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COMPARISON OF THE PHYSICAL ANISOTROPY OF MULTIGRID METHOD FOR TWO-DIMENSIONAL DIFFUSION EQUATION

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Abstract. *This work aims to analyze and optimize several multigrid parameters, while minimizing the CPU time in heat transfer problems that have anisotropy in the coefficients (physical anisotropy). The models considered are pure diffusion, that is, Poisson equation with anisotropy aligned with coordinate axis. For the discretization of the equation is used the finite difference method with uniform grids and numerical scheme of the second order (CDS) for the diffusive terms. The problems will be solved with the geometric multigrid method with Full Approximation Scheme (FAS) and V-Cycle with standard coarsening ratio (r=2), the restriction is given by injection and the prolongation by bilinear interpolation. In the solution of the equation system resulting from the discretization is used the MSI and GS as smoothing method. Several numerical experiments were carried out, varying the number of unknowns, inner iterations numbers and number of levels in the multigrid method. The main conclusion is that the anisotropy studied affects the performance of multigrid method, the CPU time and the order p of MSI and GS solvers.*

Keywords: *Physical anisotropy, Multigrid parameters, Diffusion*

1. INTRODUCTION

The Computational Fluid Dynamics (CFD) is an area of scientific computing that explores numerical methods for simulating problems involving moving fluids. These methods often require a high computational cost. The reason this occurs is because many times the problems you want to solve involve many variables and are associated with large coefficients and sparse matrices

The discretization of a mathematical model in the area of CFD can be performed by the Finite Difference Method (FDM) (Ferziger and Peric, 2002), which consists of, by approximate algebraic equations, each term of the mathematical model for each node point, or grid. This process leads to a system of algebraic equations of the form

$$AT = b. \tag{1}$$

Multigrid method is one of the most effective methods to accelerate the convergence of iterative methods and solve linear and non-linear systems, anisotropic problems, and more. However, for problems with strong anisotropy of the efficiency multigrid method may not be fully achieved. Anisotropic problems are very common in engineering and come in many physical simulations for several reasons. This phenomenon may reflect a physical model, such as material having anisotropic elasticity or anisotropic heat conduction. Anisotropy may also arise from the discretization grid where the coarsest grid is chosen in a coordinated way, to study physical characteristics (e.g., boundary layer).

The coefficients of differential equations, unit distinct and distincts from each other, may also generate physical anisotropy. For this type of problem, iterative methods will reduce efficiency.

Anisotropy physical problems were investigated by Rabi and Lemos (2001), which discretized the equation of two-dimensional advection-diffusion, with zero and constant coefficients. For the discretization of the equations the finite volume method was used. The multigrid method was employed with the CS scheme, Gauss-Seidel solver, using V and W-cycles. The authors presented a study for different velocity fields (nulls and constant), grid number, number of smoothing steps in each level and different solvers.

Karaa and Zhang (2002) presented a study of two-dimensional advection-diffusion equations with constant coefficients and variables. They concluded that the multigrid method degrades where the Reynolds number is high, i.e.; when there is strong anisotropy.

Fischer and Huckle (2006) developed the multigrid method for algebraic equations, where the anisotropy does not occur along the coordinate axes, but in arbitrary directions. Fischer and Huckle (2008) used the multigrid method developed by Fischer and Huckle (2006) and tested different methods for solving the algebraic equations. Further standard coarsening and semi-coarsening algorithms were used.

Suero *et al*. (2008) presented a study of the effect on the physical anisotropy with multigrid method to solve the two-dimensional advection-diffusion equation. They used the geometric multigrid method with the FAS correction scheme, V-cycle and MSI solver for solving algebraic equations. Different velocity fields were considered (null, constants and variables) and studied the influence of the number of inner iterations of the solver, grid number and number of variables. The main conclusion is that the physical anisotropy has a little influence on performance of multigrid because the greater the number of nodes of the grid, closer to the isotropic becomes a problem.

