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COMPARISON OF EFFICIENCY BETWEEN THE *TIME STEPPING* AND *WAVEFORM RELAXATION* METHODS FOR TWO-DIMENSIONAL FOURIER EQUATION

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Abstract. *This paper presents a methodology and comparisons of parameters between Time Stepping and Waveform Relaxation methods, used to solve the two-dimensional linear time-dependent heat diffusion problem, governed by Fourier equation. The numerical model is obtained by employing finite-difference method, using central second order approximation, Crank-Nicolson and one variant, designated herein as Modified Crank-Nicolson method for discretization in space and time, respectively. In the solution of the system of equations that resulted from discretization, we used the geometric multigrid method with Correction Scheme, V-cycle, Gauss-Seidel solver, restriction by injection, half weithing and full weithing, prolongation by bilinear interpolation and standard coarsening ratio in the spatial coordinates directions. Tests were accomplished to optimize the relationship between time and spatial discretizations, as well as multigrid parameters. We concluded that, for finer grids (greater reduction in discretization error), Time Stepping method is more efficient than Waveform Relaxation method, and the Modified Crank-Nicolson solver is more efficient than Crank-Nicolson, in relation to the analyzed parameters.*

Keywords: *Multigrid, Waveform Relaxation, Time Stepping, Fourier equation, CPU time.*

1. INTRODUCTION

The resolution of physical problems of fluid mechanics and heat transfer through numerical methods often requires a very high computational cost. One way of accelerating the delivery of the approximate solution is by using the multigrid method (Briggs *et al.*, 2000; Trottenberg *et al.*, 2001). The multigrid method is one of the most efficient and general iterative methods known for solving systems of linear and nonlinear equations. According to Briggs *et al.* (2000) and Wesseling (1992), this method consists in choosing a solution scheme that absorbs the errors associated with high frequencies, while the errors associated with low frequencies are absorbed by using coarse grids that make them high frequencies. Thus, we used auxiliary coarse grid (with fewer nodes) along with restriction and prolongation operators.

According to Stüben (2001), the efficiency of the multigrid method has not been totally reached in realistic applications regarding computational fluid dynamics. Thus, this paper presents a methodology and comparisons of parameters between Time Stepping (TS) and Waveform Relaxation (WR) methods combined to the multigrid method, used to solve the two-dimensional linear time-dependent heat diffusion problem, governed by Fourier equation.

According to Vandewalle (1993); Choptuik (2008) and Van Lent (2006), the TS method considers the parabolic differential equation as a sequence of elliptic equations, solving at each time step, using explicit or semi-implicit methods, which leads us to the approximation of the solution in the desired time step. The WR method, also known as Dynamic Iteration or Picard Iteration (Miekkala and Nevanlinna, 1997), is a technique for solving ordinary differential systems of initial value. The WR method was initially studied by Lelarasmee *et al.* (1982) as a method of practical solution in the context of integrated circuit simulation. Its convergence was described by Miekkala and Nevanlinna (1997). The combination of the WR method with multigrid techniques was first studied by Lubich and Ostermann (1987) and Van Lent and Vandewalle (2002). This method consists in solving a set of ordinary differential equations (ODE), that is, for every point of the space discretization one time ODE is solved.

This paper is organized as follows. Section (2) describes the problem to be solved and introduces the TS and WR

methods. In Section (3) the TS and WR methods are combined with multigrid, using Crank-Nicolson (CN) as smoother. We also propose a new method of smoothing, the Modified Crank-Nicolson (MCN). Section (4) presents a study on the validation of computer codes developed in Fortran 90. In section (5) we conduct a study of the parameters that reduce CPU time (t_{CPU}) for each method solution and smoother. With this study, one can compare the TS and WR methods with smoothers CN and MCN, admitting the best parameters to perform the simulations with the lowest t_{CPU} for each case studied. Section (6) presents the conclusions.

