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L-scheme and modified Picard with multigrid method for a 1D two-phase problem in rigid porous media with analytical solution

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Abstract. Applications of two-phase problems in porous media are common in Geomechanics, Hydrogeology, Engineering and Biomedicine. There are different formulations when working on two-phase problems, in this work we have chosen to use the pressure-pressure formulation. The equations system generated is a strongly non-linear system of coupled partial differential equations. Thus, the modified Picard and L-scheme to perform its linearization, the Finite Volume Method for the discretization of the equation in space and implicit Euler scheme for the discretization of the equation in time were used. The systems of linear equations generated were solved by the lexicographic Gauss-Seidel solver in a coupled way. In this work, we proposed to use multigrid method with the Correction Scheme and W-cycle, in order to accelerate the convergence of this solver. Based on the tests performed using an example with a known analytical solution, it was possible to notice the convergence to the solution with a few iterations and little computational time.

Keywords: Two-phase flow, Linearization methods, Coupled problem, Finite Volume Method, Implicit Euler.

1 Introduction

Problems with the two-phase flow in porous media are found in Engineering, Geomechanics, Hydrogeology and Biomedicine applications. For the study of these problems, different mathematical models are generated to represent them, depending on the pressure, saturation and relative permeability, being subsequently solved by numerical simulations. Independent of how these models are formulated, coupled differential equations and highly non-linear systems are generated. Therefore, the challenge is to find robust and efficient methods for the numerical solution.

Many articles are found in the literature involving two-phase flow in porous media, using different methods and approaches to variables. A numerical algorithm, based in modified Picard linearization is proposed by Celia and Binning [1] for simulation of these problems, considering unsaturated soils and pressure-pressure formulation. Kvashchuk and Radu [2] presented a new implicit scheme based in IMPES (Implicit Pressure Explicit Saturation), that obtained a superior performance in relation to the standard IMPES.

Considering the two-phase flow in porous media with dynamic capillarity effects, Karpinski et al. [3] proposed a linearization scheme, called L-scheme, that does not require regularization step, besides not using derivate calculations like the Picard and Newton method. Proved that the scheme is robust and linearly convergent. That procedure linearization was also presented by Pop et al. [4] to solve non-linear elliptical problems.

Illiano et al. [5] applied three techniques, Newton, modified Picard and L-scheme methods, for linearization of the surfactant transport in porous media. They concluded that monolithic Newton is the only method with

quadratic convergence, modified Picard and Newton generated ill-conditioned matrices and that solvers based on L-scheme were the most robust because produce well-conditioned linear systems.

Because this problem is strongly non-linear, some works consisted of manipulating the expressions that interfere in this fact. Li and Horne [6] compared some methods, for example, Purcell and Brooks-Corey methods, to calculate the relative permeability of the capillary pressure in a consolidated wetting porous media. Being that, permeability can be satisfactory if a suitable model is chosen for the problem under study.

Most of the literature found has the main focus on the analysis of different linearization methods. But, thinking about the solver convergence, Franco et al. [7] used a new approach with the use of space-time multigrid method for solving poroelasticity equations, obtaining excellent results.

Therefore, in this work, we study a problem involving the flow of two incompressible and immiscible fluids in rigid porous media. Using pressure-pressure formulation modeling, where the variables of interest are the pressures of each both phases, thus, relative permeability and saturation were calculated by numerical expressions that depend on the pressures. Discretizations in time and space were carried out by the Implicit Euler and Finite Volumes Methods (FVM) (Ferziger et al. [8]), respectively. As we have a non-linear system, so we have chosen to apply and compare two methods of linearization, modified Picard (Celia and Binning [1]) and L-scheme (Karpinski et al. [3], Pop et al. [4]), and later to solve the linear system, used iterative method, coupled Gauss-Seidel (Gaspar et al. [9]). To accelerate the convergence of solver, we proposed to use multigrid method (MG) (Briggs et al. [10]).