Oliveira *et al*. (2012) analyzed the geometric anisotropy for multiple threads and aspect ratios. And also analyzed some multigrid method parameters such as restriction types, number of levels and numbers of inner iterations between various coarsening algorithms. They conclude that the partial weighting (PW) coarsening performed well.

Dedner *et al*. (2014) showed the geometric efficiency multigrid method preconditioned to problems in geophysic anisotropic models. Good results were achieved for solving the pressure correction, found in the differential equation.

In this study, we intend to analyze the influence of physical anisotropy of the CPU time (t_{CPU}) to various parameters of the geometric multigrid method such as: solvers, number of inner iterations in the solver (*iti*), number of grids (*L*) and number of variables (*N*). The mathematical model considered is two-dimensional diffusion, where physical anisotropy appears in the coefficients and will be denoted in the text by diffusive anisotropy. For the discretization of the equations, a second order Central Difference Scheme (CDS) will be used in diffusive terms. The systems of equations obtained of the discretization are resolved using, Strongly Modified Implicit-MSI (Schneider and Zedan, 1981) and Gauss-Seidel (Briggs *et al*., 2000) solvers.

This paper is organized as follows: in Section 2 the mathematical and numerical models are presented. Section 3 presents the results of numerical experiments. Section 4 shows the conclusion of the work.

2. MATHEMATICAL AND NUMERICAL MODELS

As for the problems discussed below, the calculation domain used is given by $0 \le x, y \le 1$ and the discretization of the equations performed using uniform grid with a number of nodes given by $N = N_x . N_y$, where N_x e N_y are the numbers of nodes in the directions *x* and *y* coordinates, respectively, including the boundaries.

2.1 Mathematical models and discretization

Two problems at diffusive anisotropy will be analyzed from the two-dimensional diffusion equation given by

$$\begin{cases} -T_{xx} - \varepsilon T_{yy} = S \\ T(0,y) = T(x,0) = T(x,1) = T(1,y) = 0 \end{cases}, \tag{2}$$

where T is the variable and represents the temperature, T_{xx} and T_{yy} represent the second derivative of T as a function of *x* and *y*, respectively and $\varepsilon > 0$.

For the first problem, denoted by Diffusion1 in this text, you have the source term and the analytical solution given by

$$S = 2(y - y^2) + 2\varepsilon(x - x^2) \text{ and } T(x,y) = (x - x^2)(y - y^2). \tag{3}$$

For the second problem, denoted by Diffusion2 in this text, you have the source term and the analytical solution given by

$$S = 2\left[\left(1 - 6x^2\right)y^2\left(1 - y^2\right) + \varepsilon\left(1 - 6y^2\right)x^2\left(1 - x^2\right)\right] \text{ and } T(x,y) = (x^2 - x^4)(y^4 - y^2). \tag{4}$$

The discretization of the equations are obtained with a second order scheme (CDS) in diffusive terms as are given

$$a_P T_P + a_W T_W + a_N T_N + a_E T_E + a_S T_S = b_P. \tag{5}$$

The discretization of Eq. (2) results in Eq. (5) and for inner nodes, we have

$$a_P = \left(\frac{2}{h_x^2} + \frac{2\varepsilon}{h_x^2} \right),\ a_W = a_E = -\frac{1}{h_x^2},\ a_N = a_S = -\frac{\varepsilon}{h_y^2},\ b_p = S_p . \tag{6}$$

For north, south, east and west boundary conditions: $a_P = 1,\ a_W = a_E = a_N = a_S = 0$.

It´s been observed that the two Diffusion1 and Diffusion2 problems present diffusive anisotropy, but with different solutions.

2.2 *Multigrid* method and computational details

The system of linear equations given by Eq (1) is solved using the multigrid geometric method (Briggs *et al.*, 2000; Trottenberg *et al.*, 2001) with the FAS correction scheme, V-cycle and the null-value was taken as the initial guess for the whole domain.