2. MATHEMATICAL AND NUMERICAL MODELS

The mathematical model describes a linear two-dimensional problem of heat conduction in time-dependent regime governed by Fourier's equation

$$\dot{\mathbf{u}} = \mathbf{u}_{xx} + \mathbf{u}_{yy} + \mathbf{f}, \tag{1}$$

where $\mathbf{u}$ and $\mathbf{f}$ are functions of the variables (x, y, t), with $(x, y) \in [0, 1]$, $t \in [0, t_f]$ where t_f represents the final time, $\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t}$ represents the time derivative of $\mathbf{u}$, $\mathbf{u}_{xx}$ and $\mathbf{u}_{yy}$ represent the second spatial derivative of $\mathbf{u}$ as a function of x and y, respectively. In order to solve this problem, we defined the initial and boundary conditions as in Incropera *et al.* (2008), thus

$$\mathbf{u}(x, y, 0) = \sin(\pi x)\sin(\pi y), \tag{2}$$

and

$$\mathbf{u}(0, y, t) = \mathbf{u}(1, y, t) = \mathbf{u}(x, 0, t) = \mathbf{u}(x, 1, t) = 0. \tag{3}$$

The source term is given by

$$\mathbf{f}(x, y, t) = \pi^2 \sin(\pi x)\sin(\pi y)e^{-\pi^2 t}, \tag{4}$$

and the analytical solution is

$$\mathbf{u}(x, y, t) = \sin(\pi x)\sin(\pi y)e^{-\pi^2 t}. \tag{5}$$

In the discretization of spatial variables we used the finite-difference method in structured quadrangular meshes with central approximation of second order accuracy. To approximate the time variables we used the Crank-Nicolson method, also of second order accuracy (Tannehill *et al.*, 1997; Burden and Faires, 2003),

$$\mathbf{u}_{xx} + \mathbf{u}_{yy} = \mathbf{L}\mathbf{u}_{i,j} = \frac{\mathbf{u}_{i-1,j} - 2\mathbf{u}_{i,j} + \mathbf{u}_{i+1,j}}{(\Delta x)^2} + \frac{\mathbf{u}_{i,j-1} - 2\mathbf{u}_{i,j} + \mathbf{u}_{i,j+1}}{(\Delta y)^2}, \tag{6}$$

where $\mathbf{L}$ is a second order differential operator and the sub-indices (i, j) are related to the spatial variable in the position (x, y). In this case $(i-1, j), (i+1, j), (i, j-1)$ and $(i, j+1)$ are related to the west, east, south and north points of the position (i, j), respectively. $\Delta x = \frac{1}{N_x - 1}$ and $\Delta y = \frac{1}{N_y - 1}$ represent the distance between successive nodes, whereas N_x and N_y represent the number of nodes (or points) in the directions x and y, respectively.

Replacing the Eq. (6) in Eq. (1) one can achieve a stiff system of ODEs, as is typical for discretized parabolic equations. Since the use of explicit methods leads to severe restrictions on the size of the step time, only implicit methods are considered (Van Lent, 2006). Thus, for the approximation of the time variable, we used the Crank-Nicolson method as described in Vargas (2013). So, the Eq. (1) becomes

$$\frac{\mathbf{u}_{i,j}^k - \mathbf{u}_{i,j}^{k-1}}{\Delta t} = \frac{1}{2}\left[\mathbf{L}\mathbf{u}_{i,j}^k + \mathbf{f}^k\right] + \frac{1}{2}\left[\mathbf{L}\mathbf{u}_{i,j}^{k-1} + \mathbf{f}^{k-1}\right], \tag{7}$$

where k indicates the current time, $k-1$ indicates the previous time, and $\Delta t = \frac{t_f}{N_t}$, the time step, in which N_t expresses the number of steps in the time until reaching the desired final time, t_f.

By combining Eqs. (7), (6) and (1), one can achieve a linear system with $N_x \times N_y \times N_t$ unknowns that may be solved by using Gauss-Seidel and Crank-Nicolson smoothers in space and time, respectively, along with the TS and WR methods.

