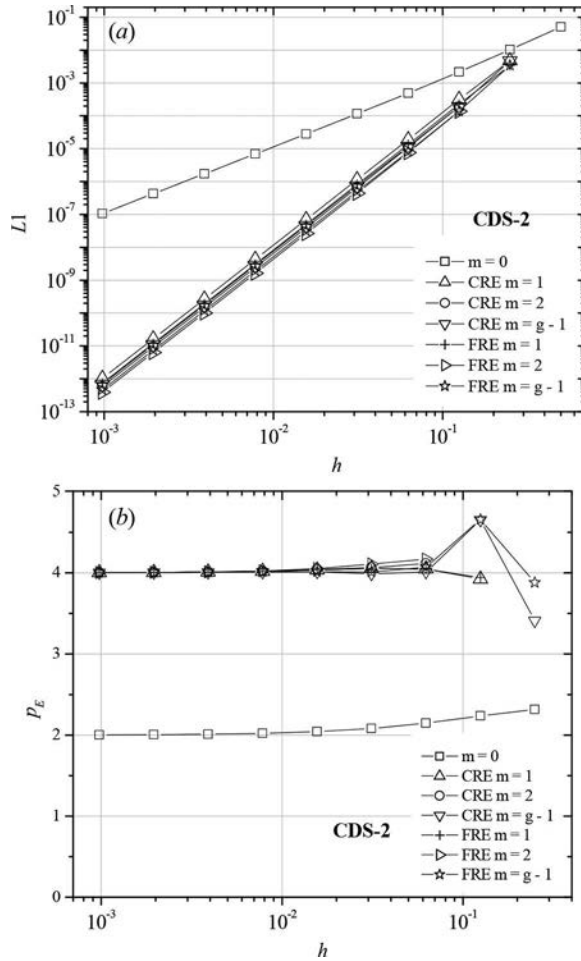


Figure 6. Results for the 2D Laplace's equation.

CRE and FRE methods, respectively. For a given grid size  $h$ , the most significant reduction in  $L_2$  and  $L_1$  (compared to  $m = 0$ ) is achieved with  $m = 1$  for both CRE and FRE methods.

Figure 6(b) shows the  $p_E$  values corresponding to each error curve in Figure 6(a). For  $m = 0$ , the values of  $p_E$  tend toward 2, which is the expected theoretical value of  $p_0$  in the numerical solution for  $u$  with CDS-2. For CRE and FRE methods, the values of  $p_E$  tend toward 4 for  $m \geq 1$ .

Unexpectedly, the results for the 2D Laplace's equation with FRE are quite different from those for the 1D Poisson's equation discretized with CDS-2, perhaps owing to the 2D nature of the problem. The CRE and FRE results for the 2D Laplace's equation are qualitatively similar to those for the 1D Burgers' and advection-diffusion equations discretized with CDS-2/CDS-2.

### 2D Burgers' equations

Figure 7 shows the results for the 2D Burgers' equations with  $Re = 1$ . The  $L_1$  results for the variable  $u$  obtained with UDS-1/CDS-2 are shown in Figure 7(a). For the CRE method with a given grid size  $h$ , the most significant error reduction relative to  $m = 0$  (no extrapolation) is obtained with  $m = 2$ , and the smallest error occurs for  $m = 3$ . For  $m > 3$ , there is a small increase in the error. For FRE with a

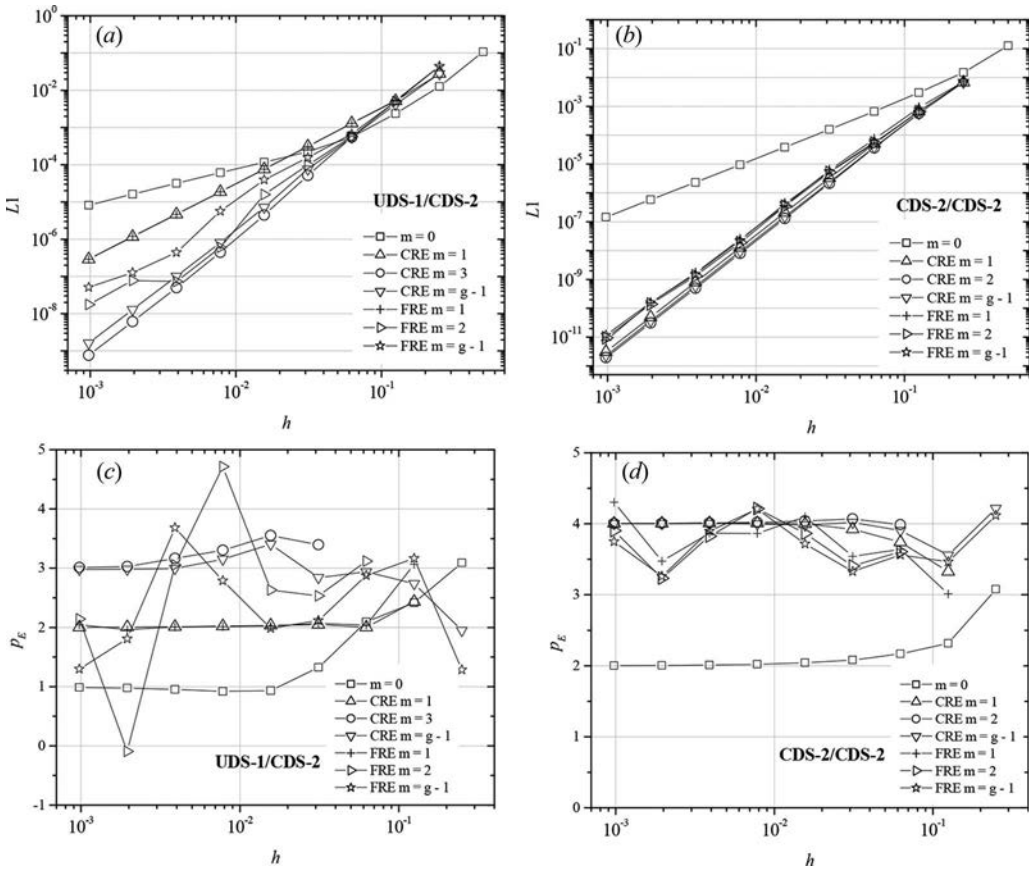


Figure 7. Results for the 2D Burgers' equations.