The rest of the paper is organized as follows. In Section 2, the one-dimensional porous media equations are introduced together with their linearization and discretization techniques in time and space. Solver and multigrid method for the equation of porous media is detailed in Section 3. The code verification and results are demonstrated in Section 4. Finally, conclusions are drawn in Section 5.

2 Mathematical Model and Discretization

In this section, we present the mathematical model and its discretization, for the two-phase flow in a rigid porous medium.

2.1 Government equations

The fluids considered were immiscible and incompressible with the flow in a rigid porous medium. For each α phase of the fluid, the mass equation can be written as:

$$\frac{\partial\left(\rho_\alpha \theta_\alpha\right)}{\partial t} + \bigtriangledown \cdot \left(\rho_\alpha \vec{q_\alpha}\right) = F_\alpha, \quad \text{in} \quad \Omega \times [0, T]. \tag{1}$$

where $\alpha = w, n$ represents the fluid phases (w wetting, n non-wetting), $\theta_\alpha = \phi S_\alpha$, ϕ is the porosity, S_α is the saturation, ρ_α is the density, $\vec{q_\alpha}$ is the volumetric flux vector and F_α is the source term of phase α. The domain is $\Omega \subset \mathbb{R}^+$ and T is the final time. The volumetric flow is given by the generalized Darcy's Law for the multiphase case

$$\vec{q_\alpha} = -\lambda_\alpha \mathbf{K}\left(\bigtriangledown p_\alpha - \rho_\alpha \vec{g}\right), \tag{2}$$

where λ_α is the mobility, $\lambda_\alpha = k_{r\alpha}/\mu_\alpha$, $k_{r\alpha} = k_{r\alpha}(S_\alpha)$ is the relative permeability, μ_α is the viscosity, $\mathbf{K}$ is the intrinsic permeability tensor (see Bastian [11]), p_α is the pressure, $\vec{g}$ is the gravitational acceleration vector.

Substitution of eq.(2) into eq.(1), and consider incompressible case and null gravitational acceleration vector, the equation can be simplified:

$$\frac{\partial \theta_\alpha}{\partial t} - \lambda_\alpha \bigtriangledown \cdot \left(\mathbf{K} \bigtriangledown p_\alpha\right) = \frac{F_\alpha}{\rho_\alpha}. \tag{3}$$

In addition to these differential equations, we have the auxiliary relations (Bastian and Helmig [12]): capillary pressure $p_c = p_n - p_w$ and saturation $S_w + S_n = 1$, so, $\theta_w + \theta_n = \phi$.

2.2 Discretization

We begin with a implicit Euler time discretization, use the modified Picard linearization (Celia and Binning [1]) to linearize the equations and consider pressure-pressure formulation (Ataie-Ashtiani and Raeesi-Ardekani

[13]). It lets superscripts n and m denote time level and iteration number, respectively. The distance between points of temporal approximation is $\tau = \frac{T}{N_t}$, where, T is final time and N_t is points number in temporal discretization. Then discretizing the eq. 3 for wetting and non-wetting phases, we get eqs. (4, 5), respectively,

$$C_w^{n+1,m}\frac{\delta p_n^{n+1,m+1}-\delta p_w^{n+1,m+1}}{\tau} - \frac{\partial}{\partial x}\left[K_w^{n+1,m}\frac{\partial}{\partial x}\left(\delta p_w^{n+1,m+1}\right)\right] = \frac{\partial}{\partial x}\left[K_w^{n+1,m}\frac{\partial}{\partial x}\left(p_w^{n+1,m}\right)\right] + F_w^{n+1} - \frac{\theta_w^{n+1,m}-\theta_w^n}{\tau}, \tag{4}$$

$$-C_w^{n+1,m}\frac{\delta p_n^{n+1,m+1}-\delta p_w^{n+1,m+1}}{\tau} - \frac{\partial}{\partial x}\left[K_n^{n+1,m}\frac{\partial}{\partial x}\left(\delta p_n^{n+1,m+1}\right)\right] = \frac{\partial}{\partial x}\left[K_n^{n+1,m}\frac{\partial}{\partial x}\left(p_n^{n+1,m}\right)\right] + F_n^{n+1} - \frac{\theta_n^{n+1,m}-\theta_n^n}{\tau}, \tag{5}$$

where $K_\alpha^{n+1,m} = \mathbf{K}\lambda_\alpha$, $\delta p_\alpha^{n+1,m+1} = p_\alpha^{n+1,m+1} - p_\alpha^{n+1,m}$ and $C_w = \frac{\partial \theta_w}{\partial p_c} = -\frac{\partial \theta_n}{\partial p_c}$. But, for L-scheme using L_s, large enough, in place of C_w, that is $L_s \geq |C_w|$ (Illiano et al. [5]).