2.1 Time Stepping method (TS)

Some studies have focused on algorithms which treat the parabolic equation as a sequence of elliptic equations at each time step; this method is called Time Stepping method (Choptuik, 2008; Van Lent, 2006). As from Eq. (7), we have

$$\mathbf{u}_{i,j}^k = \frac{\Delta t}{2}\left(\mathbf{L}\mathbf{u}_{i,j}^k + \mathbf{f}^k\right) + \mathbf{S}^{k-1}, \tag{8}$$

where

$$\mathbf{S}^{k-1} = \frac{\Delta t}{2}\left[\mathbf{L}\mathbf{u}_{i,j}^{k-1} + \mathbf{f}^{k-1}\right] + \mathbf{u}_{i,j}^{k-1}. \tag{9}$$

The approximations $\mathbf{u}_{i,j}^{k}$ are obtained by solving the above system for each time step k in order. Thus, we can use exactly the same methods as for stationary problems and the structure of the systems at each time step. The same happens with the corresponding discretized elliptic equation.

By using the lexicographical order (Burden and Faires, 2003) to update the values in space and applying the splitting $\mathbf{L} = \mathbf{L}^{+} + \mathbf{L}^{-}$, of the Gauss-Seidel method, we have

$$\mathbf{u}_{i,j}^{k,(\nu)} = \frac{\Delta t}{2}\left(\mathbf{L}^{+}\mathbf{u}_{i,j}^{k,(\nu)} + \mathbf{L}^{-}\mathbf{u}_{i,j}^{k,(\nu-1)} + \mathbf{f}_{i,j}^{k,(\nu)}\right) + \mathbf{S}^{k-1}, \tag{10}$$

where (ν) indicates values in the current iteration and $(\nu-1)$ indicates values in the previous iteration.

Note that in Eq. (10) the loop over the time steps k forms the outer loop and for each time step k there is an inner loop for the iterations (ν); this characterizes the Time Stepping method, as in Fig. 1(a).

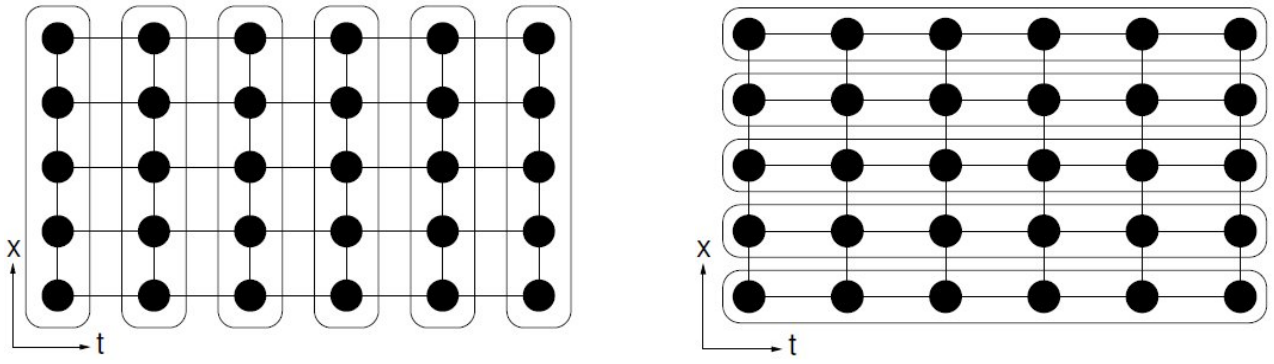
2.2 Waveform Relaxation method (WR)

We can apply a splitting method $\mathbf{L} = \mathbf{L}^{+} + \mathbf{L}^{-}$ directly to a system of ODEs given in the Eq. (1), resulting in

$$\dot{\mathbf{u}}^{(\nu)} = \mathbf{L}^{+}\mathbf{u}_{i,j}^{k,(\nu)} + \mathbf{L}^{-}\mathbf{u}_{i,j}^{k,(\nu-1)} + \mathbf{f}_{i,j}^{k,(\nu)}, \tag{11}$$

where the iterations of $\mathbf{u}^{(\nu)}$ are functions of the time. Thus, the WR method consists of discretizing the equation in space and approximating the solution in the direction of the time variable by means of some method to solve the ODEs. By using the Crank-Nicolson method to discretize the Eq. (11), it becomes identical to the Eq. (10). In that sense, Van Lent (2006) states that in the WR case, the outer loop is over the iterations and for each iteration (ν) there is an inner loop over the time steps k, as seen in Fig. 1(b).