given grid  $h$ , the most significant error reduction and the smallest absolute error are achieved with  $m = 2$ . For  $m > 2$ , the error increases. For the finest grid, the error is reduced by factors of 10,691 and 463 (relative to  $m = 0$ ) using the CRE and FRE methods, respectively. The CRE method therefore reduces the error by a much greater factor compared to the FRE method. For the finest grid, the  $L_1$  ratio between  $m = 2$  and  $m = 3$  is 1.19 for the CRE method. For a given grid size  $h$ , the most significant reduction in  $L_2$  and  $L_i$  (relative to  $m = 0$ ) is achieved with  $m = 2$  for CRE and with  $m = 1$  for FRE.

Figure 7(c) shows the  $p_E$  values corresponding to each error curve in Figure 7(a). For  $m = 0$ , the values of  $p_E$  tend toward 1, which is the theoretically expected value of  $p_0$  for the numerical solution of  $u$  with UDS-1/CDS-2. For CRE, the values of  $p_E$  tend toward 2 for  $m = 1$  and 3 for  $m > 1$ . For FRE, the values of  $p_E$  also tend toward 2 for  $m = 1$ , but they do not exhibit a clear trend for  $m > 1$ .

Figure 7(b) shows the  $L_1$  results for the variable  $u$  obtained with CDS-2/CDS-2. For the CRE method with a given grid size  $h$ , the most significant error reduction relative to  $m = 0$  is achieved with  $m = 1$ , and the smallest error occurs with  $m = 2$ . For  $m > 2$ , there is a small increase in the error. For FRE with a given grid size  $h$ , the most significant error reduction and the smallest absolute error are achieved with  $m = 1$ . For  $m > 1$ , there is a small increase in the error. For the finest grid, the error is reduced by factors of 74,422 and 18,793 (relative to  $m = 0$ ) using the CRE and FRE methods, respectively. The CRE method therefore yields a greater error reduction compared to the FRE method. For the finest grid, the  $L_1$  ratio between  $m = 1$  and  $m = 2$  is 1.60 for the CRE method. For a given grid size

$h$ , the most significant reduction in  $L2$  and  $Li$  (relative to  $m = 0$ ) is achieved with  $m = 1$  for both CRE and FRE methods.

Figure 7(d) shows the  $p_E$  values corresponding to each error curve in Figure 7(b). For  $m = 0$ , the values of  $p_E$  tend toward 2, which is the theoretically expected value for  $p_0$  in the numerical solution of  $u$  with CDS-2/CDS-2. For the CRE method, the values of  $p_E$  tend toward 4 for  $m \geq 1$ . For FRE, the values of  $p_E$  do not exhibit a clear trend for  $m \geq 1$ , but they oscillate around 4.

Comparison of Figure 7(a) and 7(b) reveals that for each grid size  $h$ , the  $L1$  values are smaller for CDS-2/CDS-2 compared to UDS-1/CDS-2 both for  $m = 0$  and for CRE and FRE methods with any  $m$ , as expected.

The results for the variable  $v$  are qualitatively similar to those shown above for the variable  $u$ .

## Conclusion

The results of this study can be summarized as follows:

1. Through extrapolations performed during post-processing, the CRE and FRE methods significantly reduce the discretization error in the numerical solutions for all nodes of the grid. The more refined the grid, the greater the error reduction.
2. For all nodes of the grid, the ideal theoretical performance of RRE was only obtained using FRE for the numerical solution of the 1D Poisson's equation. The FRE method can increase the order of accuracy ( $p_0$ ) by up to 16 units for numerical solutions with  $p_0 = 2$  or  $p_0 = 4$ .
3. The CRE method increases the value of  $p_0$  by 2 units for numerical solutions with  $p_0 = 1, 2$ , or 4 for all nodes of the grid. This conclusion holds for the five problems addressed in this study, namely the 1D Poisson's, advection-diffusion and Burgers' equations, and the 2D Laplace's and Burgers' equations. This result is also valid for the FRE method, except in the cases of the 1D Poisson's and 2D Burgers' equations.
4. The FRE method yields a greater reduction in the field discretization error compared to the CRE method for the 1D Poisson's and advection-diffusion equations and the 2D Laplace's equation. For the 1D Poisson's equation with  $p_0 = 2$  or  $p_0 = 4$ , the smallest  $L1$  value is obtained with the greatest possible number of extrapolations, that is,  $m = g - 1$ . For the 1D advection-diffusion equation with  $p_0 = 1$ , the smallest  $L1$  value is obtained with  $m = 3$ . For the 1D advection-diffusion and 2D Laplace's equations with  $p_0 = 2$ , the smallest  $L1$  value is obtained with  $m = 2$ .
5. The CRE method produces a greater reduction in the field discretization error compared to the FRE method for the 1D and 2D Burgers' equations. For the 2D Burgers' equation with  $p_0 = 1$ , the smallest  $L1$  value is obtained with  $m = 3$ . For the 1D and 2D Burgers' equations with  $p_0 = 2$ , the smallest  $L1$  value is obtained with  $m = 2$ .
6. In general, the results for  $L2$  and  $Li$  are qualitatively similar to those for  $L1$ .

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