The coarsening ratio $r = p/q$ is given by $r = 2$ (standard coarsening) (Wesseling and Oosterlee, 2001). The transfer of information among grids is made by injection in restriction and prolongation by bilinear interpolation. Systems of equations obtained discretization were resolved using the MSI and Gauss-Seidel (GS) solvers.

Each V-cycle is repeated until the achievement of a given stop criterion, which, in this work, is based on the non-dimensional norm, given below. The residue of the system of algebraic equations is defined by

$$R^n = b - AT^n , \tag{7}$$

where T^n is the variable solution in the *n* iteration.

Considering L^n how the norm 1 of residue in the n iteration and L^0 how the norm 1 of residue in the initial guess, if $\frac{L^n}{L^0} \le tol$ the iterative process is stopped. The norm used is given by $L^n = \sum_{P=1}^{N} \left| R_P^n \right|$ and tolerance is established by $tol = 10^{-7}$.

For the simulations we used double-precision arithmetic. The numeric codes were implemented using Fortran 2003 language, with the Intel Visual Fortran 9.1 application. All numeric results were obtained in a processor with the IntelCorei7 2.66 GHz, 16 GB of RAM and Windows 8 operating system, 64-bit computer.

The optimum value of a parameter is obtained when the solution of the problem has the lowest CPU time for fixed values of the other parameters. In this work, denoted by $iti_{optimum}$ the optimum number of inner iterations of the solver and by $L_{optimum}$ the optimal number of grid levels. Each problem size (*N*) has a $iti_{optimum}$ and $L_{optimum}$. The parameter $L_{maximum}$ is the maximum number of grid that can be used for each given grid being coarser than the grid with only one inner node. For example, if N=33×33 nodes, the grid are 33×33, 17×17, 9×9, 5×5 e 3×3 nodes, therefore, $L_{maximum} = 5$.

In this study, one of the goals is to check the influence of anisotropy in the performance of the multigrid method.

3. NUMERICAL RESULTS

Presented in this section, the results obtained for this study were the following parameters: the number of inner iterations (*iti*) and number of levels (*L*). These are considered N=129x129, 513x513 and 2049x2049. For each parameter, the optimal value is obtained, i.e.; the value which results in less CPU time. To analyze the number of variables (*N*), the problems N=5x5, N=9x9, N=17x17, N=33x33,..., N=4097x4097 are considered. It also presents a performance comparison of solvers MSI and GS. The notations used to present the results are as follows: Diffusion1-MSI, Diffusion1-GS, Diffusion2-MSI and Diffusion2-GS.

Equation (2) was analyzed for the following values of ε: 0.0001, 0.001, 0.01, 0.1, 1, 10, 100, 1000, 10000. When $\varepsilon = 10^a$ and $\varepsilon = 10^{-a}$, com $a \in Z$ in this study, we denote symmetric anisotropy. For example, if $\varepsilon = 10$ and $\varepsilon = 0.1$, we have a symmetric anisotropy.

3.1 Number of inner iterations (*iti*)

To obtain the optimum number of inner iterations, each problem was solved by using the maximum number of possible levels, i.e., $L_{maximum}$. Values of *iti* were tested between 1 and 20 for the solver MSI. To solver GS we tested

between 1 and 5000, since, for strong anisotropy, this solver reaches convergence with a large number of inner iterations.

It can be seen in Fig. 1(a) and Fig. 1(b) that, the isotropic problem ($\varepsilon = 1$), Diffusion1-MSI, Diffusion 2-MSI and Diffusion 2-GS, $iti_{optimum} = 3$; and for Diffusion1-GS $iti_{optimum} = 4$. For symmetric anisotropy $iti_{optimum}$ are symmetrical with GS. It is also observed that as the problem becomes more anisotropic with $\varepsilon \to 0$, Diffusion1 and Diffsion2 with both solvers have a great number to $iti_{optimum}$. It is also observed that, $iti_{optimum}(GS) >> iti_{optimum}(MSI)$. For anisotropies with $\varepsilon \to \infty$, inner number of iterations tends to oscillate for MSI (e.g., ranges between 6 and 9, for the ε values) and tends to grow too large to GS. This behavior occurs, as the fact is that the MSI is considered a strongly implicit method for the system of equations and is solved with little iteration with similar behavior to that of a direct method.