Figures 1(a) and 1(b) obtained from Van Lent (2006) show the difference between the TS and WR methods. In TS, Fig. 1(a), all spatial unknowns for a time step are smoothed simultaneously until reaching some stopping criteria; this is repeated for all time steps until the final time. In WR, Fig. 1(b), all the values for one spatial grid point are updated simultaneously in the time direction; this is repeated for all spatial points until reaching some stopping criteria.



(a) Time Stepping method (b) Waveform Relaxation method

Figure 1. Update procedure of the unknowns in Time Stepping e Waveform Relaxation methods (Van Lent, 2006).

3. MULTIGRID METHOD

The system of algebraic equations represented by the systems of Equations (10) or (11) are solved with the geometric multigrid method, as described by Wesseling (1992), by using Correction Scheme (CS). We used a V-cycle with standard coarsening ratio, $q = 2$, in spatial directions (Van Lent and Vandewalle, 2002). The restriction processes used are injection (Inj), half weighting (HW) and full weighting (FW). The prolongation is done by bilinear interpolation (Trottenberg *et al.*, 2001).

3.1 Multigrid with the Crank-Nicolson method (CN)

The following algorithm, adapted from Van Lent and Vandewalle (2002), describes how each V-cycle occurs for the TS and WR methods. The algorithm computes a new iteration $\mathbf{u}^{(\nu)}$ starting from a previous iteration $\mathbf{u}^{(\nu-1)}$.

- $\mathbf{v} \leftarrow \mathbf{u}^{(\nu-1)}$
- pre-smoothing using CN: do (ν_1) times $\{\mathbf{v}' \leftarrow \mathbf{v}$, solve $\mathbf{v}_{i,j}^k = \frac{\Delta t}{2}\left(\mathbf{L}\mathbf{v}_{i,j}^k + \mathbf{f}^k\right) + \mathbf{S}^{k-1}\}$
  - calculate defect: $\mathbf{d} \leftarrow \frac{1}{2}\left(\mathbf{L}\mathbf{v}_{i,j}^k + \mathbf{f}^k\right) + \frac{1}{2}\left(\mathbf{L}\mathbf{v}_{i,j}^{k-1} + \mathbf{f}^{k-1}\right) - \frac{\mathbf{v}_{i,j}^k - \mathbf{v}_{i,j}^{k-1}}{\Delta t}$
  - solve for coarse grid correction of the residual equation using the Crank-Nicolson method:

$$\bar{\mathbf{e}}_{i,j}^k = \frac{\Delta t}{2}\left[\overline{\mathbf{L}}\bar{\mathbf{e}}_{i,j}^k + \mathbf{R}\mathbf{d}^k\right] + \frac{\Delta t}{2}\left[\overline{\mathbf{L}}\bar{\mathbf{e}}_{i,j}^{k-1} + \mathbf{R}\mathbf{d}^{k-1}\right] + \bar{\mathbf{e}}_{i,j}^{k-1} \tag{12}$$

  - correct: $\mathbf{v} \leftarrow \mathbf{v} + \mathbf{P}\,\bar{\mathbf{e}}$
- post-smoothing using CN: do (ν_2) times $\{\mathbf{v}' \leftarrow \mathbf{v}$, solve $\mathbf{v}_{i,j}^k = \frac{\Delta t}{2}\left(\mathbf{L}\mathbf{v}_{i,j}^k + \mathbf{f}^k\right) + \mathbf{S}^{k-1}\}$
- $\mathbf{u}^{(\nu)} \leftarrow \mathbf{v}$

where $\mathbf{R}$ is a restriction operator, $\mathbf{P}$ is an prolongation operator and $\bar{\cdot}$ indicates that the variable is restricted in the coarse grid.