Subsequently, space discretization was done using the FVM. For that, our domain will be a segment of lenght L and considering uniform mesh, $D_h = \{(x_i); x_i = (i-1/2)h, i = 1, ..., N_x\}$, with N_x volume number in space and $h = \frac{L}{N_x}$ distance between volumes of the space approximation, given in Fig. 1.

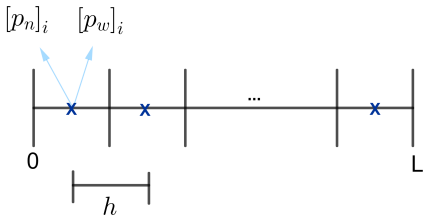


Figure 1. Space discretization

Thus, the system that we should result in each time step is described in eq. (6).

$$\begin{bmatrix} A_w & B \\ B & A_n \end{bmatrix}\begin{bmatrix} \delta p_w \\ \delta p_w \end{bmatrix} = \begin{bmatrix} f_w \\ f_n \end{bmatrix}, \tag{6}$$

being that,

$$A_\alpha = \begin{bmatrix} bc & 0 & & & & & \\ [a_\alpha]_{i-1} & [a_\alpha]_i & [a_\alpha]_{i+1} & 0 & & & \\ 0 & [a_\alpha]_{i-1} & [a_\alpha]_i & [a_\alpha]_{i+1} & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & & 0 & [a_\alpha]_{i-1} & [a_\alpha]_i & [a_\alpha]_{i+1} \\ & & & & & 0 & bc \end{bmatrix}, B = \begin{bmatrix} bc & 0 & & & & & \\ 0 & c_i & 0 & & & & \\ & 0 & c_i & 0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & 0 & c_i & 0 \\ & & & & & 0 & bc \end{bmatrix},$$

where, bc are boundary conditions and:

$$[a_\alpha]_i = -\left[C_w^{n+1,m}\right]_i + \frac{\tau}{h^2}\left(\left[K_\alpha^{n+1,m}\right]_{i+\frac{1}{2}} + \left[K_\alpha^{n+1,m}\right]_{i-\frac{1}{2}}\right),$$
$$[a_\alpha]_{i+1} = -\frac{\tau}{h^2}\left[K_\alpha^{n+1,m}\right]_{i+\frac{1}{2}},$$
$$[a_\alpha]_{i-1} = -\frac{\tau}{h^2}\left[K_\alpha^{n+1,m}\right]_{i-\frac{1}{2}},$$
$$c_i = \left[C_w^{n+1,m}\right]_i$$
$$[f_\alpha]_i = \frac{\tau}{h^2}\left[K_\alpha^{n+1,m}\right]_{i+\frac{1}{2}}\left[p_\alpha^{n+1,m}\right]_{i+1} - \frac{\tau}{h^2}\left(\left[K_\alpha^{n+1,m}\right]_{i+\frac{1}{2}} + \left[K_\alpha^{n+1,m}\right]_{i-\frac{1}{2}}\right)\left[p_\alpha^{n+1,m}\right]_i + \frac{\tau}{h^2}\left[K_\alpha^{n+1,m}\right]_{i-\frac{1}{2}}\left[p_\alpha^{n+1,m}\right]_{i-1} + \tau F_\alpha - \left[\theta_\alpha^{n+1,m}\right]_i + \left[\theta_\alpha^n\right]_i,$$

where $\left[K_\alpha^{n+1,m}\right]_{i-\frac{1}{2}}$ and $\left[K_\alpha^{n+1,m}\right]_{i+\frac{1}{2}}$ denote the interblock conductivities of each phase, calculated by arithmetic mean.