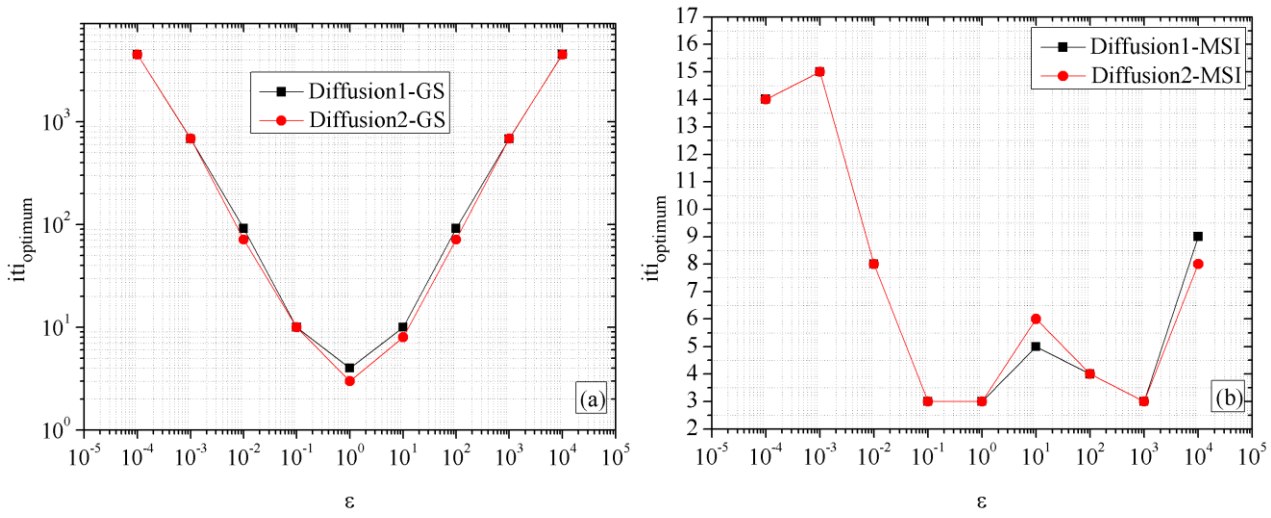

Figure 1. Optimum number of inner iterations ($iti_{optimum}$) *versus* anisotropy coefficients (ε)

3.2 Number of levels (*L*)

To determine the optimal number of levels, were employed optimum inner iterations values obtained in the previous section. Figure 2 shows $L_{optimum}$ some of the studied problems. It is observed that the isotropic problem Diffusion1-GS and Diffusion2-GS $L_{optimum} = L_{maximum}$; Diffusion1-MSI e Diffusion2-MSI $L_{optimum} = L_{maximum} - 2$.