For the TS method with multigrid, we followed the methodology described in Fig. 1(a), that is, in each fixed time k, i and j are varied to apply the algorithm of the multigrid and to conduct a V-cycle. This process is performed until reaching the desired approximation to the solution in the respective time k. The procedure must be repeated until t_f. In the WR method with multigrid, we followed the methodology described in Fig. 1(b), that is, successively in each node (i, j) of the spatial discretization, we smooth across the time domain and apply the multigrid algorithm to conduct a V-cycle. This process is repeated until reaching the desired final solution.

3.2 Multigrid with Modified Crank-Nicolson method (MCN)

The multigrid method with modified Crank-Nicolson method (MCN) consists in smoothing Eq. (10) in the finer grids with the Crank-Nicolson method (CN), and using a modification for smoothing residual equations in other grids (coarser). For this, it is necessary to substitute

$$\bar{\mathbf{e}}_{i,j}^k = \bar{\mathbf{e}}_{i,j}^{k-1} + \frac{\Delta t}{2}\left[\overline{\mathbf{L}}\bar{\mathbf{e}}_{i,j}^k\right] + \frac{\Delta t}{2}\left[\overline{\mathbf{L}}\bar{\mathbf{e}}_{i,j}^{k-1}\right] + \mathbf{R}\mathbf{d}^k \tag{13}$$

in Eq. (12) of the algorithm in section 3.1 Note that the residues are used only in the current time step and not do the arithmetic average of its value in the current time step and earlier, for the smoothing of the residual equation.

4. VERIFICATION OF THE SOLUTIONS

In order to verify the numerical solutions, according to Marchi (2001), we used the Richardson estimator based on the apparent order of the numerical error, $p_U = \frac{log\left(\frac{\phi_2-\phi_3}{\phi_1-\phi_2}\right)}{log(q)}$ and on the effective order, $p_E = \frac{log\left(\frac{E(\phi_2)}{E(\phi_1)}\right)}{log(q)}$, where ϕ_1, ϕ_2 and ϕ_3 indicate the solution of the variable of interest in the fine, coarse and extra-coarse grids, respectively. Variable q indicates the coarsening ratio between the grid levels related to ϕ_1, ϕ_2 and ϕ_3. Variable E indicates numerical error, which is the numerical error (round-off error, truncation error, iteration error and programming error) of the variable related to the computation, ϕ.

The problem to be solved is given in Eq. (1), which after discretization becomes Eq. (10). The initial and boundaries conditions and source term, are given by Eqs.(2), (3) and (4), respectively. The spatial domain is a unitary square, so $(x, y) \subset [0, 1] \times [0, 1]$, and the final time used is $t_f = 0.1$ s. The discretizations of $N_x \times N_y \times N_t$ are $5 \times 5 \times 5$, $9 \times 9 \times 9$, $17 \times 17 \times 17$ until $513 \times 513 \times 513$, corresponding to a coarsening ratio $q = 2$. In each simulation, the iterative process is interrupted when the machine error is achieved; in other words, iteration errors are minimized, prevailing basically the discretization errors. The variables analyzed at the final time are: temperature at the average point, $u(0.5, 0.5, t_f)$, average temperature, $\overline{u(x, y, t_f)}$, and the infinity norm of the numerical error, $||E||_\infty$.

Algorithms were compiled using Intel® Fortran® 2003 release version 11.1 compiler, with quadruple precision, a Console Application project. The simulations occurred in a Workstation with 2 Intel Xeon X5690 (6 core) processors, 3.5 GHz clock speed, 2.4 TB HD and 192 GB RAM.