3 Solver and Multigrid Method

To solve the system of linear equations that appear in each linearization step, there are direct and iterative methods. Direct methods are not efficient in cases where matrices are large and sparse (Burden et al. [14]). In this article, we use the coupled Gauss-Seidel method.

However, these methods present high-frequency error smoothing properties (oscillatory components), while low-frequency errors are virtually unchanged. Therefore, in the first iterations, the error is reduced quickly and later very slowly, with a predominance of smooth modes (Briggs et al. [10], Trottenberg et al. [15], Oliveira et al. [16]). So the MG technique is based on solving the problem in several grids, because after the smoothing the oscillatory modes in a refined grid, these modes are transferred for a coarse-grid, where they become more oscillatory and the convergence of the iterative method is efficient. That procedure can be repeated up to coarsest or desired grid. For the transfer process between grids, restriction and prolongation operators are required with their respective pre- (ν_1) and post-smoothing (ν_2) numbers.

There are several ways to go through the various involved grids, which we call a cycle. In this work, we solve the eq. 6 with the MG method, using the W-cycle and null initial estimative. The ratio between the size of the volumes of the fine grid (Ω^h) and the immediately coarse-grid Ω^H is defined as the grid coarsening ratio (r). In this work, we use $r = 2$ (standard coarsening) (Wesseling [17]). For the stop criterion we use $||R^m||_\infty/||R^0||_\infty \leq TOL_{MG}$, where R^m is the residual in the iteration m, R^0 is the residual in the initial guess and TOL_{MG} is the tolerance for the W-cycle.

4 Results

In this work we use the analytical solution of the problem proposed by Illiano [18] (section 5.1, p.37). In that work, Illiano [18] considers the pressure-saturation formulation of the $\bar{p}$–S_w, where $\bar{p} = \frac{p_w+p_n}{2}$. In this case, we have the analytical solution $f(x,t) = \bar{p}(x,t) = S_w(x,t) = xt(1-x)$, whose spatial x temporal domain is $D = [0, L] \times [0, T], L = T = 1$, with initial and boundary conditions $f(x, 0) = f(0, t) = f(1, t) = 0$. As we have opted for pressure-pressure formulation of the form p_w–p_n, we have to make some adaptations to use p_w and p_n instead of $\bar{p}$. For this, using the capillary pressure equation ($p_c = p_n - p_w$) and $\bar{p}$, we have obtained that $p_w = \bar{p} - \frac{p_c}{2}$ and $p_n = \bar{p} + \frac{p_c}{2}$, where $p_c(S_w) = 1 - \frac{1}{2}S_w^2$. Additionally, we use $\theta_\alpha = \phi S_\alpha$, thus $\theta_w = \phi\sqrt{2 - 2p_c}$ and $\theta_n = \phi - \theta_w$. As a consequence, we have: $C_w = \frac{\partial \theta_w}{\partial p_c} = -\frac{\phi}{\sqrt{2-2p_c}}$, for $p_c \neq 1$.

Using these expresisons we have found the source terms:

$$F_w = -\frac{1}{2}\rho_w[2\phi(x-1)x + K\lambda_w t(-4 + t - 6tx + 6tx^2)], \tag{7}$$

$$F_n = \frac{1}{2}\rho_n[2\phi(x-1)x + K\lambda_n t(4 + t - 6tx + 6tx^2)]. \tag{8}$$

According to Illiano [18] data were used to obtain easy calculations, which are presented in test 1 of Table 1. Table 1 we also find data from test 2.