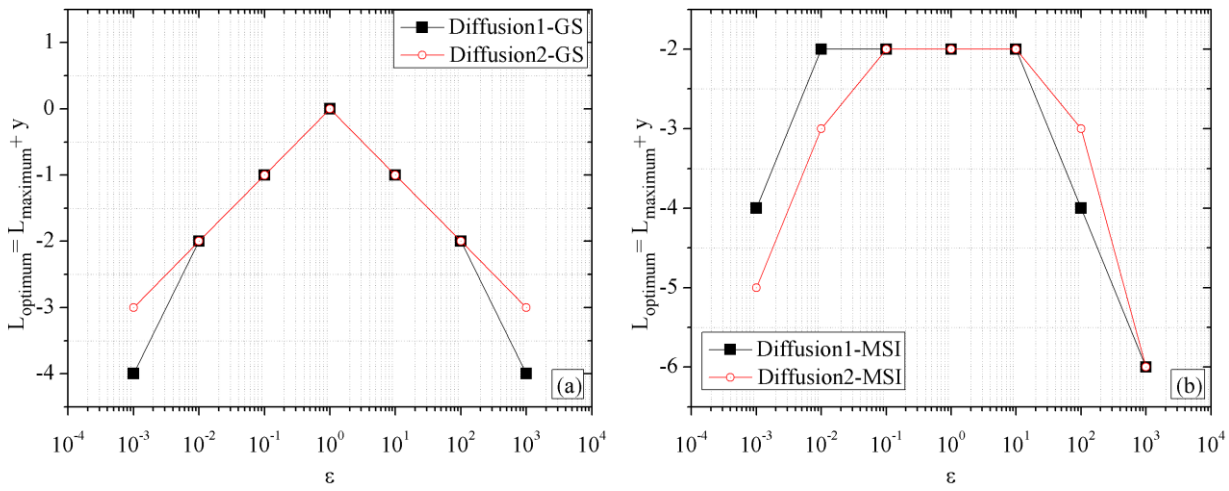

Figure 2. Optimum number of levels ($L_{optimum}$) *versus* anisotropy coefficients (ε)

As the problem becomes more anisotropic, $L_{optimum}$ becomes $L_{maximum}-1$, $L_{maximum}-2$, $L_{maximum}-3$, and so on, depending on the anisotropy. If we adopt $L_{optimum}=L_{maximum}$ for all these problems we will have a small difference in CPU time for those cases where $L_{optimum}$ is not $L_{maximum}$.

In relative terms $\left(E(\%)=\frac{t_{CPU}(L_{maximum})-t_{CPU}(L_{optimum})}{t_{CPU}(L_{maximum})}\times 100\%\right)$, the biggest error was observed for Diffusion2-MSI in grid with 2049x2049 nodes, a 9.8% difference for $\varepsilon=10000$.

One can make a geometric fit given by

$$t_{CPU}=cN^{p}. \tag{8}$$

The *c* parameter is a coefficient that depend of the method and the solver, *p* is the order of complexity of the solver associated with the employee and the *N* is the dimension of the problem. The closer the unit is to the exponent *p*, the better the performance of the method. The ideal multigrid method is one in which *p=1*, that is, the CPU time increases linearly with the number of variables (*N*).

The order of complexity (*p*) comparing the use of $L_{optimum}=L_{maximum}$ and $L_{optimum}=L_{maximum}-7$ was calculated. The particular case N=2049x2049 with $\varepsilon=10000$ can be seen in Tab. 1 the parameter *p* has a little variation when used $L_{optimum}=L_{maximum}$. Therefore, for the programming facility and the favorable data results in Fig. 2, will be adopted $L=L_{maximum}$ for all problems.

Table 1. Comparison of complexity order (*p*), N=2049x2049 for $\varepsilon=10000$

Number of levels	*p*
$L_{optimum}=L_{maximum}$	1.16405
$L_{optimum}=L_{maximum}-7$	1.15041

3.3 Numbers of variables (N)

To evaluate the effect of the number of variables of the CPU time, optimal parameters determined above were used. It can be seen in Fig. 3(a) with the GS solving the problems of Diffusion1 and Diffusion2 isotropic ($\varepsilon=1$) present t_{CPU} less for size studied problems. It is observed that, as the problem becomes more anisotropic ($\varepsilon\to 0$ and $\varepsilon\to\infty$), t_{CPU} it increased. However, the anisotropy $\varepsilon=0.01, 0.1, 10, 100$, t_{CPU} is approximately the same as the isotropic problem.

In Fig. 3(b) it's shown t_{CPU} for all anisotropy studied in Diffusion1 and Diffusion2 problems with MSI solver. For $\varepsilon\to\infty$, t_{CPU} decrease to the analyzed *N* values. However, for $\varepsilon\to 0$, increase t_{CPU}. For both problems studied, Diffusion1 and Diffusion2, and for all anisotropy coefficients, t_{CPU} it is much less using the solver MSI that the solver GS.