Table 1 presents the values of $h = \frac{1}{(N_x-1)} = \frac{1}{(N_y-1)} = \frac{t_f}{N_t} = \Delta t$ used in each numerical simulation (with mutigrid) and the respective results of p_U, p_E, assuming the infinity norm of the numerical error, $||E||_\infty$, as the variable of interest. For this, we considered the TS method with CN smoother (TS-CN), the TS method with MCN smoother (TS-MCN), the WR method with CN smoother (WR-CN) and the WR method with MCN smoother (WR-MCN). Values of the

Table 1. Apparent order (p_U) and effective order (p_E) when there is grid refinement (h) for the variable of interest, $||E||_\infty$.

h	TS-CN e WR-CN		TS-MCN e WR-MCN	
	p_U	p_E	p_U	p_E
0.1250	2.669141	—-	2.669141	—-
0.0625	2.162556	2.787051	2.162556	2.787051
0.0313	2.041061	2.199479	2.041061	2.199479
0.0156	2.010300	2.051075	2.010300	2.051075
0.0078	2.002577	2.012859	2.002577	2.012859
0.0039	2.000644	2.003221	2.000644	2.003221
0.0020	2.000161	2.000805	2.000161	2.000805
0.0010	2.000040	2.000201	2.000040	2.000201

other variables of interest and numerical simulations without the use of multigrid (singlegrid), are similar to Tab. 1 and, therefore, they will not be exposed here.

Based on the results depicted in Tab. 1, it is possible to verify that with grid refinement, the effective and apparent order tend to the value of the asymptotic order $p_L = 2.0$, which is the order of the CDS and CN method (spatial and time discretization, respectively). These numerical tests show the coherency between numerical and analytical results, demonstrating the validity of the code. Note that the alteration in the algorithm presented in section 3.2 (MCN) maintain the order of the CN method.

5. RESULTS

In this section we considered the TS-CN, TS-MCN, WR-CN and WR-MCN cases. Spatial discretizations are 5×5, 9×9 until 513×513 and the final time is $t_f = 0.1$ s. The number of presmoothing is equal to the number of postsmoothing, namely, $\nu = \nu_1 = \nu_2$. Stopping criteria adopted for the iterative process is $||\mathbf{r}||_2 < 10^{-7}$, where $\mathbf{r}$ is the defect vector in the current iteration. The grid coarsening ratio of the multigrid method is $q = 2$, applied only in spatial directions and used in both directions (Van Lent and Vandewalle, 2002). In all cases we used the bilinear interpolation prolongation (Trottenberg *et al.*, 2001).

5.1 Analysis of the variable N_t

The first parameter studied is the relationship between $N_x = N_y$ and N_t. We conducted several tests with different N_t values in each fixed $N_x = N_y$ value. In these tests, besides the predefined parameters, we adopted the number of inner iterations as $\nu = 1$, restriction by injection (Inj) and the number of grid levels $l = L_{\max}$, where $L_{\max}$ indicates the maximum number of possible grids, for instance, $N_x = N_y = 33 \times 33$, where 33×33, 17×17, 9×9, 5×5, 3×3, or else $L_{\max} = 5$.

We verified that for all the cases of spatial discretization studied, t_{CPU} is smaller when N_t is reduced, however, numerical errors values might increase or decrease when N_t values vary. Therefore, we opted for assuming that the optimum value for each discretization level is the value that provides the smallest value of the infinity norm of the error ($||E||_\infty$). Table 2 shows values used in the spatial discretization and the respective N_t that reduces the values of numerical errors associated to the iterative process.

Table 2. Relationship between $N_x = N_y$ e N_t that optimize numerical errors.

$N_x = N_y$	9	17	33	65	129	257	513
N_t	3	5	8	15	29	58	115

5.2 Comparison of the CN and MCN smoother

To compare the solvers, CN and MCN, using the TS and WR methods, we used the average convergence factor (ρ_m) with the residual norm and t_{CPU} (Briggs *et al.*, 2000; Janssen, 1997). Therefore, we made simulations considering restriction by injection, as well as prolongation by bilinear interpolation, number of grids in the multigrid method, $l = L_{\max}$, $V(1,1)$ cycle and N_t given by Tab. 2. The results are shown in Tab. 3. This table shows that with the spatial refinement, the value of $\rho_m(r)$ increases to CN method ($\rho_m(r) \to 1$), thus the multigrid method loses its efficiency and t_{CPU} drastically increase. However, MCN is more efficient than CN method, because besides solving problems with a smaller t_{CPU}, the ratio of average convergence remains constant, which is one of the main features of the multigrid method (Wesseling, 1992; Briggs *et al.*, 2000; Trottenberg *et al.*, 2001).