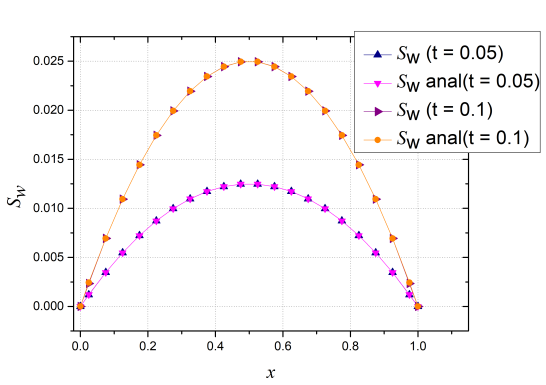
Table 1. Properties

	λ_w	λ_n	K	ϕ	ρ_w	ρ_n
Test 1 (Illiano [18])	1	2	1	1	1	1
Test 2	1	2	1	0.9	1	1

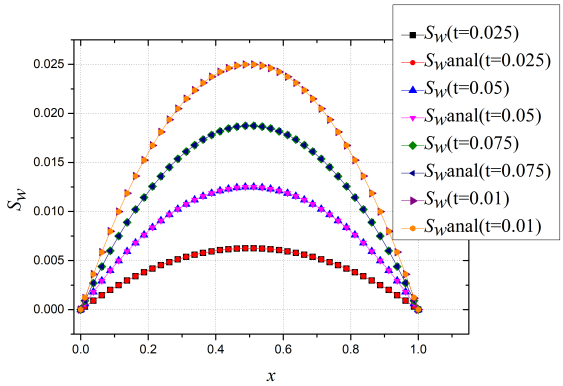
We implemented the algorithms in the Scilab 6.1.0 language on a computer with an Intel Core i7 2.6 GHz processor, 8 GB of RAM, and Windows 10 operating system, with 64 bits.

4.1 Code Verification

In order to verify our code, we reproduce the figure 5.1 of Illiano et al. [5], which presents the numerical and analytical solution obtained for the saturation for the test 1 data of Table 1 for several steps in time. These generated graphs are presented in Fig. 2, where is possible to see that all of our solutions (different grids size and in different time steps) coincide with the Illiano's analytical solutions (Illiano [18]).



(a) S_w for $h = \tau = 0.05$ with $TOL_L = 10^{-9}$



(b) S_w for $h = \tau = 0.025$ with $TOL_L = 10^{-9}$

Figure 2. Analytical and numerical saturation obtained in first step time with different meshes: (a) $N_x = N_t = 20$ and (b) $N_x = N_t = 40$

In order to analyze the MG implementation we made a comparison with Singlegrid (SG) (single grid method). For the SG and MG, we used the coupled Gauss-Seidel solver. Besides, for MG we used the following transfer operators between grids: restriction by full weighting and prolongation by linear interpolation (Trottenberg et al. [15], Rutz et al. [19]). In this work, we have used the maximum number of levels and data from the Table 2, being that, ν_1 and ν_2 is the pre- and post-smoothing number, respectively, ν_0 the number of smoothing in the coarsest grid, TOL_{MG}, the tolerance for the W-cycle and TOL_L tolerance for the linearization.

Table 2. Implementation data

ν_1	ν_2	ν_0	TOL_{MG}	TOL_L
5	5	5	$10E-5$	$10E-8$

In Table 3 we presented the required total CPU time (t_{CPU}) (which takes into account all the time steps and their respective necessary linearizations) for each method and the speedup ($S = \frac{t_{CPU}\text{-SG}}{t_{CPU}\text{-MG}}$, that is, S represents how many times the MG is faster than the SG). Here, we opted only for spatial refining, which is already sufficient to verify the desirable properties of MG concerning SG. Note that $S > 1$ in all cases, that is, MG is always faster than SG. Even more, as we refine the grid, S gets bigger and MG gets more efficient.

Table 3. CPU time for MG and SG

N_t	N_x	t_{CPU} - SG	t_{CPU}- MG	S
16	16	24.860	3.630	6.848
16	32	175.394	10.013	17.516
16	64	1310.780	27.618	47.460
16	128	9892.466	62.690	157.799

We performed a geometrical adjustment of the type $t_{CPU} = c(N_x)^p$ to analyze the performace of MG, where c is a constant relative to the method and p represents the order of the algorithm. Considering the first time step and the first linearization in different loops, $N_x = 4, 8, 16, 32, 64, 128, 256, 512$ and 1024, we obtained $c = 0.0127$ and $p = 1.1103 \approx 1$, according to the literature (Trottenberg et al. [15]).