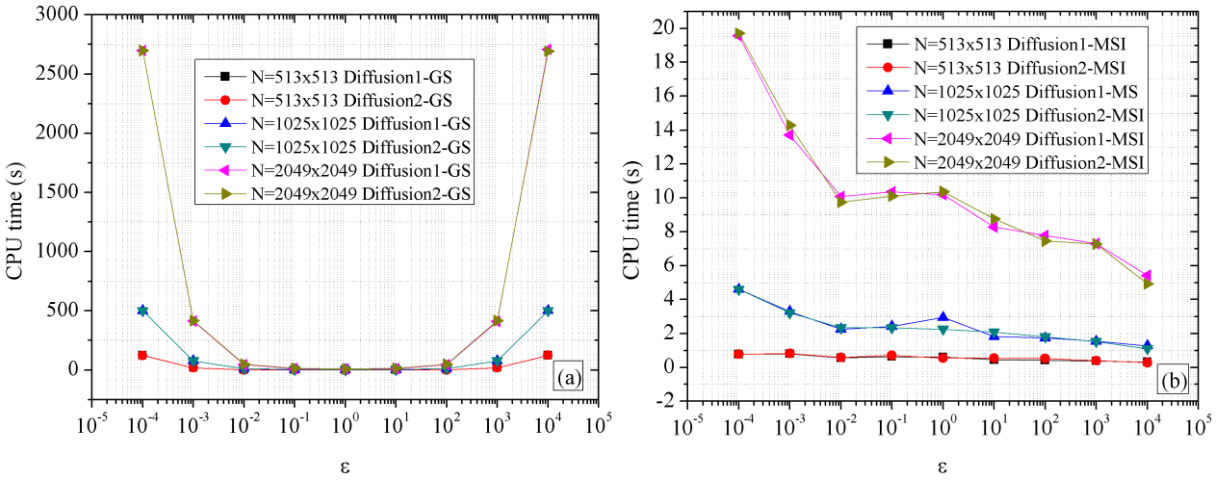

Figure 3. CPU time *versus* anisotropy coefficients (ε) for N=513x513, N=1025x1025 and N=2049x2049

It can be seen in Fig. 4(a) and Fig. 4(c) that, for the isotropic problem ($\varepsilon = 1$), t_{CPU} is the lowest for all values of N studied. For symmetric anisotropy, t_{CPU} are very close. For example, for the $\varepsilon = 0.1$ e $\varepsilon = 10$ lines almost overlap in Fig. 4(a) and Fig. 4(c). It is also observed that how much more anisotropic the problem, greater t_{CPU} for all *N* values studied.

Figure 4(b) and Fig. 4(d) with the MSI solver show that the isotropic problem ($\varepsilon = 1$), t_{CPU} is not the smallest for all N values studied. There seems to be a symmetry between the amounts of CPU time with respect to the symmetric anisotropy. For example, to $\varepsilon = 0.0001$ and $\varepsilon = 10000$, the lines relating to the CPU times are the most spaced ones and, besides, are symmetrical with respect to the line $\varepsilon = 1$. It is also noted that for all values of *N* studied, if $\varepsilon \to \infty$, we have t_{CPU} less than $\varepsilon \to 0$, i.e., $t_{CPU}(\varepsilon \to \infty) < t_{CPU}(\varepsilon \to 0)$, confirming the appointed behavior in Fig. 3(b).