Table 3. Comparison between CN and MCN smoothers for TS and WR methods using multigrid.

$N_x = N_y$	TS				WR			
	CN		MCN		CN		MCN	
	ρ_m	t_{CPU}	ρ_m	t_{CPU}	ρ_m	t_{CPU}	ρ_m	t_{CPU}
9	0.229	0.00	0.063	0.00	0.268	0.02	0.096	0.02
17	0.402	0.02	0.085	0.00	0.495	0.06	0.159	0.03
33	0.567	0.22	0.097	0.06	0.680	0.73	0.171	0.16
65	0.679	2.53	0.094	0.50	0.828	12.57	0.174	1.48
129	0.764	29.53	0.088	4.07	0.914	223.43	0.174	12.00
257	0.827	347.33	0.091	34.88	0.963	5445.54	0.172	107.59
513	0.875	4032.09	0.087	269.41	0.987	119668.80	0.169	913.26

Results from Tab. 4 show that the CN smoother lose efficiency when mesh is refined. When comparing smoothers, we verified that the MCN smoother has a t_{CPU} smaller than the CN smoother for the TS and WR methods, indicating a better efficiency of the MCN smoother than the CN smoother.

Table 4 describes the acceleration factor, or *speed up* (S), which in this case represents the ratio between the t_{CPU} of CN and MCN smoothers, for TS and WR methods in simulations involving the results described in Tab. 3. This results indicate how much faster the MCN smoother is than the CN smoother for TS and WR methods, at different levels of spatial mesh discretization. For example, in the WR method with spatial discretization given by $N_x = N_y = 513$, the *speed up* is $131,04$, which means that the MCN smoother is 131.04 times faster than the CN smoother.

Table 4. *Speed up* (S) between CN e MCN smoothers for TS and WR methods.

N_x	33	65	129	257	513
$S = \dfrac{t_{CPU}(\text{TS}-\text{CN})}{t_{CPU}(\text{TS}-\text{MCN})}$	3.50	5.06	7.26	9.96	14.97
$S = \dfrac{t_{CPU}(\text{WR}-\text{CN})}{t_{CPU}(\text{WR}-\text{MCN})}$	4.70	8.48	18.62	50.61	131.04

5.3 Parameter optimization and comparison of the methods

In order to verify the parameters of the multigrid method that minimize t_{CPU}, we considered values of $N_x = N_y$ and its respective N_t as described in Tab. 2. For this simulations, in each value of N_x, N_y and N_t, we varied the number of presmoothing and postsmoothing $\nu = \nu_1 = \nu_2$, considering $\nu = 1, \nu = 2$ ou $\nu = 3$, the number of grids used in the multigrid process, where $l = L_{\max}, L_{\max} - 1, L_{\max} - 2, L_{\max} - 3$ and the restriction process, as the injection (Inj), half weighting (HW) and full weighting (FW). The optimum values for these parameters are depicted in Tab. 5.

Table 5. Optimum parameters for restriction, number of grids used in the multigrid (l) and number of presmoothing and postsmoothing (ν) for methods TS and WR and considering MCN as the smoother.

restriction	number of grids used in the multigrid	number of inner iterations
Inj	$L_{\max}$	2

Table 5 describe the optimum parameters for all levels of spatial discretization. One can observe that for all cases, the parameters that return t_{CPU} minimum are the restriction by injection (Inj) and the grid level number $L_{\max}$ and the optimum number of inner iteration is $\nu = 2$.

Now we evaluate herein the effect of the number of unknowns in relation to t_{CPU}, with the TS-CN, TS-MCN, WR-CN and WR-MCN methods and with the optimum parameters suggested in Tab. 5.

Figure 2 shows t_{CPU} as a function of the number of unknowns in each numerical simulation performed. As stopping criteria of the iterative process, we used $||\mathbf{r}||_2 < 10^{-7}$. When comparing methods, one can verify that TS presents a smaller t_{CPU} than WR.