4.2 Results in a specific porous medium

In order to obtain the following results, we performed several simulations, comparing the linearization methods, modified Picard and L-scheme, with $L_s(t) = max(|C_w|)$. The data used were those in the Table 1, others data in the Table 2 and maximum number of linearization iterations, $itmax_L = 500$.

Table 4 shows $itme_L$, arithmetic mean linearization iterations and $itme_{MG}$, arithmetic mean of MG cycles. Thus, we can see that the number of iterations in MG is always a small number, concerning linearization methods. We also realized that for this specific problem, modified Picard needs fewer iterations that the L-scheme, becoming the most efficient.

Table 4. Modified Picard and L-scheme with Multigrid

$N_x = N_t$	Modified Picard				L-scheme			
	Test 1		Test 2		Test 1		Test 2	
	$itme_L$	it_{MG}	$itme_L$	it_{MG}	$itme_L$	it_{MG}	$itme_L$	it_{MG}
4	5.25	1.50	5.25	1.50	16.25	1.12	16.00	1.13
8	4.63	2.50	4.63	2.50	34.13	1.12	33.75	1.12
16	4.31	2.50	4.31	2.50	65.19	1.06	64.69	1.06
32	3.75	2.67	3.72	2.67	117.66	1.03	117.00	1.03
64	3.22	2.67	3.22	2.67	228.94	1.02	244.20	1.02

Figure 3 presents the infinity norm of the difference between the analytical and numerical solution of p_α *versus* $N_x = N_t = 4$ up to 64 for linearization schemes studied, the results obtained show that, regardless of the linearization scheme used, the errors were essentially the same and decrease with the grid refinement.

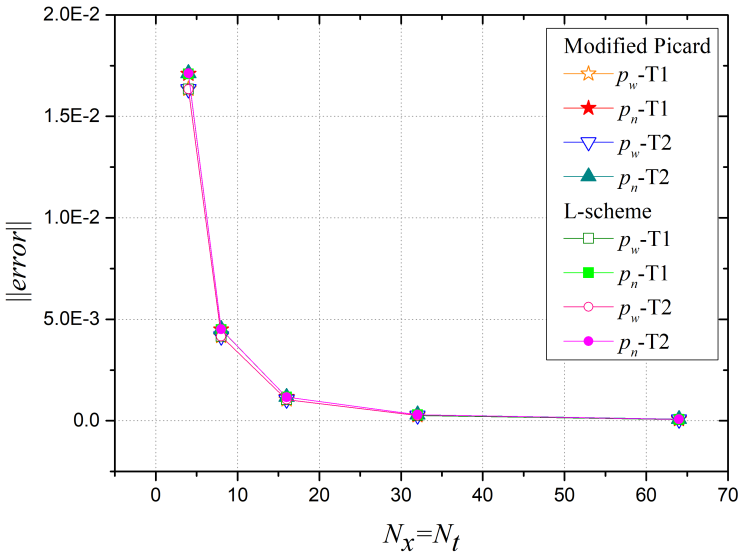


Figure 3. Infinity norm of the numerical error *vs.* $N_x = N_t$ for Tests 1 and 2, with modified Picard and L-scheme

5 Conclusions

In this work, we have analyzed two linearization methods, together with MG, for a two-phase flow problem in a rigid porous media. Initially, we carried out some tests to verify our code, compared the numerical with analytical solutions, then analyzed the speedup of MG in relation to SG, where we obtained good results. With that, we were able to generate some results and to realize the use of the linearization methods, modified Picard and L-scheme, with the MG method, generated good results, because the numbers of iterations necessary for both linearization and MG, were low numbers. The numerical error performed well because as we refined the grid, the error decreased. Therefore, we can conclude that the combination we have used in this article is efficient. Now, among the two linearization methods studied, the modified Picard with MG method was the only which presented the best performance to the problem studied.

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Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors.

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