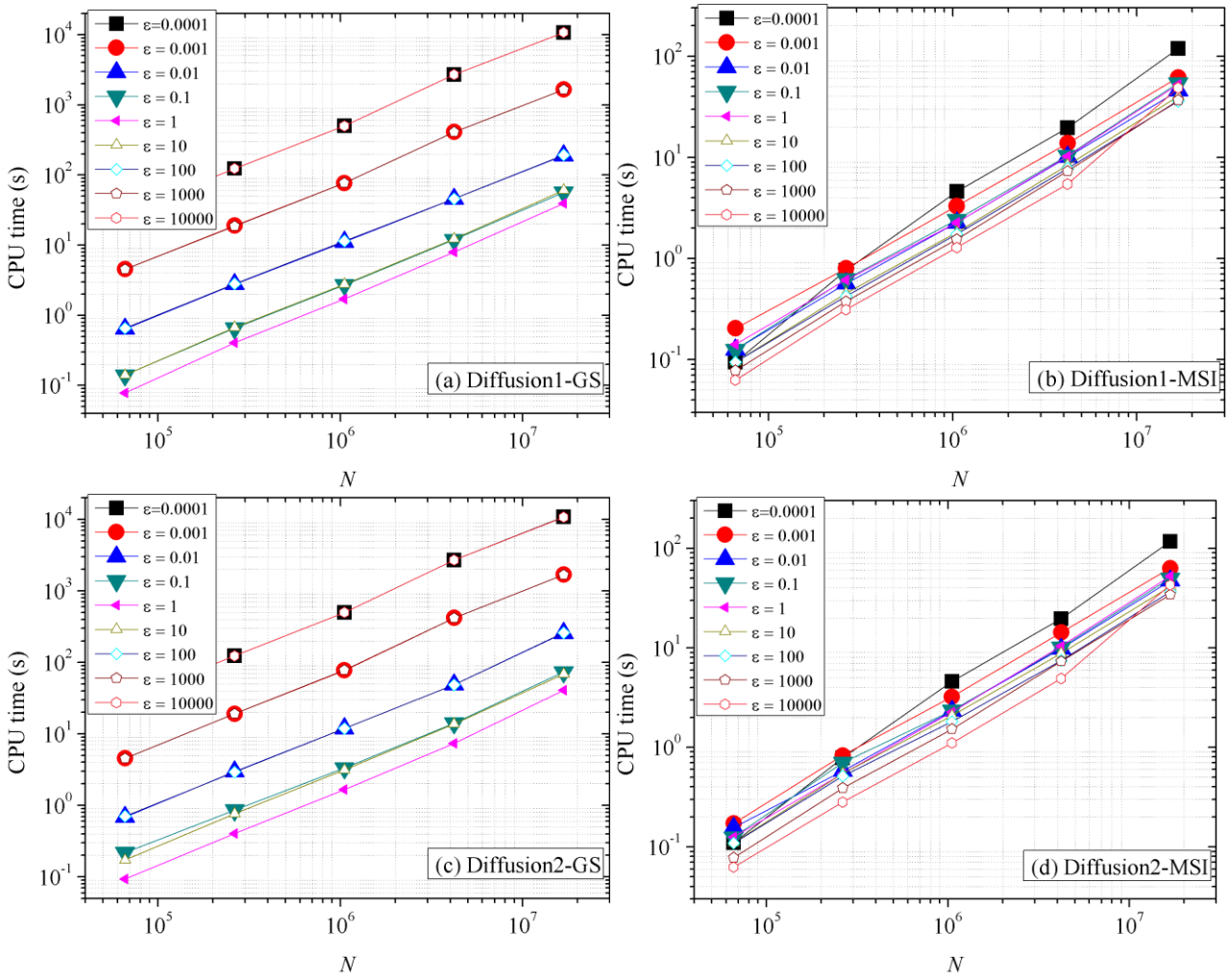

Figure 4. CPU time *versus* number of nodes (*N*)

In the same way as presented in Tab. 1, one can make a geometric fit $t_{CPU} = cN^p$, for different values of *N* for a given ε, and the various values to ε. The result is shown in Fig. 5, considering the two problems addressed (Diffusion1 and Diffusion2) and the two solvers (GS and MSI). Remember that the ideal multigrid method is one whose *p=1*. It can be seen from Fig. 5 that, the intermediate anisotropies multigrid method has a good performance, but a strong anisotropy, the order of complexity (*p*) moves away from the unit and the method performance becomes unsatisfactory. This behavior was expected, as for strong anisotropy, iterative methods reduce the efficiency.