In order to determine the order of the algorithms complexity, we made a geometric adjust type $t_{cpu}(N) = c \cdot N^p$,

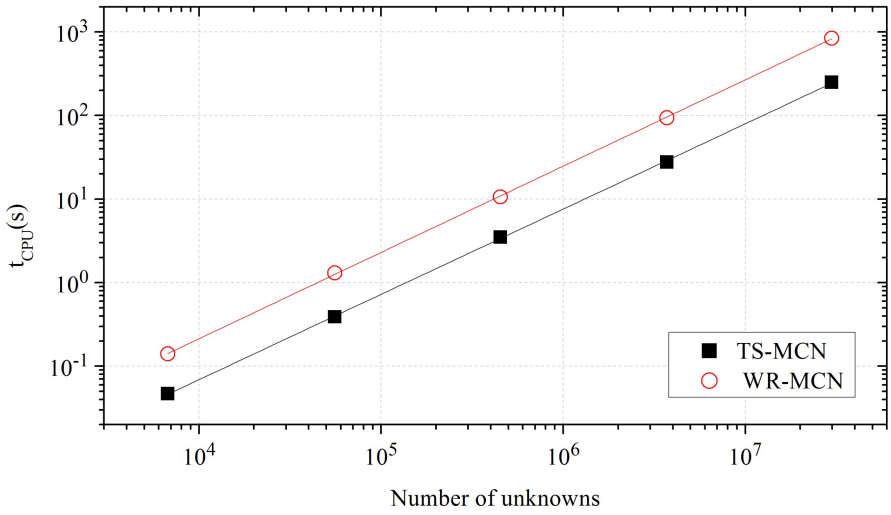


Figure 2. Comparison of t_{CPU} with parameters described in Tab. 2 and Tab. 5, for the TS and WR method, with CN and MCN smoothers.

where N represents the number of unknowns of the problem to be solved, p represents the order of the algorithm (slope in *log log* scale) and c is a constant that depends on the method. The closer the exponent p is to the unit, the better the performance of the multigrid method, that is ideal when $p = 1$ (Trottenberg *et al.*, 2001), thus, CPU time increases linearly to the size of the problem. Tab. 6 depicts the values of c and p obtained by geometric adjust of the slopes described in Fig. 2.

Table 6. Coefficients c e p of $t_{cpu}(N) = c \cdot N^p$.

Method	TS-MCN	WR-MCN
c	5,73215E-6	1,57979E-5
p	1,02059	1,03287

Based on the values in Tab. 6, we concluded that TS presents an order of complexity smaller than WR, what indicates a superiority of the TS method in relation to the WR method for solving this type of problem.

6. CONCLUSION

In this work, the two-dimensional Fourier equation with initial and boundary conditions was solved using the multigrid method with CS scheme and V-cycle. Discretization of spatial variables was performed with the finite-difference method (FDM), in quadrangular structured grids with central differencing scheme (CDS) approximation. We used lexicographic point-wise Gauss-Seidel solver. The discretization of the time variables occurred by means of the Crank-Nicolson and Modified Crank-Nicolson methods (a version proposed herein). The order of updating of the unknowns in the iterative process was determined by Time Stepping (TS) and Waveform Relaxation (WR) methods.

We verified the parameter combination that makes the multigrid method more efficient. Assuming $t_f = 0.1$ s, optimun N_t is given in Tab. 2, and the other parameters of the multigrid are exposed in Tab. 5.

The discovery of the optimum parameters made it possible to compare TS and WR methods along with the CN and MCN smoothers with minimum t_{CPU} for each case. With the aid of a geometric adjust, we were able to analyze the order of complexity of the TS-CN, TS-MCN, WR-CN and WR-MCN methods and conclude that the TS method is more efficient than the WR method. The superiority of the MCN smoother over the CN smoother was verified with the computation of *speed up*. However, we found that when the spatial mesh is refined, t_{CPU} greatly increases, probably because the optimal relation between $N_x = N_y$ and N_t given in Tab. 2 is lost during the coarsening process of the meshes. This will be topic of further study.

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