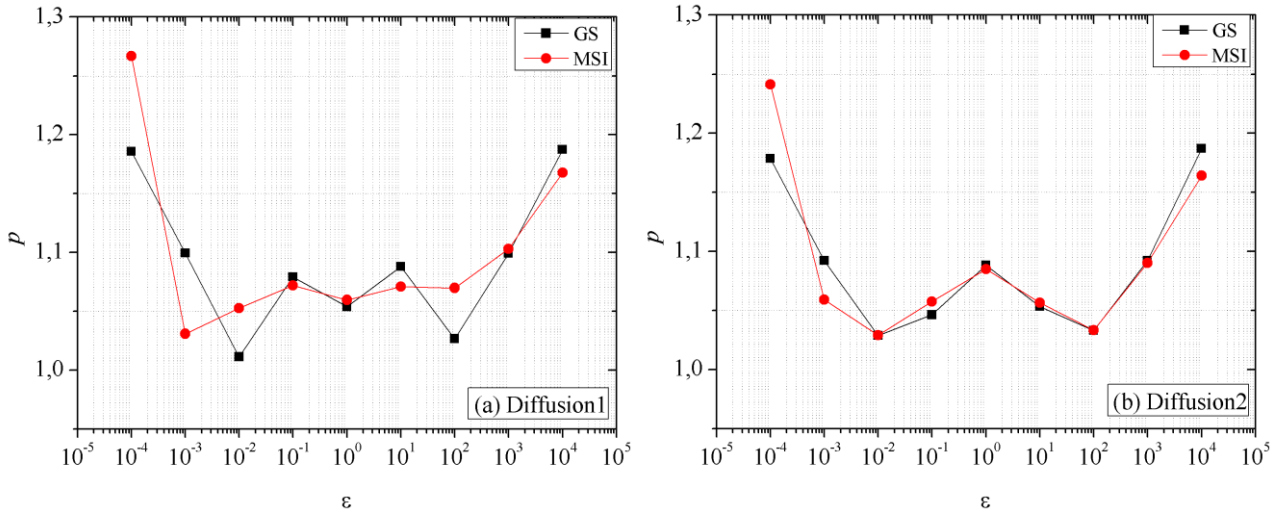

Figure 5. Complexity order (p) *versus* anisotropy coefficients (ε)

4. CONCLUSION

This work examined the effect of physical anisotropy of the multigrid method for two-dimensional problems of diffusion anisotropy with the diffusive terms (diffusive anisotropy), denoted by Diffusion1 and Diffusion2. The following multigrid parameters were studied: solver, inner iterations numbers, number of levels and effects of variables numbers in CPU time. The geometric multigrid method was employed with the FAS correction scheme, V-cycle, null-value for the initial guess, and *r=2*. The solvers studied were Modified Strongly Implicit (MSI) and Gauss-Seidel (GS). The conclusions reached are presented below:

- The studied problems show that $iti_{optimum} \cong 3$ the isotropic case ($\varepsilon = 1$).

- For the problems studied, both with the solver GS, as MSI, the optimal number of inner iterations grows as the problem becomes more anisotropic ($\varepsilon \to 0$ and $\varepsilon \to \infty$).

- This may be concluded $iti_{optimum}(MSI) << iti_{optimum}(GS)$ for all ε values studied.

- $t_{CPU}(MSI) << t_{CPU}(GS)$ for all ε values, except for the isotropic problem ($\varepsilon = 1$).

- For the isotropic problem ($\varepsilon = 1$) the order of complexity p of the multigrid method for Diffusion1 with both solvers is $p \cong 1.08$ as well as Diffusion2 is $p \cong 1.05$. For strong anisotropy ($\varepsilon \to 0$ and $\varepsilon \to \infty$) the order of complexity p moves away from the unit and method performance becomes unsatisfactory.

- $p(L_{maximum}) \approx p(L_{optimum})$ both of the solvers as to both problems, namely, physical anisotropy does not greatly influence the optimal number of grids.

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